

Linear Algebra (Spring 2005, Prof. Aitken)

Problems 1–3: *Existence of eigenvalues.* Let V be an n dimensional vector space with scalar field F where n is a positive integer. Let $f : V \rightarrow V$ be an endomorphism.

Definition. A field F is called *algebraically closed* if every non-zero polynomial $f \in F[t]$ factors into linear factors. A famous theorem, *the fundamental theorem of algebra*, says that \mathbb{C} is algebraically closed. Now \mathbb{R} is not quite algebraically closed: every polynomial factors into linear *and quadratic* factors. In \mathbb{Q} or \mathbb{F}_p the situation is even worse: there are polynomials of arbitrarily large degree which are irreducible.

1. Show that if $F = \mathbb{C}$, or if F is algebraically closed, then f has at least one eigenvalue.
2. Suppose that n is odd and that $F = \mathbb{R}$. Show that f has at least one eigenvalue.
3. Use rotations of \mathbb{R}^2 to show there are endomorphisms of \mathbb{R}^2 with no eigenvalues. Observe that rotations of \mathbb{R}^3 have eigenvalues. In fact, look at a vector in the axis of rotation to conclude that 1 is an eigenvalue.

Problem 4: *Not all matrices can be diagonalized: even when F is algebraically closed.* Let V be a two dimensional vector space with basis v_1, v_2 , and let f be an endomorphism of V .

4. Show that if $\text{Mat}_{(v_i)}(f) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then f is not diagonalizable.

Problems 5–13: *Triangulating Matrices.* Let V be an n -dimensional vector space with scalars in a field F . Let $f : V \rightarrow V$ be an endomorphism. Our goal is to show that if $F = \mathbb{C}$, or more generally if F is algebraically closed, then there is an ordered basis v_1, \dots, v_n such that $\text{Mat}_{(v_i)}(f)$ is upper triangular.

Lemma. *With the above assumptions, there is an ordered basis v_1, \dots, v_n such that, for all $1 \leq k \leq n$, the vector $f(v_k)$ is in the span of v_1, \dots, v_k .*

5. Show that the above lemma is true if $n = 1$.
6. Show that if $c \in F$ is an eigenvalue, the image of $f - c \cdot \text{Id}$ is a subspace W of V of dimension less than n .
7. (Continued). Show that $f(W) \subseteq W$, so f restricts to a linear transformation $f|_W : W \rightarrow W$.
8. (Continued). Let m be the dimension of W . By induction, we can suppose that W has basis w_1, \dots, w_m such that $f(w_k)$ is a linear combination of w_1, \dots, w_k . Recall that w_1, \dots, w_m can be extended to a basis $w_1, \dots, w_m, v_{m+1}, \dots, v_n$ of V .
9. (Continued). Show that if $v \in V$ then $f(v) = (f - c \cdot \text{Id})(v) + cv$. In other words, $f(v)$ is equal to a vector in W plus cv . Conclude that v_i is a linear combination of w_1, \dots, w_m, v_i .
10. Complete the proof of the above lemma.
11. Use the above lemma to prove the following.

Theorem. *Suppose that F is an algebraically closed field, and that V is a non-trivial finite dimensional vector space with scalar field F . Then, for every endomorphism $f : V \rightarrow V$, there is an ordered basis v_1, \dots, v_n of V such that $\text{Mat}_{(v_i)}(f)$ is upper-triangular.*

12. Similarly, show that there is an ordered basis (v_i) so that $\text{Mat}_{(v_i)}(f)$ is lower-triangular. Hint: reverse order.

13. Prove the following:

Theorem. *Suppose F is algebraically closed. Then every matrix in $M_n(F)$ is similar to an upper-triangular matrix. Likewise, every matrix in $M_n(F)$ is similar to a lower-triangular matrix.*

Theorem. *If a matrix $A \in M_n(F)$ has a characteristic polynomial with n distinct roots, then A is similar to a diagonal matrix. (But the converse may fail).*

Remark. So the only way a matrix $A \in M_n(\mathbb{C})$ is not diagonalizable is if its characteristic polynomial $\chi(t)$ has repeated roots. (But $\chi(t)$ having multiple roots does not guarantee that A is not diagonalizable.)