NEUTRAL GEOMETRY

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1. Introduction

In the previous document on IBC Geometry we saw that much of traditional Euclidean geometry can be given a careful and rigorous treatment in IBC Geometry. There are, however, some principles of geometry that require more axioms. For example, in IBC Geometry we cannot count on lines and circles intersecting as we expect them to. In addition, in IBC Geometry we cannot expect to be able to associate real numbers to segments and angles, and segments and angles to real numbers. These defects can be remedied by adding one or more *continuity* or *completeness axioms*. Dedekind's Axiom, although harder to use than the other axioms of geometry, is powerful enough to prove all the continuity principles needed in geometry, so it is often used as the single continuity axiom. Finally, there is much in geometry that depends on a *parallel axiom*.

In this document, we will discuss a geometry that has all the axioms except for a parallel axiom. It is called *Neutral Geometry* since it is neutral concerning the truth or falsity of the traditional parallel axiom. We will implement Neutral Geometry simply by adding Dedekind's Axiom to the axioms of IBC Geometry. In the next document, we will add a parallel axiom which will complete Euclidean Geometry.

Due to the complexity of using Dedekind's Axiom, and the subtleties of the real number system, we will confine some of the proofs of Neutral Geometry to appendices, or skip them entirely. Thus this document is not as self-contained as the others in this series.

2. Neutral Geometry

Neutral Geometry is used to answer the question *what can be proved without using a parallel axiom?* For example, you can prove that the angles of a triangle add up to at most 180 degrees (Saccheri-Legendre Theorem), but you cannot prove that the angles add to exactly 180 degrees. This geometry is ideal for analyzing historic attempts to prove Euclid's Fifth Postulate (E5P). After all, if you want to prove E5P or an equivalent parallel axiom, you should be able to use everything except that axiom. In other words, you should be able to use Neutral Geometry.

If you add E5P (or something equivalent) to Neutral Geometry, then you get Euclidean Geometry. But if you add the negation instead, you get Hyperbolic Geometry. So Neutral Geometry gives the theorems that are common to both of these important geometries.[1]

Neutral Geometry remedies some of the weaknesses of IBC Geometry. For example, in IBC Geometry you cannot use Euclid's proof of the existence of equilateral triangles (Prop. I-1) since it uses a fact about circles intersecting that is not available in IBC geometry In Neutral

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[1]Neutral Geometry does not cover all geometries. For example, results in Neutral Geometry do not necessarily hold in Elliptic or Spherical Geometry.

Geometry the required circles can be shown to intersect and so the proof of the existence of equilateral triangles is valid.

We now begin the official development of Neutral Geometry. Neutral Geometry consists of 5 undefined terms, 15 axioms, and anything that can be defined or proved from these.

Primitive Terms. The five primitive terms are *point*, *line*, *betweenness*, *segment congruence*, and *angle congruence*. We will adopt all the notation and definitions from IBC Geometry, so terms such as *line segment* or *triangle congruence* are ultimately defined in terms of the primitive terms.

The Primitive Term Axiom for Neutral Geometry is a preliminary axiom telling us what type of objects all the primitive terms are supposed to represent.

Axiom (Primitive Terms)**.** *The basic type of object is the point. Lines are sets of points. Betweenness is a three place relation of points. If P, Q, R are points, then $P * Q * R$ denotes the statement that the betweenness relation holds for (P, Q, R). Segment congruence is a two place relation of line segments, and angle congruence is a two place relation of angles. If $\overline{AB}$ and $\overline{CD}$ are line segments, then $\overline{AB} \cong \overline{CD}$ denotes the statement that the segment congruence relation holds between $\overline{AB}$ and $\overline{CD}$. If α and β are angles, then $\alpha \cong \beta$ denotes the statement that the angle congruence relation holds between α and β.*

The axioms of Neutral Geometry include the above Primitive Term Axiom together with the axioms I-1, I-2, I-3, B-1, B-2, B-3, B-4, C-1, C-2, C-3, C-4, C-5, C-6, and Dedekind's Axiom discussed below.

Since the axioms of IBC Geometry are a subset of the axioms of Neutral Geometry, all the propositions of IBC Geometry will hold in Neutral Geometry, and we will make free use of any previously proved proposition of IBC Geometry.

Dedekind's Axiom basically says lines have no "holes". Before we can state this axiom, we need to define the notion of a Dedekind cut.

Definition 1 (Dedekind Cut)**.** A *Dedekind cut* of a line l is a partition of l into two nonempty convex subsets Σ_1 and Σ_2. The sets Σ_1 and Σ_2 are called *slices*.

