LA 15

April 12, 2005

Linear Algebra (Spring 2005, Prof. Aitken).

Problems 1–3: Uniqueness of alternating n-linear functionals. Let R be a field (or commutative ring), and let $V = R^n$. Suppose that $f : V^n \to R$ is an alternating n-linear functional. (Review even and odd permutations from abstract algebra.)

1. (Continued) Let $w_1, \ldots, w_n$ be n vectors of V. Suppose $u_j = a_{1j}w_1 + \ldots + a_{nj}w_n$. Show that when you expand $f(u_1, \ldots, u_n)$ using linearity, you get n^n terms, but all but $n!$ will be zero. To choose a term of the expansion which is not automatically zero, you must pick a term $a_{\sigma(j)j}w_{\sigma(j)}$ from each $u_j = \sum a_{ij}w_i$, where $\sigma(1), \ldots, \sigma(n)$ are *distinct*. So each choice gives a permutation σ. Thus,

$$f(u_1, \ldots, u_n) = \sum_{\sigma \in S_n} a_{\sigma(1)1} \cdots a_{\sigma(n)n} f\big(w_{\sigma(1)}, \ldots, w_{\sigma(n)}\big).$$

2. (Continued) Show that

$$f(u_1, \ldots, u_n) = \left(\sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} \right) f(w_1, \ldots, w_n)$$

where $\epsilon(\sigma) = \pm 1$ depending on whether σ is even or odd.

3. (Uniqueness) Suppose that f is an alternating n-linear functional on V such that $f(e_1, \ldots, e_n) = c$. Show

$$f\Big((a_{11}, \ldots, a_{n1}), \ldots, (a_{1n}, \ldots, a_{nn})\Big) = c \sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n}.$$

Problems 4–6: Existence. Let R be a field (or commutative ring), and let $V = R^n$.

4. Let $\sigma \in S_n$ be a permutation. Show that

$$f\Big((a_{11}, \ldots, a_{n1}), \ldots, (a_{1n}, \ldots, a_{nn})\Big) = a_{\sigma(1)1} \cdots a_{\sigma(n)n}.$$

defines an n-linear functional on V. (But not alternating).

5. Show that the set of n-linear functional on V forms an R-module. Conclude that the formula of problem 3 defines an n-linear functional on V. In the next several problems we will see why it is alternating. Hint: modify LA13, number 7.

6. Let $\tau \in S_n$ be the transposition (2 cycle) switching i and j. Show that for every odd permutation π there is a unique even permutation α such that $\pi = \alpha\tau$.

7. Let $\tau \in S_n$ be the transposition (2 cycle) switching i and j. Let f be as in problem 3. Show that

$$f\Big((a_{11}, \ldots, a_{n1}), \ldots, (a_{1n}, \ldots, a_{nn})\Big) = c \sum_{\alpha \in A_n} a_{\alpha(1)1} \cdots \cdots a_{\alpha(n)n} - c \sum_{\alpha \in A_n} a_{\alpha(\tau(1))1} \cdots a_{\alpha(\tau(n))n}.$$

Conclude that $f(u_1, \ldots, u_n) = 0$ if $u_i = u_j$. Thus f is alternating.

Problems 8–9: Another formula. Let R be a field (or commutative ring), and let $V = R^n$.

8. Let $\sigma \in S_n$ be permutation, and let $[a_{ij}]$ be an n by n matrix with entries in R. Show that

$$a_{\sigma(1)1} \cdots a_{\sigma(n)n} = a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)}.$$

9. Let $[a_{ij}]$ be an n by n matrix with entries in R. Show that

$$\sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} = \sum_{\sigma \in S_n} \epsilon(\sigma^{-1}) a_{1\sigma^{-1}(1)} \cdots a_{n\sigma^{-1}(n)} = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$

10. Prove the following:

Theorem. *Let $V = R^n$ where R is a commutative ring. There is a unique alternating n-linear functional $\Lambda : V^n \to R$ that is normalized with $\Lambda(e_1, \ldots, e_n) = 1$. Furthermore,*

$$\Lambda\Big((a_{11}, \ldots, a_{n1}), \ldots, (a_{1n}, \ldots, a_{nn})\Big) = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n} = \sum_{\sigma \in S_n} \epsilon(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}.$$