MODELS OF HYPERBOLIC AND EUCLIDEAN GEOMETRY

MATH 410. SPRING 2007. INSTRUCTOR: PROFESSOR AITKEN

The goal of this handout is to discuss models of Hyperbolic and Euclidean Geometry, and the consistency of Hyperbolic Geometry.

1. Some concepts from Euclidean Geometry

We will use circles in Euclidean Geometry to build up models for Hyperbolic Geometry. So we will review some basic facts about circles. (Mostly without proofs). In this section assume that we are working in Euclidean Geometry.

Definition 1 (Chord)**.** Let γ be a circle. If A and B are distinct points on γ, then the set $\{C \mid A * C * B\}$ is called an open *chord* of γ. The points A and B are called *endpoints* of the chord (but A and B are not themselves in the open chord).

Definition 2 (Diameter)**.** Let γ be a circle. An open chord containing the center of γ is called an *open diameter*.

Definition 3 (Tangent)**.** Let γ be a circle. A line that intersects γ in exactly one point is called a *tangent* to γ.

Proposition 1. *If A is a point on a circle γ, then there is exactly one tangent l to γ passing through A.*

Lemma 1. *Let γ be a circle, and let r be an open chord with endpoints A and B. Let l be the tangent containing A and let m be the tangent containing B. If r is not an open diameter, then l and m intersect at a point P outside γ.*

Definition 4 (Pole)**.** Let r and P be as in the above lemma. Then P is called the *pole* of r.

Lemma 2. *Let r and s be two chords of a circle γ that do not intersect. Suppose the endpoints of r and s are distinct. Let P be the pole of r and let Q be the pole of s. Then the line $\overleftrightarrow{PQ}$ intersects the circle γ in two points, and intersects both r and s. Thus $\overleftrightarrow{PQ}$ contains a chord that intersects both r and s.*

Definition 5 (Orthogonal Circles)**.** Let γ and β be two circles intersecting in exactly two points P_1 and P_2. Suppose the tangent to γ containing P_1 and the tangent to β containing P_1 are perpendicular. Suppose this condition holds for P_2 as well. Then we say that γ and β are *orthogonal*.

The following (whose proof we skip) shows the existence of orthogonal circles. In fact, if we fix two points, we get uniqueness as well.

Theorem 1. *Let γ be a circle, and let A and B be two distinct points in the interior of γ. Then there is a unique circle β containing A and B that is orthogonal to γ.*

Date: Spring 2006 and Spring 2007. Version of May 8, 2007.

2. The Beltrami-Klein Model of Hyperbolic Geometry

This model is usually called the *Klein Model*. In this model, you start with a circle γ in the Euclidean plane. Call points in the Euclidean plane *E-points*. Not all E-points will be points in the Klein Model.

The points in the Klein Model, called *K-points*, are the E-points inside γ. The lines in the model, called *K-lines*, are the open chords. Betweenness works as follows: if A, B, C are K-points, then $A * B * C$ holds in the Klein Model if and only if $A * B * C$ in Euclidean Geometry.

Finally, we should define congruence in the Klein Model. Actually, one usually first defines a distance function (which is different from the Euclidean distance) and an angle measure function (which is different from the Euclidean version). Then one defines congruence in terms of these functions. For example, two K-segments are congruent if and only if the distance between their respective endpoints is equal. I will skip the definitions of the distance and angle measure function since they are not all that entertaining. But realize that there is an infinite distance from the center to the boundary circle γ with this new distance, even though this Euclidean distance is finite (just the radius r of γ).[1]

One then has to verify that the Klein Model is a model for Hyperbolic Geometry. For instance, to verify axiom I-1 one needs to show that for any two K-points, there is a unique chord (K-line) containing them. Some of the axioms require quite a bit of work to verify (which we skip due to lack of time). However, you should be able to verify easily (with a drawing, say) that the final axiom of Hyperbolic Geometry holds: the Hyperbolic Parallel Postulate (HPP).

Although we skipped the angle measure definition for lack of time, we can explain what perpendicular K-lines are in the Klein Model. Two K-lines r and s are perpendicular if either (i) r is an open diameter of γ and the angles produced by r and s are right angles in the Euclidean sense (in other words, that the E-line containing r is perpendicular to the E-line containing s), or (ii) the E-line containing s goes through the pole of r.

Warning: the pole of a K-line is not a K-point. It is an E-point that is outside of the model. From the point of view of the Klein Model, it is an imaginary point. Likewise, the endpoints of a K-line (chord) are imaginary from the point of view of the Klein Model: they are E-points on γ, but they are not K-points.

