LEGENDRE'S DEFECT ZERO THEOREM

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The goal of this handout is to prove Legendre's famous Defect Zero Theorem of Neutral Geometry: if there is one defect zero triangle, then all triangles have defect zero. In this handout, assume all the axioms, definitions, and previously proved results in Neutral Geometry.

1. Useful Lemmas

We begin with some lemmas that will help in the proof.

Recall that given a line l and a point P not on l, there is a unique line m containing P and perpendicular to l. We sometimes say that we "drop" the perpendicular m from P to l. (This is a metaphor from daily life. Think of l as the ground, and P as a point above l). The intersection of l and m is called the "foot".

Lemma 1. *Let $\triangle ABC$ be a triangle with acute angles $\angle A$ and $\angle C$. Drop a perpendicular from B to the base line $\overleftrightarrow{AC}$, and let F be the foot. Then $A * F * C$.*

Proof. Observe that $F \neq A$ since $\angle BAC$ is not a right angle. Likewise $F \neq C$. So by Axiom B-3, one of A, F, C is between the other two.

If $F * A * C$ then $\angle BAC$ is external to $\angle BFA$, so $\angle BAC > \angle BFA$ which contradicts the assumption that $\angle A$ is acute.

Likewise, $A * C * F$ gives a contradiction. Thus $A * F * C$. □

Lemma 2. *Every triangle has at least two acute angles.*

Proof. Let $\triangle ABC$ have a non-acute angle $\angle C$. We must show that all the other angles are acute. Let D be such that $B * C * D$. So $\angle ACD$ is right or acute since it is supplementary to $\angle C$ of the triangle. By the Exterior Angle Theorem, $\angle ACD$ is greater than $\angle A$ and $\angle B$. Thus these angles are acute. □

Remark. These lemmas are actually valid in IBC Geometry.

2. A Chain of Implications

In this section, we will give a series of implications. Together these will prove Lagrange's Zero Defect Theorem.

We can always split up a triangle into two right triangles. If the big triangle has defect zero, then the smaller ones must as well. This is the main idea of the following.

Proposition 1. *If there exists a triangle with defect zero, then there exists a right triangle with defect zero.*

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Proof. Let $\triangle ABC$ be a triangle with $\delta ABC = 0$. This triangle has two acute angles (Lemma 2). So, without loss of generality, we can assume that $\angle A$ and $\angle C$ are acute. Drop a perpendicular from B to the base line $\overleftrightarrow{AC}$, and let F be the foot. Then $A * F * C$ by Lemma 1.

By the additivity of defect and the Saccherri-Legendre Theorem, we have that $\delta ABF = 0$ and $\delta FBC = 0$. Observe that $\triangle ABF$ and $\triangle FBC$ are right triangles. $\square$

By "gluing together" two congruent right triangles of defect zero, we can form a rectangle. This is the basic idea of the following.

Proposition 2. *If there exists a right triangle with defect zero, then there exists a rectangle.*

Proof. Let $\triangle ABC$ be a right triangle with right angle $\angle A$ and defect zero. Let $\beta = |\angle B|$ and $\gamma = |\angle C|$. Since $\delta ABC = 0$, it follows that $\beta + \gamma = 90$. Let D be a point such that $\angle ACD$ is right and such that B and D are on the same side of $\overleftrightarrow{AC}$. By Axiom C-2 we can choose D so that $\overline{CD} \cong \overline{AB}$. Observe that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel since they are perpendicular to the same line $\overleftrightarrow{AC}$.

By an earlier result (in the Quadrilateral Handout) $\square ABDC$ is a regular quadrilateral. In particular, B is interior to $\angle ACD$. Thus $|\angle ACB| + |\angle BCD| = |\angle ACD| = 90$. Since $|\angle ACB| = \gamma = 90 - \beta$, we see that $|\angle BCD| = \beta$. By SAS, we have $\triangle ABC \cong \triangle DCB$. So $\angle D$ is right and $|\angle CBD| = \gamma$.

Finally, C is interior to $\angle ABD$ (since $\square ABDC$ regular), so

$$|\angle ABD| = |\angle ABC| + |\angle CBD| = \beta + \gamma = 90.$$

So all four angles of $\square ABDC$ are right angles. $\square$

If we have rectangles, we can "stack them" to form a large rectangle. This is the idea of the following:

Proposition 3. *Suppose there exists a rectangle. Then there exist arbitrarily large rectangles in the following sense. If M is any real number, regardless of how large, then there is a rectangle whose sides all have length bigger than M.*

Proof. See the Quadrilateral Handout. $\square$

Any right triangle that fits inside a rectangle must have defect zero. This is the idea of the following.

