LA 12

March 21, 2005

Linear Algebra (Spring 2005)

Problems 1-7: Inverses of matrices. In problems 1-4 assume R is a field or a commutative ring. In problems 5-7 we will play it safe and assume that $R = F$ is a field. Use the ring isomorphism $\Phi : M_n(R) \to \mathrm{End}(R)$.

1. Let $A \in M_n(R)$. Then A *invertible* means it is has a multiplicative inverse in the ring $M_n(R)$. In other words, there exists a matrix $B \in M_n(R)$ such that $AB = BA = I$ where I is the identity matrix. From ring theory we know that the inverse of A, if it exists, is unique and $(A^{-1})^{-1} = A$. Use the isomorphism from $\Phi : M_n(R) \to \mathrm{End}(R^n)$ to observe that if $A = \mathrm{Mat}(f)$ then B is the inverse of A if and only if $B = \mathrm{Mat}(f^{-1})$. So inverting matrices is equivalent to finding inverses of linear maps.

2. Show that if a matrix in $M_n(R)$ has a zero row, or a zero column, then it is not invertible. Hint: interpret the problem in terms of linear maps that can be shown not to be surjective or not to be injective.

3. Show that if $f : A \to B$ and $g : B \to C$ are functions with $g \circ f$ equal to the identity map, then g is surjective and f is injective. (This is just set theory: no linear algebra is required).

4. Show that if $A, B \in M_n(R)$ are matrices with $AB = I$ with I the identity matrices, then A is the matrix of a surjective linear map and B is the matrix of an injective linear map.

5. (Continued) Show that A and B are matrices of isomorphisms (hint: use LA10, problem 11). Conclude that A and B are invertible.

6. Show that if $A, B \in M_n(R)$ are matrices with $AB = I$ with I the identity matrices then A and B are inverses of each other (in other words, that $BA = I$). Hint: you know that A^{-1} exists. Show $B = A^{-1}$.

Proposition. *If $A, B \in M_n(F)$ are such that $AB = I$ then $BA = I$. (Here F is a field, or integral domain).*

7. Generalize. Show that if $A, B \in M_n(F)$ are matrices with AB invertible, then both A and B are invertible.

Problems 8-14: Row and column operations. Assume R is a field or a commutative ring.

8. Suppose that $1 \leq i < j \leq m$. Show that there is a matrix $X \in M_m(R)$ such that, for all n and for all $A \in M_{m,n}(R)$, the matrix XA is equal to the matrix A except that the ith and jth rows are switched. Show that X is its own inverse. Show that if $A \in M_m(R)$ then A is invertible if and only if XA is invertible. (General fact: for any associative operation with identity element, if x is invertible, then the product xa is invertible if and only if a is.)

9. Give a similar construction for switching columns.

10. Let $1 \leq i \leq m$ and let c be a unit in R (so if R is a field, just assume $c \neq 0$). Find a matrix $X \in M_m(R)$ such that, for all n and for all $A \in M_{m,n}(R)$, the matrix XA is equal to the matrix A except that every entry the ith row has been multiplied by c. Describe an inverse for X. Conclude that if $A \in M_m(R)$ then A is invertible if and only if XA is invertible. Give a similar construction for changing a column.

11. Let i and j be distinct row numbers, and let c be an element of R. Find a a matrix $X \in M_m(R)$ such that, for all n and for all $A \in M_{m,n}(R)$, the matrix XA is equal to the matrix A except that the ith row (considered as a vector in R^n) has been replaced by the ith row plus c times the jth row. Show that X has an inverse. Conclude that if $A \in M_m(R)$ then A is invertible if and only if XA is invertible. Give a similar construction for changing a column in this way.

12. The above three operations are called *row operations* and *column operations*. Consider the following procedure. Start with $C := A$ and $D := I$. Then in the *row operation step* choose a row operation, and perform this operation to both C and D: in other words replace C by XC and D by XD for X as above. Repeat the row operation step until $C = I$. Show that if this procedure ends then $D = A^{-1}$. Hint: show that D is invertible at each stage, and that at each stage $D^{-1}C = A$.

13. Find inverses of matrices of your choosing in $M_m(\mathbb{Q})$, $M_m(\mathbb{F}_3)$ and $M_m(\mathbb{Z})$, using this technique.

14. Describe a procedure which, given $A \in M_n(F)$, will either find an inverse for A or show you that A is not invertible. Assume $R = F$ is a field for this. Hint: if there is a dependency among columns, then C is not invertible. Why?