We can restate Lemma 2 as follows:

Lemma 3. *Let r and s be two K-lines that do not intersect. Suppose the endpoints of r and s are distinct. Let P be the pole of r and let Q be the pole of s. Then the E-line $\overleftrightarrow{PQ}$ contains a K-line that is perpendicular (in the sense of the Klein Model) to both r and s. (Warning: $\overleftrightarrow{PQ}$ will usually not be perpendicular to r and s in the Euclidean sense.)*

So parallel K-lines with distinct endpoints have a common perpendicular: they are type 1 parallels. It turns out that parallel K-lines that share an endpoint are type 2 parallels.

Exercise 1. Draw type 1 and type 2 parallels in the Klein Model. Given type 1 parallels, show with a drawing how to find the common perpendicular.

[1]See equations (25) and (26) on page 419 of the textbook if you are curious about distance and angle measure in the Klein Model.

3. The Poincaré Disk Model of Hyperbolic Geometry

The Poincaré Disk Model has the advantage that angle measure agrees with Euclidean angle measure (it is a *conformal* model). Unfortunately distance is distorted. Also the lines are not usually straight in the Euclidean sense.

In this model, you start with a circle γ in the Euclidean plane. Call points in the Euclidean plane *E-points*. Not all E-points will be points in the Poincaré Disk Model.

The points in the Poincaré Disk Model, called *P-points*, are the E-points inside γ. If β is a circle that is orthogonal to γ, then the intersection of β with the interior of γ is called a *P-line*. Open diameters are also considered to be P-lines.

We will skip the formal definition of betweenness, but it is the obvious idea. Finally, we should define congruence in the Poincaré Disk Model. Actually, one usually first defines a distance function (which is different from the Euclidean distance) and an angle measure function (which is closely related to the Euclidean version). Then one defines congruence in terms of these functions. For example, two P-segments are congruent if and only if the distance between their respective endpoints is equal. I will skip the definitions of the distance. But realize that there is an infinite distance from the center to the boundary circle γ with this new distance, even though this Euclidean distance is finite.

The definition of angle measure is very natural: given a P-angle α, one looks at the tangent E-rays to the P-rays making up the angle. Then the angle measure of α is defined to be the Euclidean angle measure of the angle made up of the associated E-rays. Two P-angles are said to be congruent if and only if they have the same angle measure.

One then has to verify that the Poincaré Disk Model is a model for Hyperbolic Geometry. For instance, to verify axiom I-1 one needs to use Theorem 1. Some of the axioms require quite a bit of work to verify (which we skip due to lack of time). However, you should be able to verify easily (with a drawing, say) that the final axiom of Hyperbolic Geometry holds: the HPP.

Exercise 2. Draw type 1 and type 2 parallels in the Poincaré Disk Model. Given a P-point and a P-line, draw the angle of parallelism.

4. General Discussion on Models

A *model* is an interpretation of the primitive or "undefined" terms of a theory. All the defined terms will then be interpreted in the model as well, since the defined terms are defined in terms of the primitive terms. For it to be a model for the axioms, all the axioms must be true under the interpretation. All the theorems that are provable from the axioms will also be true.

Models are good for seeing that your theory really does apply to *something*. For example, mathematicians became more comfortable with the complex numbers once a model (the complex plane) was discovered. Before then, imaginary numbers were just funny numbers whose squares were negative. Once the model was discovered, complex numbers could be thought of something more tangible: points in the plane. Likewise, mathematicians became more comfortable with Hyperbolic Geometry when Beltrami developed models in the 1860s.

Using a model is in some sense the opposite of developing a theory: (i) When you develop a theory, you leave all the primitive terms undefined, but when you set up a model you interpret (i.e., define) these primitive terms. (ii) When you develop a theory, you do not prove the axioms but you take then as given; in contrast, when you verify that an interpretation is a model, you must verify (i.e., prove) the axioms. (iii) When you develop a theory, you define new terms in terms of previously defined terms or undefined (primitive) terms, but when you work in a model you only need to interpret (i.e., define) the primitive terms since the interpretation of the defined terms will follow for free. (iv) When you develop a theory you like to prove theorems (including propositions, corollaries, lemmas), but when you work in a model you only have to verify (i.e., prove) the axioms, the theorems are automatically true since they are consequences of the axioms: you get them for free.

Every theorem is true in a model, but not everything that is true in a model can be proved from the axioms. For example, there are models of incidence geometry that are finite, but you cannot prove that there is a finite number of points in incidence geometry. (There are many models of incidence geometry that have an infinite number of points).

