LA 14

April 9, 2005

Linear Algebra (Spring 2005)

Definition. Let V be a vector space or module over R. A *trilinear functional* is a function $f : V \times V \times V \to R$ which is linear in each coordinate (explain what this means). If $f(u, v, w) = 0$ whenever two of $u, v, w \in V$ are equal, then we say that f is *alternating*.

Problems 1–6: Basic properties of alternating trilinear functionals. Assume that $f : V \times V \times V \to R$ is an alternating trilinear functional.

1. Give an example of an alternating trilinear functionals in terms of signed volumes in $\mathbb{R}^3$. Hint: generalize parallelogram to three dimensions. (Don't be too rigorous here).

2. Expand $f(u, v + w, v + w)$. Conclude that $f(u, w, v) = -f(u, v, w)$. Generalize: any time two input vectors are switched, then $f(u, v, w)$ is multiplied by -1.

3. Now assume that $V = R^3$. Let $u = (a_{11}, a_{21}, a_{31})$, $v = (a_{12}, a_{22}, a_{32})$ and $w = (a_{13}, a_{23}, a_{33})$. Show that

$$f(u, v, w) = \Big(a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\Big) f(e_1, e_2, e_3)$$

Hint: write $u = a_{11}e_1 + a_{21}e_2 + a_{31}e_3$ *et cetera*, and expand using linearity.

4. Assume that $V = R^3$ where R is commutative. Show that the unique normalized (with $f(e_1, e_2, e_3) = 1$) alternating trilinear functional is

$$f\Big((a_{11}, a_{21}, a_{31}), (a_{12}, a_{22}, a_{32}), (a_{13}, a_{23}, a_{33})\Big) = \sum_{\sigma \in S_3} \epsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)}$$

where σ varies over all 6 permutations of $1, 2, 3$, and where $\epsilon(\sigma)$ is $+1$ or -1 depending on σ.

Definition. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map. Define the *determinant* of L to be the oriented volume of the image of the unit cube. It turns out, but we will not prove, that for any solid C in $\mathbb{R}^3$, that the volume of $L(C)$ is the absolute value of the determinant of L times the original volume of C.

5. Show that if the matrix of L is $[c_{ij}]$ then the determinant of L is $\sum_{\sigma \in S_3} \epsilon(\sigma) c_{1\sigma(1)} c_{2\sigma(2)} c_{3\sigma(3)}$.

6. Use this formula to calculate volumes of some solids in $\mathbb{R}^3$.

Definition. Let V be a vector space or module over R. An *n-linear functional* is a function $f : V^n \to R$ which is linear in each coordinate: if $u_i = v_i + w_i$ then

$$f(u_1, \ldots, u_i, \ldots, u_n) = f(u_1, \ldots, v_i, \ldots, u_n) + f(u_1, \ldots, w_i, \ldots, u_n)$$

and, if $u_i = cw_i$ with $c \in R$,

$$f(u_1, \ldots, u_i, \ldots, u_n) = cf(u_1, \ldots, w_i, \ldots, u_n).$$

If $f(u_1, \ldots, u_n) = 0$ whenever $u_i = u_j$ for $i \neq j$, then we say that f is *alternating*.

Problems 7–9: Basic properties of alternating n-linear functionals. Assume that $f : V^n \to R$ is an alternating n-linear functional. Review permutations from abstract algebra, if necessary.

7. Show that if σ is a transposition (2 cycle), then $f(u_1, \ldots, u_n) = -f(u_{\sigma(1)}, \ldots, u_{\sigma(n)})$.

8. Show that if σ is an odd permutation, then $f(u_1, \ldots, u_n) = -f(u_{\sigma(1)}, \ldots, u_{\sigma(n)})$. Show that if σ is an even permutation, then $f(u_1, \ldots, u_n) = f(u_{\sigma(1)}, \ldots, u_{\sigma(n)})$. (The group of even permutations is called the *alternating group*: alternating functionals are invariant under the alternating group).

9. Now consider the case where $V = R^n$. Suppose that $w_1, \ldots, w_n$ are vectors (often $e_1, \ldots, e_n$). Suppose $u_j = a_{1j}w_1 + \ldots + a_{nj}w_n$. Show that when you expand $f(u_1, \ldots, u_n)$ using linearity, you get n^n terms (some will turn out to be zero). To choose a term of the expansion, pick a term $a_{\gamma(j)j} w_{\gamma(j)}$ from each $u_j = \sum a_{ij} w_i$, where $\gamma(j) \in \{1, \ldots, n\}$. Note there are n^n ways of choosing $\gamma(1), \ldots, \gamma(n)$ and

$$f(u_1, \ldots, u_n) = \sum_{\gamma} f\Big(a_{\gamma(1)1} w_{\gamma(1)}, \ldots, a_{\gamma(n)n} w_{\gamma(n)}\Big).$$