Proposition 4. *Let $\triangle XYZ$ be a right triangle with right angle $\angle X$. Let M be the maximum of $|\overline{XY}|$ and $|\overline{XZ}|$. Suppose there is a rectangle $\square ABCD$ whose sides all have length greater than M. Then $\delta XYZ = 0$.*

Proof. By assumption (and the Segment Measure Theorem), $\overline{XY} < \overline{AB}$ and $\overline{XZ} < \overline{AD}$. Thus there is a point Y' with $A * Y' * B$ and a point Z' with $A * Z' * D$ such that $\overline{AY'} \cong \overline{XY}$ and $\overline{AZ'} \cong \overline{XZ}$.

Recall from the Quadrilateral Handout that, since $\square ABCD$ is a rectangle, $\delta ABCD = 0$ and $\delta ABCD = \delta ABD + \delta BDC$. In particular, $\delta ABD = 0$. Since $A * Y' * B$, we have $\delta ABD = \delta AY'D + \delta Y'DB$. So $\delta AY'D = 0$. Since $A * Z' * D$, we have $\delta AY'D = \delta AY'Z' + \delta Z'Y'D$. So $\delta AY'Z' = 0$.

By SAS, $\triangle AY'Z' \cong \triangle XYZ$. Since $\delta AY'Z' = 0$, it follows that $\delta XYZ = 0$. $\square$

Since every triangle can be divided into right triangles, we get the following.

Proposition 5. *Suppose that every right triangle has defect zero. Then every triangle has defect zero and angle sum 360.*

Proof. Let $\triangle ABC$ be any triangle. This triangle has two acute angles (Lemma 2). So, without loss of generality, we can assume that $\angle A$ and $\angle C$ are acute. Drop a perpendicular from B to the base line $\overleftrightarrow{AC}$, and let F be the foot. Then $A * F * C$ by Lemma 1. So $\delta ABC = \delta ABF + \delta FBC$.

But $\triangle ABF$ and $\triangle FBC$ are right triangles and so, by assumption, $\delta ABF = \delta FBC = 0$. Thus $\delta ABC = \delta ABF + \delta FBC = 0 + 0 = 0$. ☐

3. Results

We put the above together, and we get the main theorem.

Theorem (Legendre's Defect Zero Theorem)**.** *If there exist a defect zero triangle, then all triangles have defect zero.*

Proof. By Proposition 1 there exists a defect zero right triangle. Thus, by Proposition 2, there exists a rectangle. So, by Proposition 3, there exists rectangles of arbitrary size. Hence, by Proposition 4, every right triangle must have defect zero. So, by Proposition 5, every triangle must have defect zero. ☐

We can also use the above propositions to prove a couple other interesting results:

Proposition 6. *If there exist a defect zero triangle, then there exists a rectangle.*

Proof. By Proposition 1 there exists a defect zero right triangle. So, by Proposition 2, there exists a rectangle. ☐

Proposition 7. *If there exists a rectangles, then all triangles have defect zero.*

Proof. By Proposition 3, there exists rectangles of arbitrary size. Hence, by Proposition 4, every right triangle must have defect zero. So, by Proposition 5, every triangle must have defect zero. ☐

Proposition 8. *If there is a zero defect triangle, then all regular quadrilaterals have defect zero.*

Proof. By Legendre's Zero-Defect Theorem, all triangles have zero defect. Let $\square ABCD$ be regular. Then

$$\delta ABCD = \delta ABC + \delta ADC = 0 + 0 = 0$$

(See the Quadrilateral Handout for the equation $\delta ABCD = \delta ABC + \delta ADC$). ☐

4. Positive Defect Situation

Above we discussed the consequence of having at least one defect zero triangle. Now we discuss the consequences of having a triangle with positive defect (as we will see, this turns out to be true in Hyperbolic Geometry).

Proposition 9. *If there exists a triangle $\triangle ABC$ with $\delta ABC > 0$ then all triangles have positive defect.*

Proof. Suppose otherwise, that there is a triangle of defect zero. Then by Legendre's Theorem (from the previous section), all triangles have defect zero. But this contradicts the hypothesis that $\delta ABC > 0$. □

Proposition 10. *If there exists a triangle* $\triangle ABC$ *with* $\delta ABC > 0$ *then there are no rectangles.*

Proof. Suppose that there are rectangles. Then, by Proposition 7, all triangles have defect zero. But this contradicts the hypothesis that $\delta ABC > 0$. □

Proposition 11. *If there are no rectangles, then all triangles have positive defect.*

Proof. Suppose otherwise, that there is a triangle of defect zero. Then by Proposition 6 there is a rectangle, contradicting our hypothesis. □

Remark. Later we will discuss the connection between zero or positive defect with UPP. We will see that zero defect corresponds to Euclidean geometry and positive defect corresponds to Hyperbolic geometry.