Remark. By the definition of *partition* (from set theory), the intersection of Σ_1 and Σ_2 is the empty set, but the union is all of l.

Definition 2 (Cut Point)**.** Suppose Σ_1 and Σ_2 are the slices of a Dedekind cut of a line l. Then a *cut point* C is a point on l such that for all $X, Y \in l$ such that $C * X * Y$ the points X and Y are in the same slice.

Theorem 1. *Suppose that Σ_1 and Σ_2 form a Dedekind cut of a line l. Suppose $X * C * Y$ where X and Y are in l and where C is a cut point of the given Dedekind cut. Then X and Y are in different slices.*

Proof. Suppose otherwise that X and Y are in the same slice, say Σ_1. By convexity, C is also in Σ_1. Let Z be a point of Σ_2. This point exists since each slice is nonempty. By properties of four point betweenness we have either (i) $Z - X - C - Y$, (ii) $X - Z - C - Y$, (iii) $X - C - Z - Y$, or (iv) $X - C - Y - Z$. Case (ii) and (iii) cannot occur because in these cases $X * Z * Y$, which violates convexity. Finally, (i) and (iv) cannot occur because they violate the definition of cut point. For example, in case (iv) we have $C * Y * Z$, but Y and Z are in different slices. So in each case we have a contradiction. $\square$

Remark. A cut point is intuitively a point touching both Σ_1 and Σ_2. It is a member of one of the two sets, and borders the other.

Exercise 1. (Easy) Let Σ_1 and Σ_2 be a Dedekind cut of a line l. Show that a cut point, if it exists, cannot be between two points of Σ_1. Likewise, it cannot be between two points of Σ_2.

Exercise 2. Let Σ_1 and Σ_2 be a Dedekind cut of a line l. Show that a cut point, if it exists, is between Σ_1 and Σ_2 in the following sense: if $X \in \Sigma_1$ and $Y \in \Sigma_2$ are not equal to the cut point C then $X * C * Y$.

Lemma 2. *A cut point for a Dedekind cut, if it exists, is unique.*

Proof. Suppose C_1 and C_2 are distinct cut points. By Axiom B-2 there are points X and Y such that $C_1 * X * C_2$ and $C_1 * C_2 * Y$. By the squeeze theorem, $C_1 - X - C_2 - Y$. Since C_1 is a cut point and $C_1 * X * C_2$, we have that X and C_2 are in the same slice. Since $C_1 * Y * C_2$ we have that Y and C_1 are in the same slice. Thus, X and Y are in the same slice. However, since $X * C_2 * Y$, this contradicts Theorem 1. □

Now we give Dedekind's Axiom, the final axiom of Neutral Geometry.[2]

Axiom (Dedekind's Axiom)**.** *Every Dedekind cut of a line has a cut point.*

Remark. As we saw above, a cut point is in some sense between Σ_1 and Σ_2. If a Dedekind cut did not have a cut point, then there would be a "hole" between Σ_1 and Σ_2. So, informally speaking, the above axiom says that lines have no holes. In other words, lines are "complete". Thus Dedekind's Axiom is sometimes called the *completeness axiom*.[3]

Dedekind cuts and cut points can be defined for line segments as well.

Definition 3. A *Dedekind cut* of a line segment $\overline{AB}$ is a partition of $\overline{AB}$ into two nonempty convex subsets Σ_1 and Σ_2 (called *slices*).

Definition 4. Suppose Σ_1 and Σ_2 are the slices of a Dedekind cut of $\overline{AB}$ Then a *cut point* C is a point on $\overline{AB}$ such that for all $X, Y \in \overline{AB}$ such that $C * X * Y$ the points X and Y are in the same slice.

There is a Dedekind Principle for segments:

Proposition 3. *Every Dedekind cut of a line segment has a cut point.*

[2]Note: the above theorem, exercises and lemma are valid in IBC geometry since they do not use the new axiom. Also, these results are valid for Dedekind cuts of segments.

[3]The notion of a cut was originally formulated by Dedekind for $\mathbb{Q}$ and $\mathbb{R}$ in order to build a foundation for the real numbers that is independent of geometry. Its later use in geometry is motivated by the concept that a traditional geometric line should be related to the real number line $\mathbb{R}$.

