TOPICS IN HYPERBOLIC GEOMETRY

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1. Introduction

We defined Hyperbolic Geometry in the last handout and considered a list of Hyperbolic Conditions, each of which is a theorem in Hyperbolic Geometry. In this handout we explore Hyperbolic Geometry further.

2. The Two Types of Parallels in Hyperbolic Geometry

There are two types of parallel lines in Hyperbolic Geometry. There are those who diverge from each other in both directions (type 1) and those that diverge in one direction but come arbitrarily close to each other in the other direction (type 2). The first type have a minimum positive distance at points on a common perpendicular. The second type have no minimum distance: the distance tends to zero in one direction. Here is the formal definition.

Definition 1. If l and m are parallel lines such that there exists a line t perpendicular to both, then we say that l and m are *type 1 parallels*. If l and m are parallel lines with no such common perpendicular, then we say that l and m are *type 2 parallels*.

We will state some results about type 1 and type 2 parallels. However, we will only prove the statements that have short proofs. The rest we will take on faith.

Let l and m be parallel. In a previous handout we established that if m has three distinct points of equal distance to l then rectangles exist. Since rectangles do not exist in Hyperbolic Geometry we cannot have three such points. It is possible that there are two points on m of equal distance to l. However, this only is possible for type 1 parallels:

Proposition 1. *Suppose that l and m are parallel lines, and suppose P and P' are two points on m that are equidistant from l. Then l and m are type 1 parallel lines.*

Corollary 2. *If l and m are type 2 parallel lines, then any two points of m have different distances from l.*

Proof. The corollary is just the contrapositive of the proposition, so we just need to prove the proposition. So let l and m be parallel lines, and let P and P' be distinct points on m that are equidistant from l. Drop a perpendicular from P to l and let Q be its foot. Drop a perpendicular from P' to l and let Q' be its foot. Observe that $\square PQQ'P'$ is a Sacchari quadrilateral.

In the Quadrilateral handout we showed that if M is the midpoint of $\overline{PP'}$ and if N is the midpoint of $\overline{QQ'}$ then $\overleftrightarrow{MN}$ is perpendicular to l and m. Thus l and m are type 1 parallels. $\square$

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Proposition 3. *Suppose that l and m are type 1 parallel lines. Then the common perpendicular to l and m is unique.*

Proof. Otherwise we could construct a rectangle. But rectangles do not exist in Hyperbolic Geometry. ☐

Proposition 4. *Suppose that l and m are type 1 parallel lines. Suppose that $P \in m$ and $Q \in l$ are chosen so that $\overleftrightarrow{PQ}$ is perpendicular to both l and m. If A and B are points on m such that $A * P * B$ and $\overline{AP} \cong \overline{PB}$, then A and B are equidistant from l.*

Proof. By SAS, $\triangle PAQ \cong PBQ$. In particular, $\overline{AQ} \cong \overline{BQ}$ and $\angle AQP \cong BQP$. Drop a perpendicular from A to l and let C be its foot. Drop a perpendicular from B to l and let D be its foot. (One can show that C and D are on opposite sides of $\overleftrightarrow{PQ}$ so $C * Q * D$).

Since $\angle AQP \cong BQP$ we get $\angle AQC \cong \angle BQD$ (complementary angles). Since $\overline{AQ} \cong \overline{BQ}$ we get, by AAS, that $\triangle AQC \cong \triangle BQD$. We conclude that $\overline{AC} \cong \overline{BD}$. ☐

From the above results, we easily get the following:

Corollary 5. *If l and m are type 1 parallels, then we can find two points on m that are equidistant from l. So two parallel lines are of type 1 if and only if there are two point on one line that are equidistant from the second. Thus, two parallel lines are of type 2 if and only any two points on one line are of non-equal distance from the other line.*

Finally, we show that as you go away from the common perpendicular, the lines get farther apart:

Proposition 6. *Suppose that l and m are type 1 parallel lines. Suppose that $P \in m$ and $Q \in l$ are chosen so that $\overleftrightarrow{PQ}$ is perpendicular to both l and m. If A and B are points on m such that $P * A * B$, then the distance from B to l is greater than the distance from A to l.*

Proof. Drop a perpendicular from A to l and let C be its foot. Drop a perpendicular from B to l and let D be its foot. Observe that $\square PQCA$ and $\square PQDB$ are Lambert quadrilaterals. Thus (by a proposition in the Quadrilateral handout and the fact that rectangles do not exist), $\angle CAP$ and $\angle DBP$ are acute. Since $\angle BAC$ is supplementary to $\angle CAP$, we have that $\angle BAC$ is obtuse. Thus $\angle BAC > \angle DBP$. This implies, by a result in the Quadrilateral handout, that $\overline{BD} > \overline{AC}$. ☐

Now we discuss some results, but skip the proof.

