LA 23

Linear Algebra (Spring 2005, Prof. Aitken)

Definition. Let V be a vector space with scalar field F, and let $f : V \to V$ be an endomorphism. If $v \in V$ is non-zero and $f(v) = cv$ with $c \in F$ then v is called an *eigenvector* and c the associated *eigenvalue*.

The *eigenvector space* for $c \in F$ is defined to be the set of vectors v satisfying $f(v) = cv$. This eigenvector space consists of the zero vector together with all eigenvectors, if any, with eigenvalue c. Thus c is an eigenvalue if and only if its eigenvector space contains a non-zero vector.

The dimension of the eigenvector space for $c \in F$ is called the *multiplicity* of the eigenvalue c.

Problems 1–4: *Eigenvalues and Eigenvectors of linear maps. Let V be a vector space with scalar field F, and let $f : V \to V$ be an endomorphism.*

1. Show that the eigenvector space for $c \in F$ is a vector subspace of V.

2. Suppose $v_1, \ldots, v_n$ is a basis for V and that the matrix $\mathrm{Mat}_{(v_i)}(f)$ is a diagonal matrix in $M_n(F)$. Describe the eigenvector spaces, the eigenvalues, and the multiplicities of the eigenvalues.

3. (Uniqueness). Use the previous problem to show that if $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ are two ordered bases for V such that $A = \mathrm{Mat}_{(v_i)}(f)$ and $B = \mathrm{Mat}_{(v'_i)}(f)$ are diagonal, then A and B are essentially the same. In fact, after reordering $v'_1, \ldots, v'_n$, they are the same: $A = B$. (But v_i does not have to be equal to v'_i).

4. Show that v is in the eigenspace for $c \in F$ if and only if it is in the kernel of $f - c \cdot \mathrm{Id}$. Conclude that $c \in F$ is an eigenvalue if and only if $f - c \cdot \mathrm{Id}$ has a non-zero kernel.

Problems 5–7: *Characteristic polyonomials of matrices. Let R be a commutative ring.*

Definition. The *characteristic polynomial* $\chi(t)$ of a matrix $A \in M_n(R)$ is defined to be $\det(tI - A)$, computed in $M_n\big(R[t]\big)$. Observe that the characteristic polynomial of A is a polynomial $\chi(t) \in R[t]$.

5. Show that similar matrices have the same characteristic polynomial. Hint: show

$$tI - BAB^{-1} = B(tI - A)B^{-1}.$$

6. Suppose that $\chi(t) \in R[t]$ is the characteristic polynomial of $A \in M_n(R)$. Show that if $c \in R$ then $\chi(c)$ is $\det(cI - A)$. Conclude that the constant term of $\chi(t)$ is just $(-1)^n \det(A)$.

7. What is the characteristic polynomial of a diagonal matrix? A triangular matrix?

Problems 8–12: *Characteristic polyonomials of linear maps. Let V be a finite dimensional vector space with scalar field F, and let $f : V \to V$ be an endomorphism. Let $v_1, \ldots, v_n$ be an ordered basis for V, and let $A = \mathrm{Mat}_{(v_i)}(f)$ be the matrix representing f.*

8. Show that $c \in F$ is an eigenvalue of f if and only if $A - cI$ is singular.

9. Show that $c \in F$ is an eigenvalue of f if and only if it is a root of the characteristic polynomial $\chi(t)$ of A.

10. Show that the characteristic polynomial of f does not depend on the choice of basis $v_1, \ldots, v_n$. This leads to the following definition:

Definition. Let $f : V \to V$ be an endomorphism of a finite dimensional vector space V over the field F. Then the *characteristic polynomial* of f is defined to be the characteristic polynomial of A where A is the matrix representing f relative to a choice of ordered basis.

11. Prove the following theorem. Hint: recall from abstract algebra that a non-zero polynomial $p(t)$ in $F[t]$ has at most $d = \deg p(t)$ roots in F.

Theorem. *Let $f : V \to V$ be an endomorphism of an n-dimensional vector space V. Then the characteristic polynomial χ of f is a monic polynomial of degree n. Its roots are the exactly the eigenvalues of f. In particular, there are at most n eigenvalues for f. The constant term of χ is $(-1)^n \det(f)$.*

12. Suppose $F = \mathbb{R}$ and $A = \begin{bmatrix} 1 & c \\ 1 & 1 \end{bmatrix}$. Find the characteristic polynomial, eigenvalues, and eigenvectors if (i) $c = 0$, (ii) $c = -1$, (iii) $c = 1$, (iv) $c = 4$.