

CHAPTER 6 SUMMARY: RATIONAL NUMBERS

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1. INTRODUCTION

This is an abridged form of Chapter 6 from the number systems course (Math 378). It will be used as part of the first unit for Math 422. In this chapter we describe the basic properties of the rational numbers \mathbb{Q} .

2. BASIC DEFINITIONS

Here we define the set of rational numbers \mathbb{Q} and the operations of addition and multiplication on this set. Before defining \mathbb{Q} we define a related set Q which informally represents quotients of integers. The difference between Q and \mathbb{Q} is that the latter consists of equivalence classes of the former.

$$Q \stackrel{\text{def}}{=} \{(x, y) \mid x, y \in \mathbb{Z} \text{ and } y \neq 0\}.$$

We define an equivalence relation on the set Q by the rule

$$(x, y) \sim (z, w) \stackrel{\text{def}}{\iff} xw = yz$$

It can be shown that \sim is an equivalence relation on the set Q . We define the fraction to be the equivalence class of (x, y) under this equivalence relation. We also write $\frac{x}{y}$ for this fraction.

Example. We have that $6/8 = 3/4$ since the pairs $(6, 8)$ and $(3, 4)$ are in the same equivalence class. This follows from that fact that $(6, 8) \sim (3, 4)$ which in turns follows from the identity $6 \cdot 4 = 8 \cdot 3$ and the definition of \sim .

Definition 1. We define \mathbb{Q} as the set of equivalence classes of the above relation \sim . In other words,

$$\mathbb{Q} \stackrel{\text{def}}{=} \{x/y \mid (x, y) \in Q\}.$$

Definition 2. We define two binary operations, *addition* and *multiplication*, for \mathbb{Q} as follows.

$$\frac{x}{y} + \frac{z}{w} \stackrel{\text{def}}{=} \frac{xw + yz}{yw}$$

and

$$\frac{x}{y} \cdot \frac{z}{w} \stackrel{\text{def}}{=} \frac{xz}{yw}.$$

Since these definitions involve equivalence classes, they cannot be used until they are shown to be well-defined.

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Lemma 1. *Addition and multiplication on \mathbb{Q} is well-defined.*

Theorem 2. *The set \mathbb{Q} has an additive identity $0/1$ and a multiplicative identity $1/1$.*

Theorem 3. *Suppose $x, y, c \in \mathbb{Z}$ where $y \neq 0$ and $c \neq 0$. Then*

$$\frac{cx}{cy} = \frac{x}{y}.$$

Theorem 4. *Addition and multiplication in \mathbb{Q} are commutative.*

Theorem 5. *Suppose x, y are non-zero integers. Then*

$$\frac{x}{y} \cdot \frac{y}{x} = \frac{1}{1}.$$

In other words, x/y has a multiplicative inverse and

$$\left(\frac{x}{y}\right)^{-1} = \frac{y}{x}.$$

Theorem 6. *Suppose x, y are integers where $y \neq 0$ Then*

$$\frac{x}{y} + \frac{-x}{y} = \frac{0}{1}.$$

In other words, x/y has additive inverse $(-x)/y$. Thus

$$-\frac{x}{y} = \frac{-x}{y}.$$

Theorem 7. *The set \mathbb{Q} is a field.*

We identify the fraction $u/1$ with the integer u . When we do so we can regard \mathbb{Z} as a subring of \mathbb{Q} .

3. DIVISION AND FRACTIONAL NOTATION

We can actually use fractional notation in any field. Suppose $x, y \in F$ where F is a field and where $y \neq 0$. Then x/y can be defined to be $x \cdot y^{-1}$. Most of the familiar identities and laws concerning fractions and division are valid for general fields, and can be easily proved using the identity $(xy)^{-1} = x^{-1}y^{-1}$, an identity that is valid in any field. Here are some examples,

Theorem 8. *Suppose that $x \in F$ and $y, z \in F^\times$ where F is a field. Then*

$$\frac{zx}{zy} = \frac{x}{y}.$$

Theorem 9. *Suppose that $x, y, z \in F$ where F is a field and $y \neq 0$. Then*

$$\frac{x}{y} + \frac{z}{y} = \frac{x+z}{y}.$$

Theorem 10. *Suppose $x, z \in F$ and $y, w \in F^\times$ where F is a field. Then*

$$\frac{x}{y} + \frac{z}{w} = \frac{xw + yz}{yw}.$$

Exercise 1. Let $x, z \in F$ and $y, w \in F^\times$ where F is a field. Prove the following

$$\frac{x}{y} \cdot \frac{z}{w} = \frac{xz}{yw}, \quad \frac{0}{y} = 0, \quad \frac{y}{y} = 1,$$

$$\frac{x}{1} = x, \quad x \frac{z}{y} = \frac{xz}{y}, \quad y \frac{x}{y} = x.$$

Exercise 2. Let $x, y \in F$ where F is a field and y is not zero. Then show that x/y and $(-x)/y$ are additive inverses. Conclude that

$$-\frac{x}{y} = \frac{-x}{y} \quad \text{and} \quad -\frac{-x}{y} = \frac{x}{y}.$$

Exercise 3. Let $x, y \in F^\times$ where F is a field. Then show that x/y and y/x are multiplicative inverses. Conclude that

$$\frac{1}{x/y} = \frac{y}{x}.$$

The proof of the following is easy, and is left to the reader.