Theorem 1. *Let l be a line, and P a point not on l. Then there are exactly two type 2 (limiting) parallels to l passing through P.*

Theorem 2. *Let l be a line, P be a point not on l, and Q the point on l such that $\overleftrightarrow{PQ}$ is perpendicular to l. On each side of $\overleftrightarrow{PQ}$ there is a smallest angle $\angle QPA$ such that $\overleftrightarrow{PA}$ is parallel to l. The smallest angles on each side are congruent. The line $\overleftrightarrow{PA}$ produced is a Type 2 parallel.*

Definition 2 (Angle of Parallelism)**.** Type 2 parallels are also called *limiting parallels* because of the these theorems. The size of $\angle QPA$ in the above theorem is called the *angle of parallelism*.

Theorem 3. *Let l be a line, and P a point not on l. Drop a perpendicular from P to l, and let Q be the foot. Then the measure of the angle of parallelism for l and P depends only on the distance $x = \left|\overline{PQ}\right|$.*

Definition 3. Let l be a line, P a point not on l. Drop a perpendicular from P to l, and call the foot Q. Let $\angle QPA$ be an angle of parallelism and let $x = \left|\overline{PQ}\right|$. We define $\Pi(x)$ to be the measure of $\angle QPA$.

The following in a famous theorem of Bolyai and Lobachevsky

Theorem 4 (Bolyai, Lobachevsky)**.** *The function $\Pi(x)$ is strictly decreasing on the interval $(0, \infty)$. Its limit as $x \to 0$ is 90 (measured in degrees). Its limit as $x \to \infty$ is 0. In fact, the function $\Pi(x)$ is given by the formula*

$$\Pi(x) = 2\tan^{-1}(e^{-x/k})$$

for some constant k. (Here the $\tan^{-1}$ is chosen so its result is in degrees.)

Remark 1. This result illustrates an important principle in Hyperbolic geometry: for short distances Hyperbolic Geometry is a lot like Euclidean Geometry. So for small x, the angle measure $\Pi(x)$ is close to 90. Observe that in Euclidean Geometry $\Pi(x) = 90$.

The type 2 parallels to a given line and through a given point P are the limiting parallels. Any line "between" these two will be a type 1 parallel. This is described in the following theorem.

Theorem 5. *Let l be a line, and P a point not on l. Drop a perpendicular from P to l, and let Q be the foot. Let $x = \left|\overline{PQ}\right|$ be the distance. Suppose A is a point such that P, Q, A are not collinear. Then $\overleftrightarrow{AP}$ is a type 1 parallel to l if and only if $\Pi(x) < |\angle APQ| < 180 - \Pi(x)$. Also, $\overleftrightarrow{AP}$ is a type 2 parallel to l if and only if $|\angle APQ| = \Pi(x)$ or $|\angle APQ| = 180 - \Pi(x)$ Finally, if if $|\angle APQ| < \Pi(x)$ or $|\angle APQ| > 180 - \Pi(x)$ then $\overleftrightarrow{AP}$ is not parallel to l.*

3. The natural unit of length

The constant k discussed in the Bolyai-Lobachevsky Theorem is an important constant of Hyperbolic Geometry. As discussed above on small scales, the Hyperbolic Plane "looks like" the Euclidean Plane, and the larger k is, the more Euclidean looking the geometry in a region of fixed size. For instance, if our universe were hyperbolic the constant k would have to be astronomical since the universe looks Euclidean at the scale of the solar system and the neighboring star systems.[1] Note that $\Pi(x)$ will be very close to 90 when x/k is very small, so if k is huge then for reasonable x we have that $\Pi(x)$ is approximately 90.

A natural unit of length can be chosen so that $k = 1$.