This observation does not just apply to geometry, it applies to any axiom system. For example, models of group theory are called "groups". Every theorem of group theory is true in every model (group), but not everything that is true in one given group is a theorem of group theory.

Models are useful for showing unprovability, independence, and/or consistency. For example, before models for Hyperbolic Geometry were developed, no one could be sure that Hyperbolic Geometry was consistent. It was still conceivable that even after Bolyai and Lobachevski a contradiction could be found in Hyperbolic Geometry (thereby proving the Euclidean Parallel Postulate with a proof by contradiction as envisioned by Saccheri). The models of Hyperbolic Geometry rely on the use of Euclidean Geometry (or the real numbers), so the consistency proof is a *relative consistency proof*: it shows Hyperbolic Geometry is consistent relative to Euclidean Geometry (or the real numbers).

Recall the idea of an isomorphism between models. This is a one-to-one and onto function (bijection) from the objects of one model to the objects of the other preserving all the primitive (undefined) relations. If an axiom system has the property that all its models are isomorphic, then we say that the axioms are *categorical*. This means that you are done, you do not need to add any more axioms to specify your geometry. Neutral geometry is not categorical: $\mathbb{R}^2$ and the Poincaré Disk are two non-isomorphic models. We know that they are non-isomorphic since there are statements that are true in one but false in the other.[2] An example of such a statement is the Euclidean Parallel Postulate: it is true in the model $\mathbb{R}^2$, but false in the Poincaré Disk.

5. Models of Euclidean Geometry

Theorem 2. *All models of Euclidean Planar Geometry are isomorphic.*

Proof. (Sketch) One defines the model $\mathbb{R}^2$ in the usual way. By setting up perpendicular coordinate axes in an arbitrary model $\mathcal{M}$, one gets an isomorphism between $\mathcal{M}$ and $\mathbb{R}^2$. Since all models are isomorphic to $\mathbb{R}^2$, all models are isomorphic to each other. (To carry out the details of this proof would require a several handouts all by itself, and some knowledge of real analysis.) $\square$

[2]If a statement, expressible in terms of the language of the theory, is true in one model it must be true in any isomorphic model.

The above theorem implies that our axioms of Euclidean Planar Geometry are *categorical*. This implies that our axioms are in some sense complete. In other words, we have no need to search for new axioms.[3]

Recall that the axioms for Euclidean Planar Geometry are all the axioms for IBC-Geometry, plus the Dedekind Axiom, plus the Parallel Postulate (or any other EC of your choice).

6. Models of Hyperbolic Geometry

We will investigate several models of Hyperbolic Geometry. It is important to realize that the models are actually isomorphic.

Theorem 3. *All models of Hyperbolic Planar Geometry are isomorphic.*

Proof. (Sketch) By setting up perpendicular coordinate axes in an arbitrary model $\mathcal{M}$, and using Beltrami coordinates[4], one gets an isomorphism between $\mathcal{M}$ and the Beltrami-Klein model (see below). Since all models are isomorphic to the Beltrami-Klein model, all models are isomorphic to each other. $\square$

The above theorem implies that our axioms of Hyperbolic Planar Geometry are *categorical*. This implies that our axioms are in some sense complete. In other words, we have all the axioms needed for Hyperbolic Planar Geometry.

The above theorem tells us that the Klein Model and the Poincaré Disk Model are isomorphic. There is a particularly nice isomorphism between the Klein Model and the Poincaré Disk Model. Let γ_1 be a unit circle in the xy-plane in Euclidean space $\mathbb{R}^3$ with center $(0, 0, 0)$, and consider the Klein Model $\mathcal{M}_1$ given by the points inside γ_1. Let γ_2 be a radius two circle in the xy-plane with the same center as γ_1, and consider the Poincaré Disk Model $\mathcal{M}_2$ given by the points inside γ_2. Let S be a sphere of radius 1 placed so that one point on S, which we call the "south pole", is the origin $(0, 0, 0)$, and the antipodal point to this point, which we call the "north pole", has coordinates $(x, y, z) = (0, 0, 2)$.

Then the function f from points of $\mathcal{M}_1$ to points of $\mathcal{M}_2$ is defined by the following rule: start with P, then find the point P' on the sphere S directly above P (with the same x and y coordinate, but with $0 \leq z < 1$). Now consider the line connecting the north pole of S with P'. Then $f(P)$ is defined to be the intersection of this line with the xy-plane.

