LA 10

March 15, 2005

Linear Algebra (Spring 2005)

Problems 1-5: More on freedom. Let F be the field of scalars.

1. Let V be a vector space with scalar field F. As we know, if you choose n vectors $v_1, \ldots, v_n$, then there is a unique linear $L : F^n \to V$ sending e_i to v_i. Show that if $v_1, \ldots, v_n$ is a linearly independent sequence, then L is injective. Hint: $L((a_1, \ldots, a_n)) = \sum_{i=1}^n a_i v_i$.

2. In the above, show that if $v_1, \ldots, v_n$ span then L is surjective. Show that if if $v_1, \ldots, v_n$ is a basis, then L is an isomorphism.

3. Show that if V is an n-dimensional vector space with scalar field F, then there is an isomorphism $F^n \to V$. Explain the following statement: *choosing a basis for V is the same as choosing an isomorphism $F^n \to V$.*

4. Prove the following:

Theorem. *Let V_1 and V_2 be two finite dimensional vector spaces over the scalar field F. Then V_1 is isomorphic to V_2 if and only if V_1 and V_2 have the same dimension.*

5. Show freedom for any finite dimensional vector space: Suppose V is a vector space with basis $v_1, \ldots, v_n$ and W is any other vector space (with the same scalar field F). Then for any choice $w_1, \ldots, w_n$ of n vectors of W, there is a unique linear $L : V \to W$ such that $v_i \mapsto w_i$. Show that $L(\sum a_i v_i) = \sum a_i w_i$. Hint: for existence, find a map $L_1 : F^n \to V$ with $e_i \mapsto v_i$ and a map $L_2 : F^n \to W$ with $e_i \mapsto w_i$ and figure out how to compose them.

Problems 6-11: Rank and nullity. Let V_1 and V_2 be finite dimensional vector spaces with scalar field F.

Definition. Let $L : V_1 \to V_2$ be a linear map. Then the dimension of the kernel (or nullspace) is called the *nullity* of L. The dimension of the image (or range) is called the *rank* of L.

If M is an m by n matrix, then its nullity and rank are defined to be the nullity and rank of the associated linear map $R^n \to R^m$.

6. Let $L : V_1 \to V_2$ be a linear map with image W. Let r be the rank and ν be the nullity. Show that if $w_1, \ldots, w_r$ is a basis of W, then we can find vectors $v_1, \ldots, v_r$ of V_1 such that $v_i \mapsto w_i$, and we can find vectors $t_1, \ldots, t_\nu$ that form a basis for the kernel.

7. (continued) Let v be a vector in V_1. Show that if $L(v) = \sum_{i=1}^r a_i w_i$ then $v - \sum_{i=1}^r a_i v_i$ is in $\ker L$.

8. (continued) Conclude that v is a linear combination of $t_1, \ldots, t_\nu, v_1, \ldots, v_r$. So these vectors span V_1.

9. Suppose that $a_1 t_1 + \ldots + a_\nu t_\nu + b_1 v_1 + \ldots + b_r v_r = 0$ is a linear dependency of $t_1, \ldots, t_\nu, v_1, \ldots, v_r$. Show that $b_1 w_1 + \ldots + b_r w_r = 0$ in W. Conclude that each b_i is zero. Conclude that each a_i is also zero.

10. Prove the following:

Theorem (Rank-Nullity Theorem). *Let $L : V_1 \to V_2$ be a linear map between finite dimensional vector spaces. Then the nullity of L plus the rank of L equals the dimension of V_1:*

$$\dim V_1 = \text{rank} L + \text{null} L = \dim \ker L + \dim \text{image} L$$

11. Show that if $L : V_1 \to V_2$ is a linear map between vector spaces *of the same finite dimension*, then L is an isomorphism if and only if it is injective. Likewise, L is an isomorphism if and only if it is surjective.