LA 20

Linear Algebra (Spring 2005, Prof. Aitken)

Problems 1–9: Matrices for different bases. So far we have been focussing on matrices from the point of view of the standard basis $e_1, \ldots, e_n$. However, sometimes a linear map is simplier from the point of view of a different basis. Also, general vector spaces V do not have a single preferred basis. For example, what is the best basis for the hyperplane $x + y + z = 0$ which is a two-dimensional subspace of $\mathbb{R}^3$? Each choice of basis will give rise to a possibly different matrix.

Assume that V, W and X below are finite dimensional vector spaces (or more generally, that R is a commutative ring and that V, W and X are R-modules with finite bases).

Definition. Let $f : V \to W$ be a linear map. Let $v_1, \ldots v_n$ be an ordered basis for V, and let $w_1, \ldots, w_m$ be an ordered basis for W. Then $f(v_j)$ can be written uniquely as

$$f(v_j) = \sum_{i=1}^{m} a_{ij} w_i.$$

The matrix $[a_{ij}] \in M_{m,n}(R)$ is called the *matrix associated with f and the ordered bases $v_1, \ldots v_n$ and $w_1, \ldots, w_m$* and is written $\mathrm{Mat}_{(v_i),(w_i)}(f)$. So the matrix depends on three pieces of information: (i) the linear transformation, (ii) the ordered basis for the domain, and (iii) the ordered basis for the codomain.

If $V = W$ then it is common to choose the same ordered basis $v_1, \ldots, v_n$ for the domain and codomain. In this case we write $\mathrm{Mat}_{(v_i)}(f)$.

1. Assume that $V = R^n$ and $W = R^m$. Show that $\mathrm{Mat}_{(e_i),(e_i)}(f) = \mathrm{Mat}(f)$ where $\mathrm{Mat}(f)$ is as in LA9. In other words, for standard bases the new definition agrees with the old.

2. (Optional) Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be counterclockwise rotation by θ. Use euclidean geometry and trigonometry to show that $R_\theta\big((a,b)\big) = (a\cos\theta - b\sin\theta, \ a\sin\theta + b\cos\theta)$. Conclude that R_θ is linear and find the matrix representing it (using the standard basis). What is its determinant?

3. Let $\theta = \pi/2$ radians. Find the matrix for R_θ if (i) you use the standard basis, (ii) if the domain has the standard basis, but the codomain has ordered basis $(1,1), (-1,2)$, (iii) if the domain has ordered basis $(1,1), (-1,2)$ but the codomain has the standard basis, (iv) if both the domain and codomain use the ordered basis $(1,1), (-1,2)$. Compute the determinant for each matrix.

4. Let V be the vector space of polynomials in $\mathbb{R}[x]$ of degree at most 3. Let $f : V \to V$ be differentiation. Is this a linear map? Find the matrix for f for (i) basis $1, x, x^2, x^3$, (ii) for basis $2 + x, x, x^2 + x$, and x^3. Do both matrices have determinant zero?

5. Show that any isomorphism $f : V \to W$ is represented by the identity matrix if you choose the correct ordered bases: there exists (v_i) and (w_i) so that $\mathrm{Mat}_{(v_i),(w_i)}(f) = I$.

6. Show that the matrix of the identity map $V \to V$ is I if and only if you use the same ordered basis for both the domain and codomain. In particular, $\mathrm{Mat}_{(v_i)}(\mathrm{Id}) = \mathrm{Mat}_{(v_i),(v_i)}(\mathrm{Id}) = I$.

7. Recall that choosing an ordered basis $v_1, \ldots, v_n$ for V gives rise to a unique isomorphism $\varphi : R^n \to V$ with $e_i \mapsto v_i$. Likewise, choosing an ordered basis $w_1, \ldots, w_m$ for W gives an isomorphism $\psi : R^m \to W$. Let $f : V \to W$ be linear and consider $\psi^{-1} \circ f \circ \varphi : R^n \to R^m$. Show that $\mathrm{Mat}_{(v_i),(w_i)}(f)$ is just $\mathrm{Mat}(\psi^{-1} \circ f \circ \varphi)$ so the new definition reduces to that of LA9.

8. Assume that V has ordered basis $v_1, \ldots, v_p$, that W has ordered basis $w_1, \ldots, w_q$, and that X has ordered basis $x_1, \ldots, x_r$. Suppose that $f : V \to W$ and $g : W \to X$ are linear. Show that

$$\mathrm{Mat}_{(w_i),(x_i)}(g) \cdot \mathrm{Mat}_{(v_i),(w_i)}(f) = \mathrm{Mat}_{(v_i),(x_i)}(g \circ f).$$

Thus *matrix multiplication corresponds to composition if we are consistent in the choice of bases*. Hint: just use Problem 7 and earlier results.

9. Let $f : V \to W$ be a linear map where $v_1, \ldots, v_n$ is an ordered basis for V and $w_1, \ldots, w_m$ is an ordered basis for W. Show that $\mathrm{Mat}_{(v_i),(w_i)}(f)$ can be used to compute f as follows:

$$\mathrm{Col}_{(w_i)}\big(f(v)\big) = \mathrm{Mat}_{(v_i),(w_i)}(f) \cdot \mathrm{Col}_{(v_i)}(v).$$

Here $\mathrm{Col}_{(v_i)}(v)$ is the *column vector* associated to v. More specifically, if $v = \sum_{k=1}^{n} c_k v_k$ then $\mathrm{Col}_{(v_i)}(v)$ is the n by 1 matrix whose kth row contains c_k. The m by 1 matrix $\mathrm{Col}_{(w_i)}\big(f(v)\big)$ is defined similarly. Hint: begin by deriving the formula $f(v) = \sum_{k=1}^{n} c_k f(v_k) = \sum_{i=1}^{m}(\sum_{k=1}^{n} a_{ik} c_k) w_i$.