It requires some work to show that the function f gives an isomorphism. One must show (we will skip the details) that $f(P)$ is inside γ_2, that f is one-to-one and onto, that f sends K-lines to P-lines, that f preserves betweenness, that f sends K-congruent K-segments to P-congruent P-segments, and that f sends K-congruent K-angles to P-congruent P-angles.

Exercise 3. Give a drawing illustrating the process of going from P to $f(P)$. (Hint: see page 236.)

7. The Poincaré Half Plane Model of Hyperbolic Geometry

The Half Plane Model is also a conformal model. Unfortunately distance is distorted. Also, the lines are not usually straight in the Euclidean sense.

[3] To explain completeness and incompleteness in more detail would require a digression into mathematical logic. There is also a sense in which our axioms, combined with the standard axioms of set theory, are incomplete. This relates to Gödel's Theorem.

[4] See pages 417-418 of the textbook for definitions if you are curious.

In this model, you start with the set of points in the (Euclidean) $\mathbb{R}^2$ plane with positive y-coordinate: $\{(x, y) \mid y > 0\}$ called the *upper half plane*. The points in the Half Plane Model, called *H-points*, are points in the upper half plane. If β is a circle whose center is contained in the x-axis, then the intersection of β with the upper half plane is called an *H-line* (note: the center of β is not an H-point. It is imaginary from the point of view of this model). Vertical lines (intersected with the upper half plane) are also considered to be H-lines.

We will skip the formal definition of betweenness, but it is the obvious idea. Finally, we should define congruence in the Half Plane Model. Actually, one usually first defines a distance function (which is different from the Euclidean distance) and an angle measure function (which is closely related to the Euclidean version). Then one defines congruence in terms of these functions. For example, two H-segments are congruent if and only if the distance between their respective endpoints is equal. I will skip the definitions of the distance. But realize that there is an infinite distance from any point in the upper half plane to the x-axis, even though this Euclidean distance is finite (just the y coordinate of the point).

The definition of angle measure is very natural: given an H-angle α, one looks at the tangent E-rays to the H-rays making up the angle. Then the angle measure of α is defined to be the Euclidean angle measure of the angle made up of the associated E-rays. A pair of H-angles are said to be congruent if and only if they have the same angle measure.

One then has to verify that the Half Plane Model is a model for Hyperbolic Geometry. Some of the axioms require quite a bit of work to verify (which we skip due to lack of time). However, you should be able to verify easily (with a drawing, say) that the final axiom of Hyperbolic Geometry holds: the HPP.

Exercise 4. Draw type 1 and type 2 parallels in the Upper Half Plane Model. Given an H-point and an H-line, draw the angle of parallelism.

8. Isometric Models: do they exist?

There is a natural model for Spherical Geometry: the points are points on a sphere in Euclidean Space. The lines are the great circles.[5] Now great circles are known to give the shortest curve between two points. Such curves are called *geodesics*. Angle measure and distance measure (along a geodesic) is natural in this model: there is no distortion.

Models where angles and distances (along geodesics) are not distorted are called *isometric*. If distances are distorted, but angles are not, the model is said to be *conformal*.

Gauss came up with the idea of *curvature* in a surface (curvature of curves was defined earlier: you might have seen it in Calculus III). Saddle points have negative curvature, mountain tops have positive curvature. Flat places have zero curvature. Even points on a cylinder have zero curvature since a cylinder can be "unrolled" without distortion.

In most surfaces, the curvature varies from point to point, but the sphere has constant positive curvature. In the 1860s, Beltrami came up with the idea that surfaces of constant *negative* curvature would make good models for Hyperbolic Geometry. Fortunately, people already knew of a surface of negative curvature. Gauss came up with the *pseudosphere*

[5]One can tweak this model to get a model for Elliptic Geometry. In this case points are pairs of antipodal points on the sphere, but lines are still great circles on the sphere.

and proved it had negative curvature It is generated by rotating the tractrix (a well-known curve).[6]

Beltrami proved that the pseudosphere was an isometric model for a piece of the hyperbolic plane. Points are interpreted as points on this surface, and lines are interpreted as geodesics. Unfortunately it is not a complete model of Hyperbolic Geometry.

Hilbert proved a difficult theorem that showed that is is impossible to find an isometric model of the (entire) Hyperbolic Plane as a surface in Euclidean space: you are forced to introduce some distortion. To learn more about curvature, geodesics, and this theorem of Hilbert, you need to learn some differential geometry.

Dr. Wayne Aitken, Cal. State, San Marcos, CA 92096, USA
E-mail address: waitken@csusm.edu

[6]See see page 396 for a nice picture of the pseudophere.