LA 16

April 12, 2005

Linear Algebra (Spring 2005, Prof. Aitken).

Problems 1–11: Determinants. Let R be a commutative ring, and let $V = R^n$. Let $\Lambda : V^n \to R$ be the unique normalized alternating n-linear functional on V. (*Normalized* means $\Lambda(e_1, \ldots, e_n) = 1$).

Definition (Determinant). Let $A = [a_{ij}]$ be an n by n matrix with entries in R. Then

$$\det(A) \stackrel{\text{def}}{=} \Lambda(u_1, \ldots, u_n)$$

where u_i is the ith column vector of A.

1. Let $A \in M_n(R)$. Why, if two columns of A are switched, does $\det(A)$ change by a factor of -1? Why is $\det(A) = 0$ if two columns are equal or a if a column is zero? Why is $\det(A)$ n-linear with respect to columns? Why does the identity matrix have determinate 1?

2. Assume A is an n by n matrix with entries in R. Observe that

$$\det(A) \;=\; \sum_{\sigma \in S_n} \epsilon(\sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

3. Show that if $A \in M_n(R)$ then $\det(cA) = c^n \det A$. Let $A = [a_{ij}] \in M_n(R)$. If A is diagonal (that is $a_{ij} = 0$ whenever $i \neq j$) then show that $\det A = a_{11} \cdots a_{nn}$. Hint: which $\sigma \in S_n$ matter? Now suppose that $a_{ij} = 0$ for all $j < i$. Show that $\det A = a_{11} \cdots a_{nn}$. (Such matrices are called *upper-triangular*: a similar result holds for lower triangular matrices).

Definition (Transpose). Let $A = [a_{ij}]$ be an m by n matrix. Then the *transpose* of A, written A^T, is the n by m matrix with (i, j) entry a'_{ij} given by $a'_{ij} = a_{ji}$.

4. Let $A \in M_n(R)$. Show that A and its transpose A^T have the same determinant. Hint: see *both* formulas in the theorem of LA15.

5. Let $A \in M_n(R)$. Show that if two rows of A are switched, then $\det(A)$ changes by a factor of -1. Show that if two rows are equal or if a row is zero, then $\det(A) = 0$. Show that $\det(A)$ is n-linear with respect to rows. Show that $\det(A) = \Lambda(u_1, \ldots, u_n)$ where u_i is the ith row of A.

6. Let $f : V \to V$ and $g : V \to V$ be linear maps with respective matrices $A = [a_{ij}]$ and $B = [b_{ij}]$. Observe that $g(e_j) = \sum_i b_{ij} e_i$, and so $f(g(e_j)) = \sum_i b_{ij} f(e_i)$. Use LA15 problem 2 to conclude that

$$\Lambda\Big((f \circ g)(e_1), \ldots, (f \circ g)(e_n)\Big) = \Lambda\left(\sum_i b_{i1} f(e_i), \ldots, \sum_i b_{in} f(e_i)\right) = \det B \cdot \Lambda\Big(f(e_1), \ldots, f(e_n)\Big).$$

7. (Continued) Observe that

$$\det(AB) = \Lambda\Big((f \circ g)(e_1), \ldots, (f \circ g)(e_n)\Big) \quad \text{and} \quad \det(A) = \Lambda\Big(f(e_1), \ldots, f(e_n)\Big).$$

8. Prove the following:

Theorem. *Let A and B be matrices in $M_n(R)$ where R is a commutative ring. Then*

$$\det(AB) \;=\; \det A \cdot \det B.$$

9. Show that if $A \in GL_n(R)$, then $\det A$ is a unit in R. (The converse is true, but harder to show: See LA18). Use this to find a matrix in $M_2(\mathbb{Z})$ which is invertible in $M_2(\mathbb{Q})$ but not in $M_2(\mathbb{Z})$.

10. Show that if $A \in M_n(F)$ where F is a field, and if $\det A = 0$, then $A \notin GL_n(F)$, so A is the matrix of a homomorphism which is not an isomorphism. Such A are called *singular*.

11. Let $U(R)$ be the unit group of R. Show that $A \mapsto \det A$ is a group homomorphism $GL_n(R) \to U(R)$. Show that this homomorphism is surjective. Let $SL_n(R)$ be the kernel. So what type of matrices are in $SL_n(R)$? Show that $GL_n(R)/SL_n(R)$ is isomorphic to $U(R)$. (You can refer to an abstract algebra text).