[1]According to a comment in *Greenberg*, if you assume that the universe is hyperbolic, then the constant k would have to be over "six hundred trillion miles" based on the parallax of stars. So, for common distances, x/k is very small, and $\Pi(x)$ is practically equal to 90. So it would be difficult to do a small scale experiment to decide if we lived in a Euclidean or a Hyperbolic universe. According to Einstein's theory of general relativity, the universe is neither hyperbolic in the sense of having constant negative curvature, nor Euclidean, but is a four-dimensional manifold of variable curvature. In his theory, curvature explains the force of gravity.

Theorem 6. *Hyperbolic Geometry has a natural measure of length. With respect to this length,*

$$\Pi(x) = 2\tan^{-1}(e^{-x})$$

The existence of a natural length is very surprising since Euclidean Geometry has no such natural length.

4. Other Results in Hyperbolic Geometry

Area can be defined in both Euclidean and Hyperbolic Geometry (we skip the formal definition). The main result, which Gauss seems to realized as a teenager, is the following.

Theorem 7. *The area of $\triangle ABC$ is proportional to δABC. In fact, if you measure δABC in radians, then*

$$\mathit{area}(\triangle ABC) = k^2 \delta ABC$$

where k is the constant occurring in Theorem 4.

Corollary 7. *The area of a triangle is bounded by πk^2.*

This is very surprising that a triangle can get no larger than a certain constant. As the triangle gets closer to having area πk^2. the defect approaches π (or 180 degrees), and the measures of the angles get very small.

Remark 2. If you use the natural unit of length with $k = 1$, the π is the bound on the area of triangles.

Remark 3. There is no bound on the length of sides of a triangle. You can have a triangle with large base and height, but the area is still bounded by πk^2. So the formula $\frac{1}{2}bh$ fails in Hyperbolic Geometry.

Remark 4. The above Theorem show that for triangles of very small area, the defect is practically 0 as in Euclidean Geometry. This in another reason we say that, on the small scale, Hyperbolic Geometry looks like Euclidean Geometry.

Since quadrilaterals are made up of two triangles, we get the following.

Corollary 8. *The area of a regular quadrilateral is bounded by $2\pi k^2$.*

This generalizes to regular polygons of any fixed number of sides (you get $(n-2)\pi k^2$ where n is the number of sides). Now you can think of a circle as the limit of polygons as the number of sides goes to infinity. This allows circles in hyperbolic geometry to get arbitrarily large.

There are nice formulas for the circumference and areas of circles.

Theorem 8. *Let γ be a circle with radius r. Let C be the circumference, and A the area of the circle. Then*[2]

$$C = \pi\big(e^r - e^{-r}\big) \qquad A = \pi\big(e^{r/2} - e^{-r/2}\big)^2.$$

(Here we are using the natural unit of measure where $k = 1$.)

[2]For those of you who are familiar with hyperbolic sines the formulas can be written as $C = 2\pi\sinh(r)$ and $A = 4\pi\sinh(r/2)^2$.

Remark 5. This formula shows that circles can have arbitrarily large area, even though triangles cannot.

Remark 6. The Taylor series of e^r is $1 + r + r^2/2 + r^3/3! + r^4/4! + \ldots$, so the Taylor series of $e^r - e^{-r}$ is $2r + 2r^3/3! + 2r^5/5! + \ldots$, which gives a good idea of what goes on when r is very small. In particular, the above formulas become

$$C = 2\pi r + 2\pi r^3/3! + \ldots \qquad A = \pi r^2 + \pi r^4/12 + \ldots.$$

In particular, *in Hyperbolic Geometry, the radius and area of a circle is larger than what we expect in Euclidean Geometry, but for small r they are approximately equal.* This gives another indication that at a small scale, Hyperbolic Geometry is very similar to Euclidean Geometry.

Here $k = 1$. So if the radius is small relative to the natural unit of length, a circle has approximately the same circumference and area as in Euclidean geometry.

Remark 7. In addition to what we have done, there is a trigonometry for Hyperbolic Geometry. It is amusing that hyperbolic trig functions are used extensively in this trigonometry since they were not invented for Hyperbolic Geometry.

Finally, there is a close but opposing relationship between Spherical (and Elliptic) Geometry and Hyperbolic Geometry. For example, in Spherical Geometry the *surplus* of the angle sum above 180° is proportional to area. Also, there are formulas for the circumference and area of a circle, but they give answers that are *smaller* than those in Euclidean Geometry. Of course, for really small circles and triangles on the surface of the sphere, one can approximate areas and such with Euclidean Geometry, just like you can in Hyperbolic Geometry.