Theorem 11. Let $x, z \in F$ and $y, w \in F^\times$ where F is a field. Then

$$\frac{x}{y} = \frac{z}{w} \iff xw = yz,$$

and

$$\frac{x}{y} = \frac{z}{y} \iff x = z.$$

Remark 1. The attentive reader will have noticed that in \mathbb{Q} we have two definitions of x/y . First, if $x, y \in \mathbb{Z}$ and $y \neq 0$ then x/y is defined as an equivalence class. The second definition of x/y is the product xy^{-1} which makes sense for all x, y in the field \mathbb{Q} with $y \neq 0$. Fortunately, both definitions define the same rational number in the case where $x, y \in \mathbb{Z}$ with $y \neq 0$, so we can safely use the same notation for both.

4. FURTHER PROPERTIES OF \mathbb{Q}

Theorem 12. If $r \in \mathbb{Q}$ then there is a unique pair a, b of relatively prime integers such that $b > 0$ and

$$r = \frac{a}{b}.$$

If we do not insist on the relatively prime condition, we can always find a common denominator for any two elements of \mathbb{Q} :

Theorem 13. If $u, v \in \mathbb{Q}$ then we can find integers a, b, d with $d > 0$ such that

$$u = \frac{a}{d} \quad \text{and} \quad v = \frac{b}{d}.$$

Division is related to divisibility:

Theorem 14. Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Then $a/b \in \mathbb{Z}$ if and only if $b \mid a$.

5. POSITIVE AND NEGATIVE RATIONAL NUMBERS

The set \mathbb{Q} is not just a field, but is an *ordered field*. We will define the notion of ordered field later, but a key part of the definition is the idea of a positive subset. In this section we define the subset of positive rational numbers.

Definition 3 (Positive and Negative). A number $r \in \mathbb{Q}$ is said to be a *positive* rational number if it can be written as a/b where a and b are positive integers. An number $r \in \mathbb{Q}$ is said to be a *negative* rational number if $-r$ is positive.

Remark 2. We already know, from Chapter 3, what positive and negative integers are. The above extends the definitions to rational numbers. Lemma 18 below shows that the new definitions truly extend the old definitions.

Theorem 15. *The set of positive rational numbers is closed under addition and multiplication: if $u, v \in \mathbb{Q}$ are positive, then so are $u + v$ and uv .*

Theorem 16. *Let $a/b \in \mathbb{Q}$ where $a, b \in \mathbb{Z}$ with $b \neq 0$. Then a/b is a positive rational number if and only if either (i) both a and b are positive integers or (ii) both a and b are negative integers.*

Theorem 17. *Let $a/b \in \mathbb{Q}$ where $a, b \in \mathbb{Z}$ with $b \neq 0$. Then a/b is a negative rational number if and only if either (i) a is a positive integer and b is a negative integer, or (ii) a is a negative integer and b is a positive integer.*

We now show that the definitions of positive and negative numbers really do extend the definitions of positive and negative integer.

Lemma 18. *Let $a \in \mathbb{Z}$. Then $a/1$ is a positive rational number if and only if a is a positive integer. Likewise, $a/1$ is a negative rational number if and only if a is a negative integer.*

Theorem 19 (Trichotomy version 1). *If $r \in \mathbb{Q}$ then exactly one of the following occurs: (i) $r = 0$, (ii) r is positive, (iii) r is negative.*

6. ORDERED FIELDS

Definition 4. An *ordered field* F is a field with a designated subset P such that (i) P is closed under addition and multiplication, and (ii) for any element $u \in F$ exactly one of the following occurs: $u = 0, u \in P, -u \in P$.

Definition 5. Let F be an ordered field with designated subset P . The elements in P are called the *positive elements*.

Theorem 20. *The field \mathbb{Q} is an ordered field.*

Definition 6. Let F be an ordered field with designated subset P . As in the case of $F = \mathbb{Q}$, if $u \in F$ is such that $-u \in P$ then u is said to be *negative*. We often write the designated subset P as $F_{>0}$.

For most of the rest of the chapter we will consider theorems about a general ordered field F . Thus everything applies to both \mathbb{Q} and \mathbb{R} .

As the name suggests, an ordered field is ordered: it has a natural order relation on it. This order is defined as follows.

Definition 7. Let F be an ordered field, and let $x, y \in F$. If $y - x$ is positive then we write $x < y$. We also write $y > x$ in this case.

Exercise 4. Show that $y - x$ is negative if and only if $y < x$. Here x and y are in an ordered field.

Theorem 21. Suppose $u \in F$ where F is an ordered field. Then u is positive if and only if $u > 0$. Similarly, u is negative if and only if $u < 0$.

Theorem 22 (Transitivity). Suppose $x, y, z \in F$ where F is an ordered field. If $x < y$ and $y < z$ then $x < z$.