To see how it applies to $\mathbb{Q}$ and $\mathbb{R}$, consider first the rational number line $\mathbb{Q}$. It has a "hole" at $\sqrt{2}$ since $\sqrt{2}$ is irrational. The rational numbers less than $\sqrt{2}$ forms one set Σ_1 and the rational number greater than $\sqrt{2}$ form another Σ_2. This gives a Dedekind cut with no cut point in $\mathbb{Q}$. If there were a cut point it would be $\sqrt{2}$, but $\sqrt{2} \notin \mathbb{Q}$. This shows that the rational numbers fail the axiom. The real numbers $\mathbb{R}$, on the other hand, satisfy the axiom. Interestingly, Dedekind constructed the real numbers out of cuts of $\mathbb{Q}$.

Proof. (Sketch) Let Σ_1 and Σ_2 be a Dedekind cut of a segment $\overline{AB}$. Without loss of generality, suppose $A \in \Sigma_1$. It follows that $B \in \Sigma_2$ (otherwise, $\overline{AB} \subseteq \Sigma_1$ by convexity, but Σ_2 is not empty). Let Σ_1' be the union of Σ_1 with $\{X \mid X * A * B\}$. Let Σ_2' be the union of Σ_2 with $\{X \mid A * B * X\}$. One can show that Σ_1' and Σ_2' form a Dedekind cut of $\overleftrightarrow{AB}$. Thus there is a cut point C by Dedekind's Axiom. One can show that C is a cut point for Σ_1, Σ_2, and that $C \in \overline{AB}$. □

Exercise 3. One step of the above proof, skipped in the sketch, is that Σ_1' and Σ_2' are convex. Prove that Σ_1' in the proof is indeed convex.

Exercise 4. Show that if the cut point of a Dedekind cut of a segment is an endpoint A, then $\{A\}$ is one of the two slices.

3. Circles: some basics

In IBC geometry one cannot in general prove that circles that one expects to intersect will actually intersect. A similar situation occurs between line segments and circles that one expects to intersect. Neutral Geometry corrects this problem. Before describing these results, we describe some preliminaries. None of the definitions and results in this preliminary section require Dedekind's Axiom or its consequences, and so they can be thought of as part of IBC geometry. The next section will, however, depend on Dedekind's Axiom.

Definition 5 (Circle)**.** Let $\overline{AB}$ be a segment. Then the *circle* with center A and radius $\overline{AB}$ is defined to be the set of all points X such that $\overline{AX} \cong \overline{AB}$. In other words, it is the set

$$\{X \mid \overline{AX} \cong \overline{AB}\}.$$

If $\overline{AX} \cong \overline{AB}$ then $\overline{AX}$ is called a *radius.*

Remark. Let γ be the circle with center A and radius $\overline{AB}$. If $\overline{AY}$ is another radius of γ, then γ can also be described as the circle with center A and radius $\overline{AY}$. This is easily proved using the fact that $\cong$ is an equivalence relation.

Definition 6 (Interior and Exterior of a Circle)**.** Let γ a circle with center A and radius $\overline{AB}$. Then the *interior* of γ is defined to be the set

$$\{X \mid \overline{AX} < \overline{AB}\} \cup \{A\}.$$

The *exterior* of γ is defined to be the set

$$\{X \mid \overline{AX} > \overline{AB}\}.$$

By the trichotomy law for segments, every point is either (i) in the interior, (ii) in the circle itself, or (iii) in the exterior; furthermore, exactly one of these possibilities occurs.

Definition 7 (Disk)**.** The union of a circle γ with its interior is called a *closed disk.* The interior is sometimes called an *open disk.*

Definition 8 (Tangent, Chord, Diameter)**.** A *tangent* to a circle γ is a line that intersects γ in exactly one point. If C, D are distinct points on a circle γ, then $\overline{CD}$ is called a *chord.* A chord that contains the center of γ is called a *diameter.*

Proposition 4. *The interior of a circle is convex.*

Proof. (sketch) Let X, Y be in the interior, we must show that $\overline{XY}$ is a subset of the interior. Let O be the center of the circle. The case where X, Y, O are colinear is easier, and is left to the reader. So suppose that X, Y, O are not collinear. Then we can use the following lemmas to show that $\overline{XY}$ is contained in the interior. □

Lemma 5. *Let X, Y, O be noncollinear points such that $\overline{OX} < \overline{OY}$ If $X * B * Y$ then $\overline{OB} < \overline{OY}$*

Lemma 6. *Let X, Y, O be noncollinear points such that $\overline{OX} \cong \overline{OY}$. If $X * B * Y$ then $\overline{OB} < \overline{OY}$*

Exercise 5. Prove the above lemmas. Hint: use a result from the IBC handout concerning the triangle $\triangle OXY$ (look in the section on inequalities involving triangles).

The following can be proved in a similar manner.

