HISTORICAL NOTES

MATH 410. SPRING 2007. INSTRUCTOR: PROFESSOR AITKEN

The goal of this handout is to describe some of the history that led up to the discovery of non-Euclidean Geometry.

The goal of many geometers before the discovery of non-Euclidean geometry was to prove Euclid's Fifth Postulate. They thought that Euclid's Fifth Postulate was too complex and not obvious enough to be an axiom, and that geometry would be better off if it could be shown to be a theorem.

Several mathematicians, including famous mathematicians such as Legendre, thought that they could prove Euclid's Fifth Postulate. But later mathematicians pointed out that in each case other assumptions were being used, often as unstated assumptions. Other mathematicians, such as Wallis, did not claim to prove Euclid's Fifth Postulate, but instead wanted to replace it with another postulate that seemed more natural.

1. Euclid (c. 300 BC)

The story begins with Euclid. In his famous book, *The Elements of Geometry*, he introduced his fifth postulate. Much credit is due to Euclid for realizing that such an axiom is needed: he avoided using it or an equivalent statement as an unstated assumption.

It is interesting to note that in Book I of the *Elements*, Euclid avoided using his Fifth Postulate as long as possible (until Proposition 29). Consequently the early propositions in the *Elements* are valid in Neutral Geometry. Perhaps Euclid himself was uncomfortable with his Fifth Postulate.

2. Proclus (AD 411 – 485)

Proclus was also an ancient Greek writer, but he lived long after Euclid. Actually he was more of a philosopher than a mathematician (he was a Neo-Platonic philosopher). He is not known for original mathematics, but rather for his commentaries on other writers.

He wrote a commentary on Book I of Euclid's *Elements*. In it he criticized Euclid for assuming the Fifth Postulate, and he gave a (flawed) proof. Actually he tried to prove the following, which he correctly observed implies Euclid's Fifth Postulate.

Claim. *If l and m are parallel lines, and if n is a third line that intersects m, then n also intersects l.*

We now look at his flawed proof. Let P be the intersection of n with m, drop a perpendicular from P to l with foot Q, and let $\overrightarrow{PY}$ be the ray of n on the side of m containing l. Drop a perpendicular from Y to m, and let X be the foot. Proclus claimed that as Y moves away

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from P, the distance from X to Y tends to infinity.[1] He asserted that when $\overline{XY}$ becomes larger than $\overline{PQ}$, the line n crosses l between P and Y.

The flaw in this proof is that Proclus assumed that just because $\overline{XY} > \overline{PQ}$ it must follow that $\overline{XY}$ crosses l. Roughly speaking, he assumed that parallel lines keep a common distance. *His assumption is just as powerful as the postulate he was trying to prove.* In fact, in Hyperbolic Geometry, it often happens that $\overline{XY}$ grows to infinity, but m never intersects l.

It is interesting to note that Proclus criticized Euclid's Fifth Postulate on the grounds that it is only plausible, but not certain. He asserted that there is no role in mathematics for plausible claims, and supported this opinion with quotes by Aristotle and Plato. He mentions that curved lines can come closer and closer to each other but not actually intersect (the word *asymptotic* comes from a Greek word meaning "non-intersecting"). He demands a proof that shows that straight lines cannot be asymptotic, but must actually intersect.[2]

3. Nasir al-Din al-Tusi (1201–1274)

(Our book calls him Nasir Eddin al-Tusi). Al-Tusi was a Persian mathematician who was famous for his work in mathematics, astronomy, philosophy, and logic. In mathematics he is known for his innovations in planar and spherical trigonometry. Some consider him to be the best astronomer between Ptolomy and Copernicus.

He was very familiar with Greek mathematics. In fact, he he wrote several commentaries on the writings of Greek mathematicians. Like other geometers, he was not content to accept Euclid's Fifth Postulate as an axiom, but wished to prove it instead. In doing so he made an assumption concerning distances: suppose m and l are lines such that a transversal t is perpendicular to l but not to m. Roughly speaking he assumed that on the side of the acute angle, points of m get closer to l and on the obtuse side they get farther away. From this assumption he was actually able to prove that the points do not just get closer on the side of the acute angle, but they actually intersect. This gives, at least one case of Euclid's fifth postulate, and the other cases are not much harder.

