# LA 18

**Linear Algebra (Spring 2005, Prof. Aitken).**

**Theorem (Inverse formula: version 2).** *Let $R$ be a commutative ring and let $V = R^n$. Let $\Lambda : V^n \to R$ be the normalized alternating n-linear functional. Then if $A$ is a matrix in $M_n(R)$ whose determinant is a unit in $R$, then $A$ is invertible. The inverse of $A$ is $B = [b_{ij}]$ where*

$$b_{ij} = (\det A)^{-1} \Lambda(w_1, \dots, w_{i-1}, e_j, w_{i+1}, \dots, w_n)$$

*where $w_j$ is the jth column of $A$ and $e_j$ is in the ith slot.*

**Corollary.** *Let $R$ be a commutative ring. Then $A \in M_n(R)$ is invertible if and only if $\det A$ is a unit in $R$.*

**1**. Use LA17 to justify these results when $R$ is a field $F$, or a subring of a field $F$. For example, they hold for $R = \mathbb{Z}$ (and any other integral domain: a ring is a subring of a field if and only if it is an integral domain). The above formula is closely related to the traditional formula for the *adjoint* matrix in terms of *minors* (and/or *cofactors*).

*Problems 2–11: Generalizing to all commutative $R$.* We want to prove the above theorem and corollary for general commutative rings such as $R = \mathbb{Z}_N$ where $N$ is composite. (Problems 2–9 are optional).

**2**. To prove this theorem let $b'_{kj} = \Lambda(w_1, \dots, w_{k-1}, e_j, w_{k+1}, \dots, w_n)$ and let $A = [a_{ij}]$. Observe that $w_k = \sum_{j=1}^{n} a_{jk} e_j$ and

$$\det A = \Lambda(w_1, \dots, w_{k-1}, w_k, w_{k+1}, \dots, w_n) = \sum_{j=1}^{n} a_{jk} \Lambda(w_1, \dots, w_{k-1}, e_j, w_{k+1}, \dots, w_n) = \sum_{j=1}^{n} b'_{kj} a_{jk}.$$

**3**. Conclude that the $(k,k)$th term of the product $B'A$ is equal to $\det A$. Here $B' = [b'_{ij}]$.

**4**. Now suppose that $k \neq l$. Then $\Lambda(w_1, \dots, w_{k-1}, w_l, w_{k+1}, \dots, w_n) = 0$ and $w_l = \sum_{j=1}^{n} a_{jl} e_j$. So

$$0 = \Lambda(w_1, \dots, w_{k-1}, w_l, w_{k+1}, \dots, w_n) = \sum_{j=1}^{n} a_{jl} \Lambda(w_1, \dots, w_{k-1}, e_j, w_{k+1}, \dots, w_n) = \sum_{j=1}^{n} b'_{kj} a_{jl}.$$

**5**. Conclude that if $k \neq l$ then the $(k,l)$th term of the product of $B'A$ is just 0.

**6**. Let $B' = [b'_{ij}]$. Conclude that $B'A = (\det A)I$. Conclude that $BA = I$. However, this does not automatically mean that $AB = I$ since $R$ is a general commutative ring.

**7**. Repeat the arguments of 2 to 6, but with row vectors $u_k$ instead of column vectors $w_k$:

$$c'_{jk} = \Lambda(u_1, \dots, u_{k-1}, e_j, u_{k+1}, \dots, u_n) \qquad C' = [c'_{jk}] \qquad C = (\det A)^{-1} C'.$$

Conclude that $AC' = (\det A)I$. Conclude that $AC = I$.

**8**. Show that $C = B$, so $A$ is invertible.

**9**. Justify the above theorem and corollary. (For one direction of the corollary, use the fact that if $AB = I$ then $\det A \cdot \det B = 1$.)

**10**. If a matrix has entries in $\mathbb{Z}_n$ where $n$ is composite, when does it have an inverse? Give a few examples of inverses of matrices in $M_2(\mathbb{Z}_n)$ and $M_3(\mathbb{Z}_n)$.

**11**. Prove the following Proposition. (It generalizes the proposition of LA12 to all commutative rings). Hint: $\det A \cdot \det B = 1$ implies that $\det A$ is a unit.

**Proposition.** *Let $R$ be a commutative ring. If $A, B \in M_n(R)$ satisfy $AB = I$, then $BA = I$. So, for square matrices, a one-sided inverse is automatically a two-sided inverse.*