5. The Beltrami-Klein Model of Hyperbolic Geometry

Now we consider models of Hyperbolic Geometry. These are based on Euclidean Geometry. Since there are models of Hyperbolic Geometry inside of Euclidean Geometry, we know that Hyperbolic Geometry is consistent (assuming, of course, that Euclidean Geometry is consistent).

We begin with the *Beltrami-Klein Model.* This model is usually called the *Klein Model.* In this model, you start with a circle γ in the Euclidean plane. Call points in the Euclidean plane *E-points.* Not all E-points will be points in the Klein Model.

The points in the Klein Model, called *K-points*, are the E-points inside γ. The lines in the model, called *K-lines*, are the open chords. Betweenness works as follows: if A, B, C are K-points, then $A * B * C$ holds in the Klein Model if and only if $A * B * C$ in Euclidean Geometry.

Finally, we should define congruence in the Klein Model. Actually, one usually first defines a distance function (which is different from the Euclidean distance) and an angle measure function (which is different from the Euclidean version). Then one defines congruence in terms of these functions. For example, two K-segments are congruent if and only if the distance between their respective endpoints is equal. I will skip the definitions of the distance and angle measure functions since they are rather technical, and not necessary for this short discussion. But realize that there is an infinite distance from the center to the boundary circle γ with this new distance, even though this Euclidean distance.[3]

[3]See *Greenberg* if you are curious about the exact definition of distance and angle measures.

One then has to verify that the Klein Model is a model for Hyperbolic Geometry. For instance, to verify axiom I-1 one needs to show that for any two K-points, there is a unique chord (K-line) containing them. Some of the axioms require quite a bit of work to verify. You should be able to verify easily (with a drawing, say) that HPP (the Hyperbolic Parallel Property) holds.

There is a nice description of perpendicular K-lines in the Klein Model. Two K-lines r and s are perpendicular if either (i) r is an open diameter of γ and the angles produced by r and s are right angles in the Euclidean sense (in other words, that the E-line containing r is perpendicular to the E-line containing s), or (ii) the E-line containing s goes through the pole of r. (See the appendix for the definition of pole).

Warning: the pole of a K-line is not a K-point. It is an E-point that is outside of the model. From the point of view of the Klein Model, it is an imaginary point. Likewise, the endpoints of a K-line (chord) are imaginary from the point of view of the Klein Model: they are E-points on γ, but they are not K-points.

We can restate Lemma 12 as follows:

Lemma 9. *Let r and s be two K-lines that do not intersect. Suppose the endpoints of r and s are distinct. Let P be the pole of r and let Q be the pole of s. Then the E-line $\overleftrightarrow{PQ}$ contains a K-line that is perpendicular (in the sense of the Klein Model) to both r and s. (Warning: $\overleftrightarrow{PQ}$ will usually not be perpendicular to r and s in the Euclidean sense.)*

So parallel K-lines with distinct endpoints have a common perpendicular: they are type 1 parallels. It turns out that parallel K-lines that share an endpoint are type 2 parallels.

Exercise 1. Draw type 1 and type 2 parallels in the Klein Model. Given type 1 parallels, show with a drawing how to find the common perpendicular.

6. The Poincaré Disk Model of Hyperbolic Geometry

The Poincaré Disk Model has the advantage that angle measure agrees with Euclidean angle measure (it is a *conformal* model). Unfortunately distance is distorted. Also the lines are not usually straight in the Euclidean sense.

In this model, you start with a circle γ in the Euclidean plane. Call points in the Euclidean plane *E-points*. Not all E-points will be points in the Poincaré Disk Model.

The points in the Poincaré Disk Model, called *P-points*, are the E-points inside γ. If β is a circle that is orthogonal to γ, then the intersection of β with the interior of γ is called a *P-line*. Open diameters are also considered to be P-lines.

We will skip the formal definition of betweenness, but it is the obvious idea. Finally, we should define congruence in the Poincaré Disk Model. Actually, one usually first defines a distance function (which is different from the Euclidean distance) and an angle measure function (which is closely related to the Euclidean version). Then one defines congruence in terms of these functions. For example, two P-segments are congruent if and only if the distance between their respective endpoints is equal. I will skip the definitions of the distance. But realize that there is an infinite distance from the center to the boundary circle γ with this new distance, even though this Euclidean distance is finite.

