LA 2

Linear Algebra (Spring 2005)

Definition. A *vector space* over a field F is a set V equipped with the following data:

D1. A function $V \times V \to V$ called *vector addition*, usually written additively.

D2. A function $F \times V \to V$ called *scalar multiplication*, usually written multiplicatively.

D3. A distinguished element $\mathbf{0} \in V$ called the *zero vector*.

D4. A function $V \to V$ called the *additive inverse function*, denoted $\mathbf{u} \mapsto -\mathbf{u}$.

In addition, V is required to satisfy the following axioms:

A1. Associativity: Vector addition is associative. In addition, scalar multiplication is associative:

$$a(b\mathbf{u}) = (ab)\mathbf{u} \quad \text{for all } a, b \in F \text{ and } \mathbf{u} \in V.$$

A2. Commutivity: Vector addition is commutative.

A3. Identity: $\mathbf{0} + \mathbf{u} = \mathbf{u}$ and $1 \cdot \mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in V$.

A4. Inverse: $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for all $\mathbf{u} \in V$.

A5. Distributivity. For all $a, b \in F$ and $\mathbf{u}, \mathbf{v} \in V$:

$$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u} \quad \text{and} \quad a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}.$$

The elements of F are often called *scalars*. If $F = \mathbb{R}$ then V is called a *real vector space* and if $F = \mathbb{C}$ then V is called a *complex vector space*.

1. In the above definition there are two types of addition: one for F and one for V. Which use of $+$ in the above axioms corresponds to vector addition? Likewise the symbol $-$, symbols for multiplication, and the symbol for zero are used in two different ways. To make things clearer, we can use bold face letter for vectors. This practice, however, will be eventually dropped.

Problems 2–7: Examples. In these problems assume that F is a field.

2. Why is a vector space an abelian group (with extra structure)? Give an alternate definition of a vector space as an abelian group V (instead of a set) with extra structure (described in terms of data and axioms).

3. One of the prototypical examples of a vector space is in terms of directed line segments. In Euclidean space (with no coordinate system needed), a *directed line segment* is a line segment with one endpoint designated as the *tail*, and the other as the *head*. The direction of the segment is from tail to head. Now identify all directed line segments if they are translations of each other (in other words, we say that AB, with tail A and head B, equals CD if the translation of AB which sends A to C is exactly CD). Call such objects *arrow vectors*. The zero arrow vector AA can be thought of as a point. Show that the set of all such arrow vectors forms a vector space V. (Here, you can do some handwaving). This notion of vector is very popular in physics to indicate forces, velocities, momentum, etc. The idea is that a particle can have a property that has a magnitude and a direction. (But in curved space, as studied in differential geometry, the notion of direction is only local, so the notion of translation is more complicated.) Draw a triangle illustrating vector subtraction $\mathbf{u} - \mathbf{v}$ (Hint: draw $\mathbf{u}$ and $\mathbf{v}$ having a common point for a tail).

4. Show that F^n is a vector space (this the other prototypical example). Is there a way to relate $\mathbb{R}^3$ to the previous example. Let F^ω be the set of infinite sequences with terms in F. Show that F^ω is a vector space.

5. The complex numbers $\mathbb{C}$ is usual regarded as a field. If, however, we forget some of its multiplicative structure it is a vector space with scalars in $\mathbb{R}$. Likewise, the set of n by n matrices (with coefficients in F) is naturally a ring, but if we forget some of its structure then it is a vector space over F. In fact, n by m matrices (with n and m fixed) form a vector space. Show that if R is a ring (or field) containing a subfield F, then R is a vector space over F. So $\mathbb{R}$ is a vector space over $\mathbb{Q}$.

6. Show that the set of polynomials $F[x]$ is a vector space over F. What about polynomials of the form $a + (a + b)x + bx^2$? What about polynomials of the form $a + abx + bx^2$? What about polynomials with no constant term? What about polynomials with constant term 1.

7. Let $[a, b]$ be an interval of $\mathbb{R}$ (much of what we say applies to other types of intervals such as (a, b) and $(-\infty, b]$). Do the set of continuous functions on $[a, b]$ form a ring? a vector space over $\mathbb{R}$? a field? What about the set of differentiable functions? What about the set of functions of the form $a\sin(x) + b\cos(x)$? What about the set of polynomial functions? linear functions? Hint: show the set of functions $a\sin(x) + b\cos(x)$ equals the set of functions $A\sin(x + c)$. Observe that $\sin(x)^2 \geq 0$.