Proposition 7. *Closed disks are convex.*

Exercise 6. Show that the exterior of a circle is not convex.

We will need the following lemma later.

Lemma 8. *Suppose γ is a circle, and that $\overline{AB}$ is a line segment with A interior to γ and B exterior to γ. Then the intersection of $\overline{AB}$ and the interior of γ is convex, and the intersection of $\overline{AB}$ with the exterior of γ is convex.*

Proof. By Proposition 4 the interior of γ is convex. From Incidence-Betweenness geometry, we know that $\overline{AB}$ is convex. Therefore, the intersection is convex.

Now we show that the intersection of $\overline{AB}$ and the exterior of γ is convex. Suppose not. Then there are points $X * Z * Y$ on $\overline{AB}$ such that X and Y are in the exterior of γ, but Z is not. In other words, Z is in the closed disk. Switching the labels X and Y if necessary, we can assume $A * X * Y$. This implies A-X-Z-Y, which in turn implies that $A * X * Z$. Since A and Z are in the closed disk, and the closed disk is convex, we have X is in the closed disk, a contradiction. □

4. Intersections with Circles

In Neutral Geometry we have the following intersection principle:

Theorem 9 (Circle-Segment Intersection Principle)**.** *Suppose that γ is a circle, and that $\overline{AB}$ is a segment such that A is in the interior of γ and B is in the exterior of γ. Then the segment must intersect the circle.*

Proof. (sketch) Suppose $\overline{AB}$ does not intersect γ. Let Σ_1 be the set of points of $\overline{AB}$ in the interior of γ, and let Σ_2 be the set of points of $\overline{AB}$ in the exterior of γ. By Lemma 8, this gives a Dedekind cut of the segment $\overline{AB}$.

By Proposition 3 there is a cut point $C \in \overline{AB}$. The cut point must be in Σ_1 or Σ_2. However, one can show that no point of Σ_1 can be a cut point.[4] Similarly, no point of Σ_2 can be a cut point. This gives a contradiction. □

[4]This can be shown using the version of the triangle inequality presented in the IBC handout. Details are left to the reader.

The proof of the following will be discussed in an appendix.

Theorem 10 (Circle-Circle Intersection Principle)**.** *Suppose that γ and δ are two circles. If γ contains both a point in the interior of δ and a point in the exterior of δ, then gamma contains a point of δ. In other words, the circles intersect.*

Here is an application of a theorem that can be proved in Neutral Geometry:

Proposition 11 (Euclid's First Proposition)**.** *If $\overline{AB}$ is a segment, then there is a point C such that $\triangle ABC$ is an equilateral triangle.*

Proof. Let γ be the circle with center A and radius $\overline{AB}$. Let δ be the circle with center B and radius $\overline{BA}$. Observe that $B \in \gamma$, so γ has a point in the interior of δ. Let D be a point such that $D * A * B$ and $\overline{AB} \cong \overline{AD}$ (Axiom B-2 and C-2). Then $D \in \gamma$. But $\overline{DB} > \overline{AB}$ so D is in the exterior of δ.

By the Circle-Circle Intersection Principle, there is a point C in the intersection of γ and δ. By the following lemma, A, B, C are non-collinear, so we get a triangle $\triangle ABC$. Since $C \in \gamma$, we must have $\overline{AC} \cong \overline{AB}$. Since $C \in \delta$ we must have $\overline{BC} \cong \overline{BA}$. Thus all three sides of $\triangle ABC$ are congruent. So $\triangle ABC$ is an equilateral triangle. $\square$

Lemma 12. *In the above proof, $C \notin \overleftrightarrow{AB}$.*

Proof. By uniqueness assertion of Axiom C-2, D is the only point of γ on $\overrightarrow{AD}$ and B is the only point of γ on $\overrightarrow{AB}$. So $\overleftrightarrow{AB} \cap \gamma = \{B, D\}$. Now $C \neq B$ and $C \neq D$ since B is interior to δ, D is exterior to δ, and $C \in \delta$. Thus $C \notin \gamma \cap \overleftrightarrow{AB}$. Hence, $C \notin \overleftrightarrow{AB}$ since $C \in \gamma$. $\square$

5. The Archimedean Principle

Definition 9. Let $\overline{EF}$ be a line segment.[5] Then $1 \cdot \overline{EF}$ is defined to be $\overline{EF}$. Let F_2 be a point such that $E * F * F_2$ and $\overline{FF_2} \cong \overline{EF}$ (Axioms B-2 and C-2). Define $2 \cdot \overline{EF}$ to be $\overline{EF_2}$. We continue recursively. Suppose we have defined $k \cdot \overline{EF}$ to be a segment $\overline{EF_k}$ where $F_k \neq E$ is on the line $\overleftrightarrow{EF}$. Let F_{k+1} be a point such that $E * F_k * F_{k+1}$ and $\overline{F_k F_{k+1}} \cong \overline{EF}$. Define $(k+1) \cdot \overline{EF}$ to be the segment $\overline{EF_{k+1}}$. In this way we define $n \cdot \overline{EF}$ for all positive integers n.

