LA 19

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Linear Algebra (Spring 2005, Prof. Aitken).

Problems 1–2: Transpose. Let $A, B \in M_{m,n}(R)$ where R is a commutative ring. Recall the definition of A^T from LA16:

Definition (Transpose). Let $A = [a_{ij}]$ be an m by n matrix. Then the *transpose* of A, written A^T, is the n by m matrix with (i, j) entry a'_{ij} given by $a'_{ij} = a_{ji}$.

1. Show that $(A + B)^T = A^T + B^T$ and $(A^T)^T = A$.

2. Show that $(AB)^T = B^T A^T$ whenever the product AB is defined. Thus the transpose operator defines an *anti-isomorphism* $M_n(R) \to M_n(R)$ which is its own inverse. (An *anti-homomorphism* $f : R_1 \to R_2$ between rings is a function such that $f(a + b) = f(a) + f(b)$ and $f(ab) = f(b)f(a)$ and $f(1) = 1$).

Problems 3–4: Cofactors (Optional). *Cofactors* occurred in the formula for inverses from LA17 and LA18. Here we discuss a shortcut for computing them (which is the method given in most texts). Let R be a commutative ring and let $V = R^n$. Let $\Lambda : V^n \to R$ be the normalized alternating n-linear functional. Then the *(i, j)th cofactor* of a matrix $A \in M_n(R)$ is defined to be $\Lambda(w_1, \ldots, w_{i-1}, e_j, w_{i+1}, \ldots, w_n)$ where $w_1, \ldots, w_n$ are the columns of A.

3. Consider the matrix A' obtained by replacing the first column of $A \in M_n(R)$ by e_1. Let A_{11} be the matrix obtained by removing the first row and first column of A. Show that in the determinant formula for A', you only need to sum over the permutations $\sigma \in S_n$ that fix 1. Show that the set of permutations in $\sigma \in S_n$ that fix 1 forms a subgroup $H \subsetneq S_n$ isomorphic to S_{n-1}. Compare the $(n-1)!$ terms of $\det A_{11}$ with the $(n-1)!$ terms of $\det A'$ corresponding to $\sigma \in H$. Show that $\det A' = \det A_{11}$.

4. Let $A = [a_{ij}]$. Use the determinant formula to show that the (i, j)th cofactor is equal to $(-1)^{i+j} \det A_{ji}$ where A_{ji} is the $n-1$ by $n-1$ matrix obtained by removing the jth row and the ith column. Hint: permute the rows and columns of A so that the new matrix A' has first column equal to e_1, and so that when you remove the first row and column you are left with A_{ji}. Show that $\det A' = \det A_{ji}$.

Problems 5–8: Row Rank and Column Rank. Assume that $R = F$ is a field and that $A \in M_{m,n}(F)$ is an m by n matrix. The dimension of the span of the columns vectors of A is called the *column rank* of A. The dimension of the span of the row vectors is called the *row rank* of A. Our goal is to show that the column rank equals the row rank.

5. Recall the definition of *rank* from LA10. If A is the matrix of a linear map $f : F^n \to F^m$, show that the rank of f is the column rank of A. Show that the column rank is at most the minimum of m and n.

6 Show that if $B \in M_n(F)$ is invertible, then AB and A have the same column ranks. Show that if $C \in M_m$ is invertible, then CA and A have the same column ranks. Using transposes, show the same holds for row ranks. Hint: B and C are matrices of isomorphisms. For example, if $A = \text{mat}(f)$ and $C = \text{mat}(\gamma)$, then show that the image $(\gamma \circ f)(F^m)$ is isomorphic to the image $f(F^m)$.

7. Suppose that $A = [a_{ij}]$ is such that (i) if $i \neq j$ then $a_{ij} = 0$, and (ii) for each i, either $a_{ii} = 1$ or $a_{ii} = 0$. Show that the column rank of A and the row rank of A are equal. Show that the rank is the number of i such that $a_{ii} = 1$.

8. Using row and column operations, show that there is an invertible matrix $B \in M_n(F)$ and an invertible matrix $C \in M_m(F)$ such that CAB is a matrix $A' \in M_{m,n}(F)$ as in problem 7. Prove the following:

Theorem. *Let F be a field and $A \in M_{m,n}(F)$ a matrix. Then the row rank of A is equal to the column rank of A. This rank is at most the minimum of m and n.*

Remark. The *rank* of a matrix is defined to be either the row or the column rank: since they are equal, it doesn't matter which we choose. A matrix in $M_{m,n}(F)$ has *maximal rank* if its rank is exactly the minimum of m and n. So for square matrices, maximal rank means invertible (or non-singular). There is a sense that a random matrix in $M_{m,n}(\mathbb{R})$ has maximal rank.

9. Let p be a prime number. How many matrices are in the ring $M_2(\mathbb{Z}_p)$? How many are invertible? In other words, what is the size of $GL_2(\mathbb{F}_p)$? What is the probability that a random matrix in $M_2(\mathbb{Z}_p)$ is invertible? What happens to this probability as p grows? Optional: generalize to $n > 2$.