The definition of angle measure is very natural: given a P-angle α, one looks at the tangent E-rays to the P-rays making up the angle. Then the angle measure of α is defined to be the

Euclidean angle measure of the angle made up of the associated E-rays. Two P-angles are said to be congruent if and only if they have the same angle measure.

One then has to verify that the Poincaré Disk Model is a model for Hyperbolic Geometry. For instance, to verify axiom I-1 one needs to use Theorem 11. Some of the axioms require quite a bit of work to verify (which we skip due to lack of time). However, you should be able to verify easily (with a drawing, say) that HPP holds.

Exercise 2. Draw type 1 and type 2 parallels in the Poincaré Disk Model. Given a P-point and a P-line, draw the angle of parallelism.

7. General Discussion on Models

A *model* is an interpretation of the primitive or "undefined" terms of a theory. All the defined terms will then be interpreted in the model as well, since the defined terms are defined in terms of the primitive terms. For it to be a model for the axioms, all the axioms must be true under the interpretation. All the theorems that are provable from the axioms will also be true.

Models are good for seeing that your theory really does apply to *something*. For example, mathematicians became more comfortable with the complex numbers once a model (the complex plane) was discovered. Before then, imaginary numbers were just funny numbers whose squares were negative. Once the model was discovered, complex numbers could be thought of something more tangible: points in the plane. Likewise, mathematicians became more comfortable with Hyperbolic Geometry when Beltrami developed models in the 1860s.

Using a model is in some sense the opposite of developing a theory: (i) When you develop a theory, you leave all the primitive terms undefined, but when you set up a model you interpret (i.e., define) these primitive terms. (ii) When you develop a theory, you do not prove the axioms but you take then as given; in contrast, when you verify that an interpretation is a model, you must verify (i.e., prove) the axioms. (iii) When you develop a theory, you define new terms in terms of previously defined terms or undefined (primitive) terms, but when you work in a model you only need to interpret (i.e., define) the primitive terms since the interpretation of the defined terms will follow for free. (iv) When you develop a theory you like to prove theorems (including propositions, corollaries, lemmas), but when you work in a model you only have to verify (i.e., prove) the axioms, the theorems are automatically true since they are consequences of the axioms: you get them for free.

Every theorem is true in a model, but not everything that is true in a model can be proved from the axioms. For example, there are models of incidence geometry that are finite, but you cannot prove that there is a finite number of points in incidence geometry. (There are many models of incidence geometry that have an infinite number of points).

This observation does not just apply to geometry, it applies to any axiom system. For example, models of group theory are called "groups". Every theorem of group theory is true in every model (group), but not everything that is true in one given group is a theorem of group theory.

Models are useful for showing unprovability, independence, and/or consistency. For example, before models for Hyperbolic Geometry were developed, no one could be sure that Hyperbolic Geometry was consistent. It was still conceivable that even after Bolyai and Lobachevski a contradiction could be found in Hyperbolic Geometry (thereby proving the Euclidean Parallel Postulate with a proof by contradiction as envisioned by Saccheri). The models of Hyperbolic Geometry rely on the use of Euclidean Geometry (or the real numbers), so the consistency proof is a *relative consistency proof*: it shows Hyperbolic Geometry is consistent relative to Euclidean Geometry (or the real numbers).

Recall the idea of an isomorphism between models. This is a one-to-one and onto function (bijection) from the objects of one model to the objects of the other preserving all the primitive (undefined) relations. If an axiom system has the property that all its models are isomorphic, then we say that the axioms are *categorical*. This means that you are done, you do not need to add any more axioms to specify your geometry. Neutral geometry is not categorical: $\mathbb{R}^2$ and the Poincaré Disk are two non-isomorphic models. We know that they are non-isomorphic since there are statements that are true in one but false in the other.[4] An example of such a statement is the Euclidean Parallel Postulate: it is true in the model $\mathbb{R}^2$, but false in the Poincaré Disk.

8. Models of Euclidean Geometry

Theorem 9. *All models of Euclidean Planar Geometry are isomorphic.*

Proof. (Sketch) One defines the model $\mathbb{R}^2$ in the usual way. By setting up perpendicular coordinate axes in an arbitrary model $\mathcal{M}$, one gets an isomorphism between $\mathcal{M}$ and $\mathbb{R}^2$. Since all models are isomorphic to $\mathbb{R}^2$, all models are isomorphic to each other. □

The above theorem implies that our axioms of Euclidean Planar Geometry are *categorical*. This implies that our axioms are in some sense complete. In other words, we have no need to search for new axioms.