Here is an important principle of Neutral Geometry that cannot be proved in IBC Geometry. Its proof will be discussed in an appendix.

Theorem 13 (Archimedean Principle)**.** *Suppose $\overline{AB}$ and $\overline{EF}$ are line segments. Then there is an integer n such that $n \cdot \overline{EF} > \overline{AB}$. (This holds no matter how small $\overline{EF}$ is or how big $\overline{AB}$ is)*

Remark. This principle, which is not a theorem of IBC geometry, is the key to introducing real valued measures of segments and angles.

Remark. In IBC Geometry we proved that every segment has a unique midpoint, so we can define $\frac{1}{2^n}\overline{EF}$ for all $n \geq 0$ (something like this is done to prove Theorem 14). A consequence of the Archimedean Principle is the following principle: *Suppose $\overline{AB}$ and $\overline{EF}$ are line segments. Then there is an integer n such that $\frac{1}{2^n}\overline{EF} < \overline{AB}$. (This holds no matter how large $\overline{EF}$ is or how small $\overline{AB}$ is.)*

[5]And fix an ordering of the endpoints.

6. Segment Measure

Up to this point we have not really needed the real numbers $\mathbb{R}$. We have only used $\mathbb{R}$ for models of geometry, but not in the theoretic development of geometry itself. This is in the spirit of Euclid who did not assume the existence of such a system of numbers.

However, to do geometry in a modern way, real numbers are useful. For example, They measure segments and angles. They can also be used to measure areas and volumes, but this is beyond the scope of these notes.

The following is a theorem of Neutral Geometry since its proof requires the Archimedean Principle.[6]

Theorem 14 (Segment Measure). *Let $\overline{OI}$ be a fixed segment (called the unit). There is a unique way to assign to each segment $\overline{AB}$ a positive real number $\left|\overline{AB}\right| \in \mathbb{R}$ (called its length) in such a way that*

(i) $A * B * C$ if and only if $\left|\overline{AC}\right| = \left|\overline{AB}\right| + \left|\overline{BC}\right|$.

(ii) $\left|\overline{AB}\right| = \left|\overline{BC}\right|$ if and only if $\overline{AB} \cong \overline{BC}$.

(iii) $\left|\overline{OI}\right| = 1$.

Furthermore

(iv) $\left|\overline{AB}\right| < \left|\overline{CD}\right|$ if and only if $\overline{AB} < \overline{CD}$.

and

(v) for each positve $x \in \mathbb{R}$, there exists a segment $\overline{AB}$ such that $\left|\overline{AB}\right| = x$.

Proof. (Sketch) Let I_1 be the midpoint of $\overline{OI}$. Let I_2 be the midpoint of $\overline{OI_1}$. In general, let I_{k+1} be the midpoint of $\overline{OI_k}$. Intuitively, $\left|\overline{OI_k}\right|$ should be $1/2^k$.

Let $\overline{AB}$ be a given segment. For each integer $i \geq 0$ let n_i be the maximum natural number such that $n_i \cdot OI_i$ is not greater than $\overline{AB}$. The Archimedean Principle guarantees that such n_i exists for each i. Define the ith approximation to the length as follows:

$$m_i \stackrel{\text{def}}{=} \frac{n_i}{2^i}.$$

One can show that $n_{i+1} = 2n_i$ or $n_{i+1} = 2n_i + 1$. In particular,

$$m_i \leq m_{i+1} \leq m_{i+1} + \frac{1}{2^{i+1}}.$$

One can also show that m_i is bounded below by m_0 and bounded above by $m_0 + 1$. One can also show, by the Archimdean Principle, that $n_i > 0$ for sufficiently large i. This means that the sequence (m_i) is monotonic and bounded. There is a property of the real numbers $\mathbb{R}$ that guarantees that the limit exists. Define the length as

$$\left|\overline{AB}\right| \stackrel{\text{def}}{=} \lim_{i \to \infty} m_i.$$

Since $m_i > 0$ for sufficiently large i, we can show that the limit is positive. (Observe that the approximation m_0 gives the integral part of the length).

