LA 24

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Linear Algebra (Spring 2005, Prof. Aitken)

Problems 1–5: *The trace of a matrix. Let $A, B \in M_n(R)$ where R is a commutative ring.*

Definition. Let $A = [a_{ij}]$ be in $M_n(R)$ where R is a commutative ring. Then the *trace of* A, written $\mathrm{Tr}A$, is defined to be $\sum_{i=1}^{n} a_{ii}$.

1. Show that $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$. Show that the trace defines an R-module homomorphism $M_n(R) \to R$. (Since homomorphisms into R are often called *functionals*, we can call this the *trace functional*).

2. Show that the trace of A is just (-1) times the t^{n-1} coefficient of the characteristic polynomial of A. Conclude that similar matrices have the same trace. (Hint: most $\sigma \in S_n$ give lower power terms).

3. For matrices in $M_2(R)$ show that you can compute the characteristic polynomial simply by finding the trace and determinant.

4. Suppose that V is a vector space or R-module that has a finite basis $v_1, \ldots, v_n$. Define a *trace map* $\mathrm{Tr} : \mathrm{End}(V) \to R$. Show that it is a homomorphism and does not depend on the choice of basis of V.

5. Show that if B is in $GL_n(R)$ then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Hint: look at $B^{-1}(BA)B$. (Optional) Generalize to the case where B is not in $GL_n(R)$. Hint: use the product formula for matrices.

Problems 6–13: *Diagonalization. Let V be a finite dimensional vector space with scalar field F. Let $f : V \to V$ be an endomorphism.*

Definition. An endomorphism $f \in \mathrm{End}(V)$ is said to be *diagonalizable* if there is an ordered basis $v_1, \ldots, v_n$ such that $\mathrm{Mat}_{(v_i)}(f)$ is a diagonal matrix.

6. Show that f is diagonalizable if and only if V has a basis of eigenvectors.

7. Suppose that $w_1, \ldots, w_k$ are eigenvectors for *distinct* eigenvalues. Show that $w_1, \ldots, w_k$ are linearly independent. Hint: take a non-trivial dependency with the fewest number of non-zero terms. Apply f, giving a second dependency. From these two, get a linear dependency with fewer terms.

8. Show that if the characteristic polynomial of f has distinct roots, then f is diagonalizable.

9. Show that if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is linear with negative determinant, then f is diagonalizable.

10. Show that if the number of eigenvalues of V, counting multiplicity, is n if and only if f is diagonalizable. Show if f is diagonalizable, the associated diagonal matrix is essentially unique: it consists of eigenvalues with the multiplicities giving the number of entries with a given eigenvalue. (Hint 1: choose a basis for each distinct non-trivial eigenvector spaces. Show that the union of these basis is a basis for V. Hint 2: to justify the previous hint, show that any non-trivial linear dependency of the vectors can be grouped into a dependency $w_1 + \ldots + w_k = 0$ where $w_1, \ldots, w_k$ are eigenvectors of distinct eigenvalues.)

11. Let $c \in F$ be an eigenvalue of multiplicity k. Show that $(t - c)^k$ divides the characteristic polynomial of f. In other words, the "algebraic multiplicity" is greater than or equal to the true multiplicity. Conclude (assuming unique factorization of polynomials) that the number of eigenvalues, counting multiplicity, is at most n. Hint: form a basis $v_1, \ldots, v_n$ that uses eigenvectors $v_1, \ldots, v_k$. What does $\mathrm{Mat}_{(v_i)}(f)$ look like?

12. Suppose that θ is not a multiple of π. Explain why the rotation R_θ in $\mathbb{R}^2$ fixing $(0, 0)$ does not have eigenvectors for $F = \mathbb{R}$. Conclude that R_θ is not diagonalizable. What if $F = \mathbb{C}$? Show that the trace of R_θ has absolute value less than 2, so the characteristic polynomial has distinct complex roots. What are the diagonal entries (eigenvalues) for $\theta = \pi/2$?

13. Let $F = \mathbb{Z}_3 = \mathbb{F}_3$. Suppose that, for some basis (v_i) and some $c \in F$,

$$\mathrm{Mat}_{(v_i)}(f) \;=\; \begin{bmatrix} 1 & c \\ 1 & 1 \end{bmatrix}.$$

Show that f diagonalizes if $c = 1$, but not if $c = 2$ or $c = 0$. How does f diagonalize if $c = 1$?