LA 3

Linear Algebra (Spring 2005)

Problems 1–2: Elementary properties of vector spaces. Let V be a vector space over F.

1. Prove the following elementary laws (for arbitrary $a, b \in F$ and $\mathbf{u}, \mathbf{v} \in V$):

$$-(-\mathbf{u}) = \mathbf{u} \tag{1}$$
$$0 \cdot \mathbf{u} = \mathbf{0} \tag{2}$$
$$a \cdot \mathbf{0} = \mathbf{0} \tag{3}$$
$$(-1)\mathbf{u} = -\mathbf{u} \tag{4}$$
$$a(-\mathbf{u}) = -(a\mathbf{u}) \tag{5}$$

2. Show that if $a\mathbf{u} = \mathbf{0}$ then $a = 0$ or $\mathbf{u} = \mathbf{0}$ (where $a \in F$ and $\mathbf{u} \in V$).

Definition. Let V be a vector space. A subset W of V is called a *vector subspace* or a *linear subspace* or just a *subspace* if all the data (D1–D4) of V is closed on W (that is, each restricts to W to produce the same type of data for W).

Problems 3–7: Subspace properties. Let V be a vector space over F, and let W be a subset.

3. Make the above definition more explicit and precise. Show that if W is non-empty, then closure of D3 and D4 follows from closure of D1 and D2. Show that the set of polynomials of degree at most k is a vector subspace of $V = F[x]$. What about the set of polynomials satisfying the equation $f(1) = c$?

4. Show that a vector subspace of a vector space is itself a vector space over F. Show that $\{\mathbf{0}\}$ is a vector subspace, and is minimal (this space is called the *zero subspace*, and is sometimes written 0 just to confuse us: $0 = \{\mathbf{0}\}$). Show that V is itself a vector subspace of V. Show that the intersection $\bigcap_{i \in I} W_i$ of vector subspaces is a vector subspace (what if I is the empty set?).

5. Suppose W is a vector subspace, $\mathbf{w} \in W$, and $\mathbf{u} \in V - W$. Show that $\mathbf{w} + \mathbf{u}$ is not in W.

6. Show that if W_1 and W_2 are vector subspaces of V, then $W_1 \cup W_2$ is a vector subspace only if one of the subspaces is contained in the other. Hint: use 5.

7. Let $a_1, \ldots, a_n, c \in F$ where at least one a_i is non-zero. The *hyperplane* determined by these scalars is the set of all elements $(x_1, \ldots, x_n)$ of F^n such that $a_1 x_1 + \ldots + a_n x_n = c$. Show that such a hyperplane is a vector subspace of F^n if and only if $c = 0$. What does this tell you about lines in $\mathbb{R}^2$ or planes in $\mathbb{R}^3$?

Problems 8–12: Spans of vectors. Let V be a vector space over F, and let S be a set of vectors in V.

Definition. The *span* of S, written span(S), is the intersection of all vector subspaces of V containing S. We say S spans V if the span of S is all of V, in other words V is the only subspace containing every element of S. A vector space V is called a *finite dimensional vector space* if there is a finite set S which spans V; otherwise V is called an *infinite dimensional vector space*.

8. Show that the span of S is a vector subspace containing S. Show that the span of S is the smallest vector subspace of V containing S in the following sense: if W is a vector subspace containing S then span(S) $\subseteq W$.

Definition. Let S be a set of vectors in V. A *linear combination* of elements of S is an element of the form $a_1 \mathbf{u}_1 + \ldots + a_n \mathbf{u}_n$ where $\mathbf{u}_1, \ldots, \mathbf{u}_n \in S$ and $a_1, \ldots, a_n \in F$. We write $\langle S \rangle$ for the set of all linear combinations of elements of S. If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a finite set, we write sometimes write $\langle S \rangle$ as $\langle \mathbf{u}_1, \ldots, \mathbf{u}_n \rangle$.

9. Show that span(S) $= \langle S \rangle$. (Assume S is non-empty). Hint: first show that $\langle S \rangle$ is a vector subspace of V.

10. Show that $V = F^n$ is a finite dimensional vector space. Show that $V = F[x]$ is infinite dimensional.

Definition. A set S is said to be a *minimal spanning set* for the vector space V if (i) it spans V and (ii) no proper subset of S spans V.

11. Show that if $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a minimal spanning set, then the only way $a_1 \mathbf{u}_1 + \ldots + a_n \mathbf{u}_n = \mathbf{0}$ holds is if each a_i is zero.

12. Show that if V is a finite dimensional vector space then it has a finite minimal spanning set. (If $V = 0$ then the empty set is the minimal spanning set). Show that given a finite spanning S set for V, the minimal spanning set can be taken to be a subset of S.

We will later show that all minimal spanning sets of such V have the same number of elements. This will be the first important theorem of the course. A minimal spanning set will be called a "basis".