LA 17

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Linear Algebra (Spring 2005, Prof. Aitken).

Problems 1–2: Non-singular matrices. Let F be a field and let $V = F^n$. Let $\Lambda : V^n \to F$ be the unique normalized alternating n-linear functional on V. (*Normalized* means $\Lambda(e_1, \ldots, e_n) = 1$).

1. Let $f : V^n \to F$ be an alternating n-linear functional. Show that if u_n is a linear combination of $u_1, \ldots, u_{n-1}$ then $f(u_1, \ldots, u_n) = 0$. Conclude that if the columns of a matrix are linearly dependent, then its determinant is 0. Show that if the rows of a matrix are linearly dependent, then its determinant is 0.

2. Show that if a square matrix $A \in M_n(F)$ has linear independent columns, then A is invertible, and $\det A \neq 0$. Hint: see LA10 problems 1 and 2, and LA16 problem 9. Now prove the following:

Theorem. *Let $A \in M_n(F)$ be a square matrix where F is a field. Then the following are equivalent:*

1. *The matrix A is invertible. (So $A \in GL_n(F)$. In other words, A is non-singular).*
2. $\det A \neq 0$.
3. *The column vectors of A form a basis for F^n.*
4. *The column vectors of A are linearly independent in F^n.*
5. *The column vectors of A span F^n.*
6. *The row vectors of A form a basis of F^n.*
7. *The row vectors of A are linearly independent in F^n.*
8. *The row vectors of A span F^n.*

Problems 3–6: Cramer's Rule. Now we know that $\det A \neq 0$ is equivalent to A being invertible, if the scalars are a field F. But what if the scalars are a commutative ring? The answer is that $A \in M_n(R)$ is invertible if and only if $\det A$ is a unit in R. In order to show this, we need Cramer's Rule, which is an important idea even over fields. Let $V = R^n$ where R is a commutative ring, and let $f : V \to V$ be invertible. Cramer's rule is a technique to find $f^{-1}(u)$ of a vector $u \in V$ if f is invertible.

3. Explain how the problem of finding $f^{-1}(u)$ is related to solving n linear equations in n unknowns:

$$a_{i1}x_1 + \ldots + a_{in}x_n = c_i \qquad i = 1, \ldots, n.$$

Conclude that Cramer's rule can be understood as a technique for solving linear equations.

4. Let $w_j = f(e_j)$ and suppose f is invertible. Show that if $f^{-1}(u) = (b_1, \ldots, b_n) = \sum b_i e_i$ then $u = \sum b_i w_i$. Show that $\Lambda(w_1, \ldots, w_{i-1}, u, w_{i+1}, \ldots, w_n) = b_i \Lambda(w_1, \ldots, w_n)$ and prove the following

Theorem (Cramer's Rule). *Let $V = R^n$ where R is a commutative ring, and let $\Lambda : V^n \to R$ be the normalized alternating n-linear functional for V. Assume that $f : V \to V$ is linear and invertible with matrix A. Then if $u \in V$, the preimage $f^{-1}(u) = (b_1, \ldots, b_n)$ is given by the formula*

$$b_i = (\det A)^{-1} \Lambda(w_1, \ldots, w_{i-1}, u, w_{i+1}, \ldots, w_n)$$

where w_j is the jth column of A and u is put in the ith slot.

5. Suppose that $f(e_1) = (1, 2, 1)$, $f(e_2) = (3, 1, -1)$, $f(e_3) = (0, 1, 1)$. (Assume $R = \mathbb{Q}$ if you wish). Use Cramer's rule to find a vector v such that $f(v) = (0, 2, 1)$. (Optional: now find it with row operations).

6. A very important special case is computing $f^{-1}(e_j)$ since it gives the matrix for f^{-1}. Prove the following. Use the theorem to find the matrix for f^{-1} where $f(e_1) = (1, 2, 1)$, $f(e_2) = (3, 1, -1)$, $f(e_3) = (0, 1, 1)$. Assume $R = \mathbb{Q}$ if you wish. (Optional: now find it using row reduction).

Theorem (Inverse formula). *Let $V = R^n$ where R is a commutative ring. Also, assume that $\Lambda : V^n \to R$ is the normalized alternating n-linear functional for V. Finally assume that $f : V \to V$ is linear and invertible with matrix A and inverse matrix B. Then $f^{-1}(e_j) = (b_{1j}, \ldots, b_{nj})$ is given by the formula*

$$b_{ij} = (\det A)^{-1} \Lambda(w_1, \ldots, w_{i-1}, e_j, w_{i+1}, \ldots, w_n)$$

where w_j is the jth column of A and e_j is put in the ith slot. Also $B = [b_{ij}]$ is the matrix for $f^{-1} : V \to V$ (so $B = A^{-1}$). Thus A^{-1} can be computed with determinants.