

Linear Algebra (Spring 2005)

Problems 1–4: Review of key algebraic structures: rings and fields.

Definition. A *ring* (with unity) is a set R equipped with the following data:

D1. A function $R \times R \rightarrow R$ called the *addition* of R , usually written with $+$.

D2. A function $R \times R \rightarrow R$ called *multiplication* of R , often written with \cdot .

D3. A distinguished element $0 \in R$ called the *zero* of R .

D4. A distinguished element $1 \in R$ called the *unity* or *one* of R .

D5. A function $R \rightarrow R$ called the *additive inverse*, denoted $a \mapsto -a$.

In addition, R is required to satisfy the following axioms:

A1. Associativity: Both addition and multiplication are associative.

A2. Commutivity: Addition is commutative.

A3. Identity: $0 + a = a$ and $1 \cdot a = a \cdot 1 = a$ for all $a \in R$.

A4. Inverse: $a + (-a) = 0$ for all $a \in R$.

A5. Distributivity. For all $a, b, c \in R$:

$$(a + b) \cdot c = a \cdot c + b \cdot c \quad \text{and} \quad c \cdot (a + b) = c \cdot a + c \cdot b.$$

Parenthesis are used in the usual way and we often drop \cdot for multiplication. The notation $a - b$ is short for $a + (-b)$. If $ab = ba = 1$ then we write $b = a^{-1}$ and $a = b^{-1}$ and say that a and b are *multiplicative inverses*. It could be that some $a \neq 0$ in R do not have multiplicative inverses. If commutativity holds for multiplication as well as addition, then R is called a *commutative ring*.

1. (You may look in an abstract algebra book). How does these definitions compare to the definitions you might learn in abstract algebra? Give examples of a commutative rings and a non-commutative rings. Give an example of an element of a commutative ring which has a multiplicative inverse, and a non-zero element of a commutative ring which does not have a multiplicative inverse. What simplifications can you make to the definition for commutative rings?

2. Prove the following elementary laws (for arbitrary a, b in a ring R)

$$-(-a) = a \quad \text{Hint: write a three term equation involving } a, -a \text{ and } -(-a). \quad (1)$$

$$0 \cdot a = 0 \quad (\text{and } a \cdot 0 = 0) \quad (2)$$

$$(-a)b = -(ab) \quad (3)$$

$$a(-b) = -(ab) \quad (4)$$

$$(-a)(-b) = ab \quad \text{Hint: use (3) and (4), then use (1).} \quad (5)$$

$$(-1)(-1) = 1 \quad (6)$$

Definition. A *field* is a set F equipped with the following data:

D1. A function $F \times F \rightarrow F$ called the *addition* of F , usually written with $+$.

D2. A function $F \times F \rightarrow F$ called *multiplication* of F , often written with \cdot .

D3 and **D4.** Distinguished elements $0, 1 \in F$ called the *zero element* and the *unity element*, respectively.

D5. A function $F \rightarrow F$ called the *additive inverse*, denoted $a \mapsto -a$.

D6. A function $F^\times \rightarrow F^\times$ called the *multiplicative inverse*, denoted $a \mapsto a^{-1}$.

Here F^\times is $F - \{0\}$. In addition, F is required to satisfy the following:

A1. Associativity: both addition and multiplication are associative.

A2. Commutivity: both addition and multiplication are commutative.

A3. Identity: $0 + a = a$ and $1 \cdot a = a$ for all $a \in F$, and $0 \neq 1$.

A4. Inverse: $a + (-a) = 0$ for all $a \in F$, and $aa^{-1} = 1$ for all non-zero $a \in F$.

A5. Distributivity: $c(a + b) = ca + cb$ for all $a, b, c \in F$.

3. If R is a ring, what extra properties must hold for R to be a field? Let F be a field, and $a, b \in F$. Show that if $ab = 0$ then $a = 0$ or $b = 0$. Is this true in general rings?

4. (You may look in an abstract algebra book). How does this definition compare to the definition you might learn in abstract algebra? Give three examples of infinite fields. Give three examples of finite fields.

We will skip some of the proofs of the basic properties of rings and fields, and go directly to vector spaces next time. In this course, fields will be used mainly as the set of scalars in a vector space. We will learn about modules which are similar to vector spaces, except the scalars are in a ring instead of a field.