In the special case that $\overline{AB} = \overline{OI}$ we see that $n_i = 2^i$, and so $m_i = 1$ for all i. This means that the length (the limit) is 1.

[6]The Archimedean Principle, IBC Geometry, and some basic properties of $\mathbb{R}$ are all that is needed for most of the proof. Part (v), however, requires more than just the Archimedean Principle. It requires a direct appeal to Dedekind's Axiom.

If $A * B * C$ then the integer n_i used for $\overline{AC}$ is approximately the sum of the integers needed for $\overline{AB}$ and $\overline{BC}$. In fact, the difference is at most two. When we divide by 2^i and take the limit we find that the length is additive.

We skip the proof of the rest of the details (see an Appendix for more information). □

Remark. Note that $<$ has two meanings: one meaning was established for line segments in IBC geometry. The other is its meaning for real numbers. The above shows that these two are compatible.

Theorem 15 (Triangle Inequality)**.** *Let $\triangle ABC$ be a triangle. Then $|\overline{AC}| < |\overline{AB}| + |\overline{BC}|$.*

Proof. Let D be a point such that $A*B*D$ and $\overline{BD} \cong \overline{BC}$ (Axiom B-2 tells us that there is a point with $A*B*X$, and Axiom C-2 tells us that there is a point D on the ray $\overrightarrow{BX}$ such that $\overline{BD} \cong \overline{BC}$). Then $\overline{AC} < \overline{AD}$ by the IBC form of the triangle inequality (See IBC handout). So $|\overline{AC}| < |\overline{AD}|$ (Theorem 14 part iv). Since $A * B * D$ we have $|\overline{AD}| = |\overline{AB}| + |\overline{BD}|$ (Theorem 14 part i). Thus $|\overline{AC}| < |\overline{AB}| + |\overline{BD}|$.

Finally, $|\overline{BD}| = |\overline{BC}|$ since $\overline{BD} \cong \overline{BC}$ (Theorem 14 part ii). □

7. Angle Measure

In a similar manner, we can define angle measure.

Theorem 16 (Angle Measure)**.** *There is a unique way to assign to each angle α a positive real number $|\alpha| \in \mathbb{R}$ (called its degree measure) in such a way that*

(i) if D is interior to $\angle BAC$ then $|\angle BAC| = |\angle BAD| + |\angle DAC|$.

(ii) $|\alpha| = |\beta|$ if and only if $\alpha \cong \beta$.

(iii) α is a right angle if and only if $|\alpha| = 90$.

Furthermore,

(iv) $|\alpha| < |\beta|$ if and only if $\alpha < \beta$.

(v) $|\alpha| < 180$ for every angle α,

(vi) if α and β are supplementary, then $|\alpha| + |\beta| = 180$, and

(vii) for each $0 < x < 180$ there exists an angle α such that $|\alpha| = x$.

Proof. (sketch) The idea is similar to segment measure, except that a fixed right angle γ_0 plays the role of $\overline{OI}$. We bisect γ_k to form γ_{k+1}. We define the ith approximation by

$$m_i \stackrel{\text{def}}{=} 90\frac{n_i}{2^i}$$

where n_i is defined in an analogues manner as in the proof of Theorem 14.

Various lemmas have to be proved, such as the fact that if you subtract a right angle from an obtuse angle, the result is acute. □

Proposition 17. *Let $\triangle ABC$ be a triangle. Then $|\angle B| + |\angle C| < 180$.*

Proof. Let D be such that $B * C * D$ (Axiom B-2). By the Exterior Angle Theorem of IBC Geometry, $\angle ACD > \angle B$. So $|\angle ACD| > |\angle B|$ (Theorem 16 part iv). Since $\angle C$ and $\angle ACD$ are supplementary, $|\angle C| + |\angle ACD| = 180$ (Theorem 16 part vi). Thus

$$|\angle B| + |\angle C| < |\angle ACD| + |\angle C| = 180.$$

□

8. Defects of Triangles

Definition 10 (Angle Sum and Defect)**.** If $\triangle ABC$ is a triangle, then the *angle sum* is defined as

$$\sigma ABC \stackrel{\text{def}}{=} |\angle A| + |\angle B| + |\angle C|.$$

The *defect* is defined as

$$\delta ABC \stackrel{\text{def}}{=} 180 - \sigma ABC.$$

In other words,

$$\delta ABC = 180 - |\angle A| - |\angle B| - |\angle C|.$$

One reason for using defects instead of simple angle sums is that fact that it adds when a triangle is subdivided:

