LA 22

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Linear Algebra (Spring 2005, Prof. Aitken)

Problems 1–5: *Homomorphism of scalars.* Let $\varphi : R \to R'$ be a ring homomorphism between commutative rings. Given $A \in M_{m,n}(R)$, let $\widehat{\varphi}(A) \in M_{m,n}(R')$ be the matrix obtained by applying φ to each entry of A.

1. Let $\varphi : \mathbb{Z} \to \mathbb{Z}_3$ be reduction modulo 3 (which is a ring homomorphism). Give examples of $\varphi(A)$ for various matrices $A \in M_n(\mathbb{Z})$. Give examples of $\widehat{\varphi}(AB) = \widehat{\varphi}(A) \cdot \widehat{\varphi}(B)$ and $\det\big(\widehat{\varphi}(A)\big) = \varphi(\det A)$.

2. Suppose $A, B \in M_{m,n}(R)$ and $c \in R$. Show that $\widehat{\varphi}(A + B) = \widehat{\varphi}(A) + \widehat{\varphi}(B)$ and $\widehat{\varphi}(cA) = \varphi(c)\widehat{\varphi}(A)$. Suppose $A \in M_{n,m}(R)$ and $B \in M_{m,l}(R)$. Show that $\widehat{\varphi}(AB) = \widehat{\varphi}(A) \cdot \widehat{\varphi}(B)$. Suppose $A \in M_n(R)$. Show that $\det\big(\widehat{\varphi}(A)\big) = \varphi(\det A)$.

3. Suppose $A_t \in M_n\big(R[t]\big)$, so $p(t) = \det A_t$ is in $R[t]$. Show that when $p(t)$ is evaluated at $t = a$, we get $\det A_a$ where A_a is the matrix obtained by evaluating each entry of A_t at $t = a$.

4. (Optional) Fact from abstract algebra: for any non-trivial commutative ring R we can find a ring homomorphism $\varphi : R \to F$ into some field F (in general, this relies on the existence of maximal ideals, which uses the axiom of choice from set theory). Suppose $A \in M_{n,m}(R)$ and $B \in M_{m,n}(R)$ are matrices such that $AB = I_n$ and $BA = I_m$ (where I_k is the k by k identity matrix). Show that $\widehat{\varphi}A \cdot \widehat{\varphi}B = I_n$ and $\widehat{\varphi}B \cdot \widehat{\varphi}A = I_m$. Conclude that $\widehat{\varphi}A$ is the matrix of an isomorphism. Conclude that $n = m$ and the following:

Proposition. *Let R be a non-trivial commutative ring. Suppose $A \in M_{n,m}(R)$ and $B \in M_{m,n}(R)$ are matrices such that $AB = I$ and $BA = I$. Then $m = n$.*

5. (Optional) The reason that we assumed $R = F$ was a field in LA21 is that we did not know if two bases have the same size for general R. Allow, for the time being, non-square change of basis matrices and extend LA21 to commutative rings R. If $B \in M_{n,m}(R)$ is a change of basis matrix from (v_i) to (v'_j), then $BB' = I$ and $B'B = I$ where $B' \in M_{m,n}(R)$ is the change of basis matrix from (v'_j) to (v_i). Conclude the following:

Proposition. *Let R be a commutative ring. If V is an R-module that possesses a finite basis, then all bases of V has the same size. (Recall, however, that some finitely generated R-modules do not possess bases).*

Problems 6–8: *Similar matrices.* Assume that R is a commutative ring.

Definition. Let R be a commutative ring. We say that two matrices A and A' in $M_n(R)$ are *similar* if there is an invertible $B \in GL_n(R)$ such that $A' = BAB^{-1}$.

6. Show that similarity is an equivalence relation on $M_n(R)$.

7. Show that $\det(B^{-1}) = (\det B)^{-1}$ for invertible $B \in M_n(R)$.

8. Show that similar matrices have the same determinant.

Problems 9–12: *Similar matrices and change of basis.* Let V be a finite dimensional vector space, or, more generally, an R-module with a finite basis.

9. Let $f : V \to V$ be an endomorphism. Let $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ be two ordered bases of V. Show that $\mathrm{Mat}_{(v_i)}(f)$ is similar to $\mathrm{Mat}_{(v'_i)}(f)$. Observe that $\det \mathrm{Mat}_{(v_i)}(f)$ does not depend on (v_i) but only on f.

Definition. Let $f : V \to V$ be an endomorphism of a vector space. Define the *determinant of f*, written $\det f$, to be the determinant of $\mathrm{Mat}_{(v_i)}(f)$ for some choice of basis (v_i). The above shows that the answer is independent of the choice of (v_i). So we get a basis-independent function $\det : \mathrm{End}(V) \to F$.

10. Let $f, g \in \mathrm{End}(V)$. Show that $\det(f \circ g) = \det(f)\det(g)$.

11. Suppose that $B = [b_{ij}] \in M_n(R)$ is invertible and that $v_1, \ldots, v_n$ is an ordered basis of V. Let $v'_j = \sum_{i=1}^n b_{ij} v_i$. By LA21 Problem 2, B represents an endomorphism $f : V \to V$ using (v_i). Observe that (i) f is an isomorphism, (ii) $f(v_j) = v'_j$, and (iii) since f is an isomorphism, $v'_1, \ldots, v'_n$ is another basis of V.

12. Suppose that $B = [b_{ij}] \in M_n(R)$ is invertible and that $v_1, \ldots, v_n$ is an ordered basis of V. Let $v'_j = \sum_{i=1}^n b_{ij} v_i$. Show that $v'_1, \ldots, v'_n$ is a basis. Show that B is the change of basis matrix from (v'_i) to (v_i). Thus, *any invertible matrix can be a change of basis matrix*. Prove the following:

Proposition. *Let $v_1, \ldots, v_n$ be an ordered basis for V, and let $f : V \to V$ be an endomorphism with matrix $A = Mat_{(v_i)}(f)$ in $M_n(R)$. Then $A' \in M_n(R)$ is similar to A if and only if there is an ordered basis $v'_1, \ldots, v'_n$ such that $A' = Mat_{(v'_i)}(f)$.*