LA 11

March 15, 2005

Linear Algebra (Spring 2005)

Problems 1-4: Matrix multiplication. Let R be a commutative ring or field of scalars.

1. Suppose $\gamma : R^k \to R^m$ is a linear map with $\mathrm{Mat}(\gamma) = [c_{ij}]$. Suppose $\delta : R^n \to R^k$ is a linear map with $\mathrm{Mat}(\delta) = [d_{ij}]$. We would like to find $\mathrm{Mat}(\gamma \circ \delta)$. Explain why $\mathrm{Mat}(\gamma \circ \delta)$ is an m by n matrix.

2. Let $\mathbf{e}_j$ be a basis vector in R^n. We already know the following (why?):

$$\delta(\mathbf{e}_j) = \sum_{i=1}^{k} d_{ij}\mathbf{e}_i = (d_{1j}, \ldots, d_{kj}).$$

Apply LA9, Problem 5, to show that the ith coordinate of $\gamma(d_{1j}, \ldots, d_{kj})$ is $\sum_{l=1}^{k} c_{il}d_{lj}$. Prove the following:

Theorem. *Let $\gamma : R^k \to R^m$ be a linear map with $\mathrm{Mat}(\gamma) = [c_{ij}]$, and let $\delta : R^n \to R^k$ is a linear map with $\mathrm{Mat}(\delta) = [d_{ij}]$. Then $\mathrm{Mat}(\gamma \circ \delta)$ is the m by n matrix $[a_{ij}]$ defined by the rule*

$$a_{ij} = \sum_{l=1}^{k} c_{il}d_{lj}.$$

This motivates the following:

Definition. Let $C = [c_{ij}]$ be a m by k matrix, and $D = [d_{ij}]$ be a k by n matrix. Then define CD to be the matrix $[a_{ij}]$ defined by the rule

$$a_{ij} = \sum_{l=1}^{k} c_{il}d_{lj}.$$

3. Give a short proof that matrix multiplication is associative (when the product is defined) using linear maps. And when is the product defined?

4. Let Let $L : R^n \to R^m$ be a linear map with $\mathrm{Mat}(L) = C = [c_{ij}]$. Let $(a_1, \ldots, a_n)$ be a vector in the domain, which we write as a n by 1 matrix $A = [a_{i1}]$ with $a_{i1} = a_i$. So CA is a m by 1 matrix $[b_{i1}]$. Show that $L((a_1, \ldots, a_n)) = (b_1, \ldots, b_m)$ where $b_i = b_{i1}$. Hint: this is just a restatement of something from LA9.

Problems 5-13: Endomorphism Rings. Let R be a commutative ring or field of scalars.

Definition. Let V be a vector space (or module). An *endomorphism* of V is an element of $\mathrm{Hom}(V, V)$. An *automorphism* of V is an endomorphism that is an isomorphism. The set of endomorphisms of V is written $\mathrm{End}(V)$. The set of automorphisms of V is written $\mathrm{Aut}(V)$.

5. Show that composition is a binary operation on $\mathrm{End}(V)$. Show that it is associative. Call this operation *multiplication*.

6. Show that the matrix of any element of $\mathrm{End}(R^n)$ is a square n by n matrix.

7. Find an identity for multiplication in $\mathrm{End}(V)$. What is its matrix if $V = R^n$.

8. Show that $\mathrm{Aut}(V)$ is the set of element of $\mathrm{End}(V)$ that have inverses (under multiplication).

9. Show the left and right distributive laws for $\mathrm{End}(V)$.

10. Show that $\mathrm{End}(V)$ is a ring. Show that $\mathrm{Aut}(V)$ is the unit group of the ring $\mathrm{End}(V)$. Hint: some steps were done in LA9, Problem 7. See LA1 for definition (but 1 is not the scalar 1 but the identity of problem 7).

11. Show that $M_n(R) \stackrel{\text{def}}{=} M_{nn}(R)$ is a ring under multiplication which is ring isomorphic to $\mathrm{End}(R^n)$. The units of $M_n(R)$ form a group called $GL_n(R)$. Show that $GL_n(R)$ is isomorphic to $\mathrm{Aut}(R^n)$. Hint: use the fact that $\mathrm{End}(R^n)$ is a ring, and that there is an addition preserving bijection between $\mathrm{End}(R^n)$ and $M_n(R)$.

12. If $c \in R$, let $\tilde{c}$ be the map $v \mapsto cv$. Show that $\tilde{c} \in \mathrm{End}(V)$. What is the matrix for $\tilde{c}$ if $V = R^n$.

13. Show that $c \mapsto \tilde{c}$ is a ring homomorphism $R \to \mathrm{End}(V)$. Show that it is injective if V is a non-zero vector space. Conclude that R is isomorphic to a subring of $\mathrm{End}(V)$ (if V is not the zero space). Conclude that R is isomorphic to a subring of $M_n(R)$. Show that $\mathrm{End}(R^1)$ is isomorphic to R. What is $\mathrm{End}(V)$ if R is zero dimensional?

14. Show that $\mathrm{End}(R^n)$ and $M_n(R)$ are non-commutative rings if $n > 1$ if R is not the trivial ring ($1 \neq 0$). Hint: you really only need to do it for $M_n(R)$.