LA 13

April 2, 2005

Linear Algebra (Spring 2005)

Definition. Let V be a vector space (or module) over R. A *bilinear functional* is a function $f : V \times V \to R$ such that (i) $f(u+v,w) = f(u,w)+f(v,w)$, (ii) $f(u,v+w) = f(u,v)+f(u,w)$, and (iii) the scalar propery: $f(cv,w) = f(v,cw) = cf(v,w)$ where $c \in R$. (I use *functional* whenever I want to emphasize that the codomain is the set of scalars. Some authors use *form* instead of *functional*).

Definition. Let V be a vector space (or module) over R. A *alternating bilinear functional* is a bilinear functional $f : V \times V \to R$ such that $f(v,v) = 0$.

Problems 1–2: Area functional. Why would we want to study such a thing as an "alternating bilinear functional"? Where does it actually come up? We will find that it comes up in euclidean geometry when computing area. Let $V = \mathbb{R}^2$ be the usual cartesian plane. This section is purely motivational, so it does not have to be done super rigorously.

Definition. Let $v, w \in \mathbb{R}^2$. Define $A(v,w)$ to be the signed area of the parallelogram with sides corresponding to the vectors v and w. (More precisely, the parallelogram with vertices 0, v, w, and $v+w$). By *signed area*, we mean area which is counted positively if the ray starting from 0 and going through v can be rotated counter-clockwise, less than 180 degrees, to give a ray starting from 0 and going through w. Otherwise, we count area negatively (if non-zero). Warning: this definition relies on the traditional representation of $\mathbb{R}^2$ in our space – a purely mathematical definition cannot use terms like "counter-clockwise". Call $A : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ the *Area functional*.

1. Show that the area functional is a bilinear functional. Explain why we need to use signed area for the bilinearity property. (Take as given euclidean geometry including the area formula for parallelograms).

2. Show that the area functional is an alternating bilinear functional such that $A(e_1, e_2) = 1$. This gives an informal existence proof for $\mathbb{R}^2$ for alternating bilinear functionals (that are not the zero map).

Problems 3–8: basic properties of alternating bilinear functionals. Assume that $f : V \times V \to R$ is an alternating bilinear functional.

3. Expand $f(v+w, v+w)$. Conclude that $f(w,v) = -f(v,w)$. This is why the word *alternating* is used: switching vectors alternates the sign.

4. Now assume that $V = R^2$. Let $v = (a,b) = ae_1 + be_2$ and $w = (c,d) = ce_1 + de_2$. Show that $f(v,w) = (ad - bc)f(e_1,e_2)$. If $f(e_1,e_2) = 1$, then we say that f is *normalized*.

5. Assume that $V = R^2$ and that R is commutative. Show $f\big((a,b),(c,d)\big) = ad - bc$ is the unique normalized alternating bilinear functional $V \times V \to R$. When $R = \mathbb{R}$ this gives an area formula! Hint: uniqueness has essentially been done.

6. Use this formula to calculate areas of three or four parallelograms in $\mathbb{R}^2$.

7. Show that for general V, the set of alternating bilinear functionals is a vector space (if R is a field) or module (if R is a commutative ring). Find a basis in the case where $V = R^2$ and R commutative.

8. What are the alternating bilinear functionals for $V = R^1$?

Problems 9–10: Introduction to determinants. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map.

Definition. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map. Define the *determinant* of L to be the signed area of the image of the unit square $\{(x,y) \mid 0 \le x, y \le 1\}$. It turns out, but we will not prove, for any region C in $\mathbb{R}^2$, that the area of $L(C)$ is the absolute value of the determinant of L times the original area of C.

9. Show that if the matrix of L is $[c_{ij}]$ then the determinant of L is $c_{11}c_{22} - c_{12}c_{21}$. Hint: what are $L(e_1)$ and $L(e_2)$? Also, use Problem 5.

10. Calculate the area inside the ellipse $4x^2 + 9y^2 = 36$. Hint: find a L that transforms a circle into this ellipse.