Theorem 23. Suppose $x, y, z \in F$ where F is an ordered field. If $x < y$ then $x + z < y + z$.

Theorem 24. Let $x, y, x', y' \in F$. If $x < y$ and $x' < y'$ then $x + x' < y + y'$.

Theorem 25 (Trichotomy version 2). Suppose $x, y \in F$ where F is an ordered field. Then exactly one of the following occurs: (i) $x = y$, (ii) $y < x$, or (iii) $x < y$.

Theorem 26. Suppose $x, y \in F$ where F is an ordered field. If x and y are positive, then xy is positive. If x is positive, but y is negative, then xy is negative. If x and y are negative, then xy is positive.

Theorem 27. Suppose $x, y, z \in F$ where F is an ordered field. If $x < y$, and if z is positive, then $xz < yz$. If $x < y$, and if z is negative, then $xz > yz$.

The follow statement is already known for $F = \mathbb{Q}$. The point of proving it here is to show that it is true of any other possible ordered field F .

Theorem 28. Suppose x is a positive element of an ordered field F . Then x^{-1} is also positive. Suppose x is a negative element of an ordered field F . Then x^{-1} is also negative.

Theorem 29. Suppose x, y are positive elements of an ordered field F . If $x < y$ then $y^{-1} < x^{-1}$.

Now we consider the special case where $F = \mathbb{Q}$. Recall that \mathbb{Z} is regarded as a subset of \mathbb{Q} . We have an order for \mathbb{Z} from Chapter 3, and an order for $F = \mathbb{Q}$ defined in the current section. We now show that the new order extends the old order.

Lemma 30. The order relation $<$ on \mathbb{Q} extends the order relation $<$ on \mathbb{Z} . In other words, if $a, b \in \mathbb{Z}$, then $a < b$ (as defined in Chapter 3) if and only if $a/1 < b/1$ (as defined in this section).

Theorem 31. Suppose a and b are integers, and d is a positive integer. Then $a/d > b/d$ if and only if $a > b$.

7. LESS THAN OR EQUAL

Let F be an ordered field. For example, F can be \mathbb{Q} .

Definition 8. If $x, y \in F$ then $x \leq y$ means $(x < y) \vee (x = y)$. We also write $y \geq x$ in this case.

Theorem 32. Let $x, y \in F$. Then the negation of $x < y$ is $y \leq x$. The negation of $y \leq x$ is $x < y$.

Theorem 33 (Mixed transitivity). Let $x, y, z \in F$. If $x < y$ and $y \leq z$ then $x < z$. Likewise, if $x \leq y$ and $y < z$ then $x < z$.

Theorem 34 (Transitivity). Let $x, y, z \in F$. If $x \leq y$ and $y \leq z$ then $x \leq z$.

Theorem 35. Let $x, y, z \in F$. If $x \leq y$ then $x + z \leq y + z$.

Theorem 36. Let $x, y, z \in F$ where $x \leq y$. If $z \geq 0$ then $xz \leq yz$. If $z \leq 0$ then $yz \leq xz$.

Theorem 37. Let $x, y, x', y' \in F$. If $x \leq y$ and $x' \leq y'$ then $x + x' \leq y + y'$.

8. ABSOLUTE VALUE

Let F be an ordered field. For example, F can be \mathbb{Q} .

Definition 9. The *absolute value* $|x|$ of $x \in F$ is defined as follows.

$$|x| \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Theorem 38. If $x \in F$ then $|x| \geq 0$.

Theorem 39. Let $x \in F$. Then

$$|x| = 0 \iff x = 0.$$

Theorem 40. Let $x \in F$. Then

$$|x| > 0 \iff x \neq 0.$$

Theorem 41. Let $x \in F$. Then $|x| = |-x|$.

Absolute value is compatible with multiplication.

Theorem 42. Let $x, y \in F$. Then

$$|xy| = |x| \cdot |y|.$$

Theorem 43. Suppose $x, y \in F$ where $y \geq 0$. Then

- (i) $|x| < y$ if and only if $-y < x < y$,
- (ii) $|x| > y$ if and only if $x > y$ or $x < -y$, and
- (iii) $|x| = y$ if and only if $x = y$ or $x = -y$.

Corollary 44. Suppose $x, y \in F$ where $y \geq 0$. Then

- (i) $|x| \leq y$ if and only if $-y \leq x \leq y$,
- (ii) $|x| \geq y$ if and only if $x \geq y$ or $x \leq -y$, and

The following is sometimes called the “triangle inequality” since the analogous vector version says that the third side of a triangle can be no larger than the sum of the lengths of the other two sides.

Theorem 45. *If $x, y \in F$ then*

$$|x + y| \leq |x| + |y|.$$

9. DENSITY

Ordered fields, including \mathbb{Q} are dense. This means that given any two elements one can find a element between them. (Observe that \mathbb{Z} is not dense).

Theorem 46. *Let $x, y \in \mathbb{Q}$ be such that $x < y$. Then we can find an element $z \in \mathbb{Q}$ with $x < z < y$. In other words, \mathbb{Q} is dense.*

The proof is based on choosing $z = (x + y)/2$.