To discuss his proof of Euclid's fifth postulate in more detail, suppose l and m are two lines cut by a transversal t such that adjacent interior angles have angle sum less than 180. Label the lines as $l = \overleftrightarrow{QR}$, $m = \overleftrightarrow{PC}$, $t = \overleftrightarrow{PQ}$, where R and C are on the same side of t. One case he considers is where $\angle PQR$ is right, and $\angle QPC$ is acute. He wishes to prove l and m intersect. So he drops a perpendicular from C to $\overleftrightarrow{PQ}$ with foot B. In a lemma he shows that as C gets farther away from P the length of segment $\overline{PB}$ grows to infinity. So if $\overline{PC}$ is large enough, then $P * Q * B$, and then you can argue that m crosses l. Unfortunately, the proof of this lemma was based on a (reasonable) assumption about decreasing distances that cannot be justified rigorously without using the theorem he was trying to prove.

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[1]Proclus did not prove this. However, it turns out to be provable in Neutral Geomety: it known as Aristotle's Hypothesis or Axiom.

[2]Proclus said it was obvious that when the interior angles are lessened from two right angles, then the distance decreases as you move along the lines, but he wanted proof that they eventually intersect and are not merely asymptotic. I don't know if al-Tusi studied Proclus, but it is interesting to note that al-Tusi makes a similar assumption about decreasing distance: that they continue to decrease. Surprisingly, with this decreasing distance assumption al-Tusi *can* prove the lines intersect. Unfortunately, in Hyperbolic Geometry, when the angles are lessened a little from a sum of 180 degrees, the lines approach each other only for a while, but eventually grow farther apart. So the lines stay parallel.

Contrast this with Hyperbolic Geometry: as $\overline{PC}$ goes to infinity, it can happen that $\overline{PB}$ stays bounded by $\overline{PQ}$.

It is interesting to note that al-Tusi used Saccheri Quadrilaterals long before Saccheri was born. Also al-Tusi's work followed similar work on Euclid's fifth postulate by the famous Persian poet and scientist Omar Khayyam (1048 - 1122).

4. Replacing the Fifth Postulate

John Wallis (1616 – 1703) was an English mathematician famous for finding methods of integration before Newton and Leibniz. In fact, some consider him to be the best English mathematician before Newton. He was a professor at Oxford University. While there, he asked another Oxford professor, a professor of Arabic, to translate al-Tusi's work. Because of this al-Tusi's work became known in Europe. Wallis concluded that al-Tusi's proof had flaws, but it caused Wallis to search for his own solution. Finally, Wallis proposed replacing Euclid's Fifth Postulate by a axiom that he found more natural:

Axiom (Wallis)**.** *If $\triangle ABC$ is a triangle, and x is a positive real number, then there is a triangle $\triangle DEF$ similar to $\triangle ABC$ such that $\left|\overline{DE}\right| = x$.*

It turns out that Euclid's Fifth Postulate can be proved from this axiom, so it replaces the fifth postulate with another axiom. This axiom turns out to be false in Hyperbolic geometry.

Another example of replacing Euclid's Postulate with another axiom is Clairaut.[3]

Axiom (Clairaut 1741)**.** *Rectangles exist.*

This is false in Hyperbolic Geometry as well.

As another example, the modern form of the parallel postulate was first officially proposed by Playfair, although others before Playfair seemed to have realized that it was equivalent to Euclid's original Fifth Postulate. For example, Proclus' claim above is very close to Playfair's Axiom.

Axiom (Playfair 1795)**.** *Let l be a line, and P a point not on the line. Then there is a unique line m that is parallel to l and contains P.*

5. Saccheri

See the textbook (pages 154–155).

6. Legendre

See the textbook (pages 157–159, and 21–23).

[3]Another example was given by Christoph Clavius (1537-1612). His axiom was that the set of points m that are (i) equidistant from a given line l and (ii) are on a given side of l, form a line. In Hyperbolic Geometry, the set m exists but is not a line.

7. Lambert and others

In 1763, Klügel wrote a doctoral thesis outlining flaws in all 28 known proofs of the Fifth Postulate. He, and others, began to express doubts that it could be proved. Each of the known proofs assumed another statement that was not justified from the other postulates. In essence, these proofs just replaced one postulate with another.

D'Alembert (1767) called the situation "the scandal of elementary geometry."

See the textbook for Lambert and Taurinus (pages 159-161). (Especially interesting is the idea that Hyperbolic Geometry is like Spherical Geometry with an imaginary radius).

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