LA 21

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Linear Algebra (Spring 2005, Prof. Aitken)

Problems 1–3: *Some non-canonical isomorphisms.* Assume that R is a commutative ring, and let V and W be R-modules that possess finite bases. (If $R = F$ is a field, then this just means that V and W are finite dimensional vector spaces). Recall that $\mathrm{Hom}(V, W)$ is an R-module, and $\mathrm{End}(V)$ is a ring.

Definition. Let V and W be as above. Suppose $v_1, \ldots v_n$ is an ordered basis for V, and $\varphi : R^n \to V$ is the linear map with $e_i \mapsto v_i$. Suppose $w_1, \ldots, w_m$ is an ordered basis for W, and $\psi : R^m \to W$ is the linear map with $e_i \mapsto w_i$. Let $f : V \to W$ be a linear map. Observe that $\psi^{-1} \circ f \circ \varphi$ maps R^n to R^m. Then $\mathrm{Mat}_{(v_i),(w_i)}(f)$ can be defined to be $\mathrm{Mat}(\psi^{-1} \circ f \circ \varphi)$. This is equivalent to the definition given at the start of LA20 (see Problem 7 of LA20).

1. Let φ and ψ be as above. Show that $f \mapsto \psi^{-1} \circ f \circ \varphi$ is an isomorphism $\mathrm{Hom}(V, W) \to \mathrm{Hom}(R^n, R^m)$ of R-modules. Observe that $\mathrm{Hom}(V, W)$ is isomorphic to $M_{n,m}(R)$ using $f \mapsto \mathrm{Mat}(\psi^{-1} \circ f \circ \varphi)$, the composition $\mathrm{Hom}(V, W) \to \mathrm{Hom}(R^n, R^m) \to M_{m,n}(R)$. Note that these isomorphisms from $\mathrm{Hom}(V, W)$ are non-canonical: they depend on the choice of bases. (In contrast, the isomorphism $\mathrm{Hom}(R^n, R^m) \to M_{m,n}(R)$, from LA9 Problem 11, is canonical since it involves only the standard bases.)

2. Let φ be as above. Show that $f \mapsto \varphi^{-1} \circ f \circ \varphi$ is a ring isomorphism $\mathrm{End}(V) \to \mathrm{End}(R^n)$. Observe that $\mathrm{End}(V)$ is isomorphic to $M_n(R)$ using $f \mapsto \mathrm{Mat}(\varphi^{-1} \circ f \circ \varphi)$, the composition $\mathrm{End}(V) \to \mathrm{End}(R^n) \to M_n(R)$. Note that these isomorphisms from $\mathrm{End}(V)$ are non-canonical: they depend on the choice of bases. (In contrast, the isomorphism $\mathrm{End}(R^n) \to M_n(R)$, from LA11, Problem 11, is canonical.)

3. Suppose that $R = F$ is a field, and V and W are vector spaces of dimension n and m respectively. Show that $\dim \mathrm{Hom}(V, W) = nm$ and $\dim \mathrm{End}(V) = n^2$.

Problems 4–11: *Change of basis matrix.* Let V be a finite dimensional vector space over the field F. Let $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ be two ordered bases for V. (These problems generalize easily to scalars in a commutative ring, except we haven't yet shown that any two bases have the same size. See LA22.)

Definition. Let V be a finite-dimensional vector space. Let $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ be two ordered bases of V. Then the *change of basis matrix* from $v_1, \ldots, v_n$ to $v'_1, \ldots, v'_n$ is defined to be $\mathrm{Mat}_{(v_i),(v'_i)}(id)$ where $id : V \to V$ is the identity map.

4. Show that $B = [b_{ij}]$ is the change of basis matrix from $v_1, \ldots, v_n$ to $v'_1, \ldots, v'_n$ if and only if $v_j = \sum_{i=1}^n b_{ij} v'_i$.

5. Let B be the change of basis matrix from $v_1, \ldots, v_n$ to $v'_1, \ldots, v'_n$. Given a vector $v = \sum_{j=1}^n c_j v_j$, the matrix B can be used to rewrite v in terms of the basis v'_i. Show that, in terms of the notation of LA20 Problem 9, $\mathrm{Col}_{(v'_i)}(v) = B \cdot \mathrm{Col}_{(v_i)}(v)$.

6. Let $V = \mathbb{Q}^2$; let $v_1 = (1, 2)$ and $v_2 = (3, 1)$; and let $v'_1 = (1, 1)$ and $v'_2 = (-1, 1)$. Write $v = 3v_1 + v_2$ in terms of the ordered basis v'_1, v'_2 using the technique of Problem 5.

7. (Transitivity) Suppose that (v_i) and (v'_i) and (v''_i) are all ordered bases of V. Suppose B is the change of basis matrix from (v_i) to (v'_i) and B' is the change of basis matrix from (v'_i) to (v''_i). Show that $B'B$ is the change of basis matrix from (v_i) to (v''_i). Hint: LA20, Problem 8.

8. Show that the change of basis matrix from $v_1, \ldots, v_n$ to $v'_1, \ldots, v'_n$ is invertible, and the inverse is the change of basis matrix from $v'_1, \ldots, v'_n$ to $v_1, \ldots, v_n$.

9. Let $f : V \to W$ be a homomorphism. Suppose that (v_i) is an ordered basis for V and (w_i) and (w'_i) are two ordered bases for W. Let B be the change of basis matrix from (w_i) to (w'_i). Show that $\mathrm{Mat}_{(v_i),(w'_i)}(f)$ is $B \cdot \mathrm{Mat}_{(v_i),(w_i)}(f)$. Now suppose that (v'_i) is another ordered basis for V. What is the relationship between $\mathrm{Mat}_{(v'_i),(w_i)}(f)$ and $\mathrm{Mat}_{(v_i),(w_i)}(f)$? between $\mathrm{Mat}_{(v'_j),(w'_i)}(f)$ and $\mathrm{Mat}_{(v_i),(w_i)}(f)$? Hint: LA20, Problem 8.

10. Let $f : V \to V$ be an endomorphism. Suppose that (v_i) and (v'_i) are ordered bases for V. Let B be the change of basis matrix from (v_i) to (v'_i). What is the relationship between $\mathrm{Mat}_{(v'_i)}(f)$ and $\mathrm{Mat}_{(v_i)}(f)$?

11. Redo Problems 3 and 4 from LA20 using change of basis matrices.