Proposition 18. *Let $\triangle ABC$ be a triangle, and let D be such that $A * D * C$. Then*

$$\delta ABC = \delta ABD + \delta DBC.$$

Proof. By the Crossbar-Betweenness Proposition, D is interior to $\angle B$. By the Angle Measure Theorem, $|\angle B| = |\angle ABD| + |\angle DBC|$. In addition, since $\angle ADB$ and $\angle BDC$ are supplementary, $|\angle ADB| + |\angle BDC| = 180$. Thus

$$\begin{aligned}
\delta ABD + \delta DBC &= 180 - |\angle A| - |\angle ABD| - |\angle ADB| \\
&\qquad +180 - |\angle DBC| - |\angle BDC| - |\angle C| \\
&= 360 - |\angle A| - |\angle C| - \big(|\angle ABD| + |\angle DBC|\big) \\
&\qquad -\big(|\angle ADB| + |\angle BDC|\big) \\
&= 360 - |\angle A| - |\angle C| - |\angle B| - 180 \\
&= 180 - |\angle A| - |\angle C| - |\angle B| \\
&= \delta ABC.
\end{aligned}$$

□

9. The Saccheri-Legendre Theorem

The Saccheri-Legendre Theorem is one of the most important theorems of neutral geometry.[7]

The following will be critical to the proof. It says that we can make an angle of a triangle smaller while preserving the angle sum and defect.

Lemma 19. *Let $\triangle ABC$ be a triangle where $\angle A$ has measure α. Then there is another triangle $\triangle XYZ$ such that $\sigma ABC = \sigma XYZ$ and $\delta ABC = \delta XYZ$, and such that $\triangle XYZ$ has an angle with measure at most $\alpha/2$.*

Proof. Let M be the midpoint of $\overline{BC}$. Let D be such that $A * M * D$ and $\overline{AM} \cong \overline{MD}$. By the Crossbar-Betweenness Proposition, M is in the interior of $\angle A$. This implies that if $\alpha_1 = |\angle BAM|$ and if $\alpha_2 = |\angle MAC|$ then $\alpha = \alpha_1 + \alpha_2$. By the Crossbar-Betweenness Proposition, M is in the interior of $\angle ACD$, so $|\angle ACD| = \gamma + \gamma'$ where γ is the angle measure of $\angle MCA$ and γ' is the angle measure of $\angle MCD$.

[7] It cannot be proved in IBC geometry due to a model given by M. Dehn.

By SAS, $\triangle AMB \cong \triangle DMC$ (using the Vertical Angle Theorem). So $\angle B \cong \angle MCD$ (where $\angle B$ is the angle in the original triangle). Thus $\gamma' = \beta$ where β is the angle measure of $\angle B$. Likewise, $\angle CDM \cong \angle BAM$. So $\angle CDM$ has measure α_1.

Observe that σABC and σACD are both $\alpha_1 + \alpha_2 + \beta + \gamma$. So $\sigma ABC = \sigma ACD$. This implies $\delta ABC = \delta ACD$ as well.

Now $\alpha = \alpha_1 + \alpha_2$, so some α_i must be less than $\alpha/2$. If $\alpha_1 \leq \alpha/2$ then $\angle ADC$ has measure $\alpha_1 \leq \alpha/2$. If $\alpha_2 \leq \alpha/2$ then $\angle DAC$ has measure $\alpha_2 \leq \alpha/2$. In either case, $\triangle ACD$ has an angle of measure $\alpha_i \leq \alpha/2$. Let $\triangle XYZ$ be $\triangle ACD$. □

Corollary 20. *Let $\triangle ABC$ be a triangle such that $\angle A$ has measure α. Let k be a positive integer. Then there is another triangle $\triangle XYZ$ such that $\delta ABC = \delta XYZ$ and such that $\triangle XYZ$ has an angle with measure at most $\alpha/2^k$.*

Proof. Use the above lemma k times. □

Lemma 21. *Suppose $\delta ABC < 0$, and let $\epsilon = -\delta ABC$. Then every angle of $\triangle ABC$ has angle measure greater than or equal to ϵ.*

Proof. Suppose, to the contrary, that $|\angle A| < \epsilon$. Now $|\angle B| + |\angle C| < 180$ by Proposition 17. Thus

$$|\angle A| + |\angle B| + |\angle C| < 180 + \epsilon.$$

So

$$-\epsilon < 180 - |\angle A| - |\angle B| - |\angle C|$$

a contradiction to the assumption that $\delta ABC = -\epsilon$. □

Now we can prove the main result.