9. Models of Hyperbolic Geometry

There are several models of Hyperbolic Geometry. It is important to realize that the models are actually isomorphic.

Theorem 10. *All models of Hyperbolic Planar Geometry are isomorphic.*

Proof. (Sketch) By setting up perpendicular coordinate axes in an arbitrary model $\mathcal{M}$, and using Beltrami coordinates[5], one gets an isomorphism between $\mathcal{M}$ and the Beltrami-Klein model. Since all models are isomorphic to the Beltrami-Klein model, all models are isomorphic to each other. □

The above theorem implies that our axioms of Hyperbolic Planar Geometry are *categorical*. This implies that our axioms are in some sense complete. In other words, we have all the axioms needed for Hyperbolic Planar Geometry.

The above theorem tells us that the Klein Model and the Poincaré Disk Model are isomorphic. There is a particularly nice isomorphism between the Klein Model and the Poincaré Disk Model. Let γ_1 be a unit circle in the xy-plane in Euclidean space $\mathbb{R}^3$ with center $(0, 0, 0)$, and consider the Klein Model $\mathcal{M}_1$ given by the points inside γ_1. Let γ_2 be a radius two circle in the xy-plane with the same center as γ_1, and consider the Poincaré Disk Model $\mathcal{M}_2$ given by the points inside γ_2. Let S be a sphere of radius 1 placed so that one point on S, which

[4] If a statement, expressible in terms of the language of the theory, is true in one model it must be true in any isomorphic model.

[5] *Greenberg* has the definition of Beltrami coordinates.

we call the "south pole", is the origin $(0,0,0)$, and the antipodal point to this point, which we call the "north pole", has coordinates $(x,y,z)=(0,0,2)$.

Then the function f from points of $\mathcal{M}_1$ to points of $\mathcal{M}_2$ is defined by the following rule: start with P, then find the point P' on the sphere S directly above P (with the same x and y coordinate, but with $0 \le z < 1$). Now consider the line connecting the north pole of S with P'. Then $f(P)$ is defined to be the intersection of this line with the xy-plane.

It requires some work to show that the function f gives an isomorphism. One must show (we will skip the details) that $f(P)$ is inside γ_2, that f is one-to-one and onto, that f sends K-lines to P-lines, that f preserves betweenness, that f sends K-congruent K-segments to P-congruent P-segments, and that f sends K-congruent K-angles to P-congruent P-angles.

Exercise 3. Give a drawing illustrating the process of going from P to $f(P)$. (Hint: such a drawing can be found in *Greenberg*.)

10. The Poincaré Half Plane Model of Hyperbolic Geometry

The Half Plane Model is also a conformal model. Unfortunately distance is distorted. Also, the lines are not usually straight in the Euclidean sense.

In this model, you start with the set of points in the (Euclidean) $\mathbb{R}^2$ plane with positive y-coordinate: $\{(x,y) \mid y > 0\}$ called the *upper half plane*. The points in the Half Plane Model, called *H-points*, are points in the upper half plane. If β is a circle whose center is contained in the x-axis, then the intersection of β with the upper half plane is called an *H-line* (note: the center of β is not an H-point. It is imaginary from the point of view of this model). Vertical lines (intersected with the upper half plane) are also considered to be H-lines.

We will skip the formal definition of betweenness, but it is the obvious idea. Finally, we should define congruence in the Half Plane Model. Actually, one usually first defines a distance function (which is different from the Euclidean distance) and an angle measure function (which is closely related to the Euclidean version). Then one defines congruence in terms of these functions. For example, two H-segments are congruent if and only if the distance between their respective endpoints is equal. I will skip the definitions of the distance. But realize that there is an infinite distance from any point in the upper half plane to the x-axis, even though this Euclidean distance is finite (just the y coordinate of the point).

The definition of angle measure is very natural: given an H-angle α, one looks at the tangent E-rays to the H-rays making up the angle. Then the angle measure of α is defined to be the Euclidean angle measure of the angle made up of the associated E-rays. A pair of H-angles are said to be congruent if and only if they have the same angle measure.