Theorem 22 (Saccheri-Legendre)**.** *If $\triangle ABC$ is a triangle, then $\delta ABC \geq 0$. In other words*

$$|\angle A| + |\angle B| + |\angle C| \leq 180.$$

Proof. Suppose not. Then $\delta ABC = -\epsilon$ for some positive ϵ.

Let $\alpha = |\angle A|$. Since the sequence $\alpha/2^i$ converges to 0, there is a k such that $\alpha/2^k < \epsilon$. By Corollary 20, there is a triangle $\triangle XYZ$ such that $\delta XYZ = \delta ABC = -\epsilon$, and such that $\angle X$ (say) has measure at most $\alpha/2^k$. So $|\angle X| < \epsilon$ contradicting the Lemma 21. □

Because of the Saccheri-Legendre Theorem, we can conclude the following.

Corollary 23. *Let $\triangle ABC$ be a triangle, and let D be such that $A * D * C$. Then $\delta ABC = 0$ if and only if $\delta ABD = \delta DBC = 0$.*

Proof. Suppose $\delta ABC = 0$. By Proposition 18,

$$0 = \delta ABC = \delta ABD + \delta DBC,$$

and by the Saccheri-Legendre Theorem all terms are non-negative. The conclusion follows.

The other direction follows from Proposition 18. □

10. Appendix: Proof of the Archimedean Principle

This appendix gives a sketch of the proof of the Archimedean Principle in Neutral Geometry.

Suppose the Archimedean Principle fails for a particular $\overline{AB}$ and $\overline{EF}$. Let Σ_1 be the set of elements $X \in \overline{AB}$ with the following property: $X = A$ or the theorem is true concerning $\overline{AX}$ and $\overline{EF}$. For example, Σ_1 contains every point D on $\overline{AB}$ such that $\overline{AD} < \overline{EF}$ (just take $n = 1$ in this case). Let Σ_2 be the set of elements $X \in \overline{AB}$ such that $D \neq A$ and such that the theorem fails of $\overline{AX}$ and $\overline{EF}$.

By assumption $A \in \Sigma_1$ and $B \in \Sigma_2$. Thus Σ_1 and Σ_2 is a partition of $\overline{AB}$ into two non-empty subsets. If we show that Σ_1 and Σ_2 are convex then we can conclude that they form a Dedekind cut. Showing convexity is pretty straightforward. For example, suppose that $X, Y \in \Sigma_1$. We discuss the case where $\overline{AY} > \overline{AX}$. Other cases are similar. If $Z \in \overline{XY}$ it follows that $\overline{AZ} < \overline{AY}$. Since the theorem of true of $\overline{AY}$ and $\overline{EF}$, it is true of $\overline{AZ}$ and $\overline{EF}$ for the same value of n. Thus $Z \in \Sigma_1$. So $\overline{XY} \subseteq \Sigma_1$. This shows Σ_1 is convex. To show that Σ_2 is convex suppose that $X, Y \in \Sigma_2$. We discuss the case where $\overline{AX} < \overline{AY}$. Other cases are similar. If $Z \in \overline{XY}$ it follows that $\overline{AX} < \overline{AZ}$. If $Z \in \Sigma_1$ then it would follow that $X \in \Sigma_1$ using the same value of n, a contradiction. Thus $Z \in \Sigma_2$. We conclude that $\overline{XY} \subseteq \Sigma_2$. Hence Σ_2 is also convex.

By Dedekind's Axiom, there is a cut point C. Either $C \in \Sigma_1$ or $C \in \Sigma_2$.

First consider the case where $C \in \Sigma_1$ and $C \neq A$. So there is an integer n such that $n \cdot \overline{EF} > \overline{AC}$. In other words, there is a point X with $A * C * X$ such that $n \cdot \overline{EF} \cong \overline{AX}$. Since $\overline{AX}$ does not intersect Σ_2, contradicting the fact that C is a cut point. The case $C = A$ is similar (take $n = 1$).

Now consider the case where $C \in \Sigma_2$. Let X be a point of $\overrightarrow{CA}$ such that $\overline{CX} \cong \overline{EF}$. Since C is a cut point, there must be a point $Y \in \Sigma_1$ in $\overline{CX}$. This means that the theorem is true for $\overline{AY}$ for some n (take $n = 1$ if $Y = A$). This implies that the theorem is true for $\overline{AC}$ for $n + 1$. So $C \in \Sigma_2$, a contradiction.

In any case, we get a contradiction.

11. Appendix: Additional Details

NEXT DRAFT?