One then has to verify that the Half Plane Model is a model for Hyperbolic Geometry. Some of the axioms require quite a bit of work to verify (which we skip due to lack of time). However, you should be able to verify easily (with a drawing, say) that HPP holds.

Exercise 4. Draw type 1 and type 2 parallels in the Upper Half Plane Model. Given an H-point and an H-line, draw the angle of parallelism.

11. Isometric Models: do they exist?

There is a natural model for Spherical Geometry: the points are points on a sphere in Euclidean Space. The lines are the great circles.[6] Now great circles are known to give the shortest curve between two points. Such curves are called *geodesics*. Angle measure and distance measure (along a geodesic) is natural in this model: there is no distortion.

Models where angles and distances (along geodesics) are not distorted are called *isometric*. If distances are distorted, but angles are not, the model is said to be *conformal*.

Gauss came up with the idea of *curvature* in a surface (curvature of curves was defined earlier: you might have seen it in Calculus III). Saddle points have negative curvature, mountain tops have positive curvature. Flat places have zero curvature. Even points on a cylinder have zero curvature since a cylinder can be "unrolled" without distortion.

In most surfaces, the curvature varies from point to point, but the sphere has constant positive curvature. In the 1860s, Beltrami came up with the idea that surfaces of constant *negative* curvature would make good models for Hyperbolic Geometry. Fortunately, people already knew of a surface of negative curvature. Gauss came up with the *pseudosphere* and proved it had negative curvature It is generated by rotating the tractrix (a well-known curve).[7]

Beltrami proved that the pseudosphere was an isometric model for a piece of the hyperbolic plane. Points are interpreted as points on this surface, and lines are interpreted as geodesics. Unfortunately it is not a complete model of Hyperbolic Geometry.

Hilbert proved a difficult theorem that showed that is is impossible to find an isometric model of the (entire) Hyperbolic Plane as a surface in Euclidean space: you are forced to introduce some distortion. To learn more about curvature, geodesics, and this theorem of Hilbert, you need to learn some differential geometry.

Appendix: Some concepts from Euclidean Geometry

Above we used circles in Euclidean Geometry to construct models for Hyperbolic Geometry. So to study models of Hyperbolic Geometry, one needs a good understanding of circles in Euclidean Geometry. This appendix is provided as a reference.

In this appendix assume that we are working in Euclidean Geometry.

Definition 4 (Chord)**.** Let γ be a circle. If A and B are distinct points on γ, then the set $\{C \mid A * C * B\}$ is called an open *chord* of γ. The points A and B are called *endpoints* of the chord (but A and B are not themselves in the open chord).

Definition 5 (Diameter)**.** Let γ be a circle. An open chord containing the center of γ is called an *open diameter*.

Definition 6 (Tangent)**.** Let γ be a circle. A line that intersects γ in exactly one point is called a *tangent* to γ.

Proposition 10. *If A is a point on a circle γ, then there is exactly one tangent l to γ passing through A.*

[6] One can tweak this model to get a model for Elliptic Geometry. In this case points are pairs of antipodal points on the sphere, but lines are still great circles on the sphere.

[7] See *Greenberg* for a nice picture of the pseudophere.

Lemma 11. *Let γ be a circle, and let r be an open chord with endpoints A and B. Let l be the tangent containing A and let m be the tangent containing B. If r is not an open diameter, then l and m intersect at a point P outside γ.*

Definition 7 (Pole)**.** Let r and P be as in the above lemma. Then P is called the *pole* of r.

Lemma 12. *Let r and s be two chords of a circle γ that do not intersect. Suppose the endpoints of r and s are distinct. Let P be the pole of r and let Q be the pole of s. Then the line $\overleftrightarrow{PQ}$ intersects the circle γ in two points, and intersects both r and s. Thus $\overleftrightarrow{PQ}$ contains a chord that intersects both r and s.*

Definition 8 (Orthogonal Circles)**.** Let γ and β be two circles intersecting in exactly two points P_1 and P_2. Suppose the tangent to γ containing P_1 and the tangent to β containing P_1 are perpendicular. Suppose this condition holds for P_2 as well. Then we say that γ and β are *orthogonal*.

The following (whose proof we skip) shows the existence of orthogonal circles. In fact, if we fix two points, we get uniqueness as well.

Theorem 11. *Let γ be a circle, and let A and B be two distinct points in the interior of γ. Then there is a unique circle β containing A and B that is orthogonal to γ.*