EUCLIDEAN AND HYPERBOLIC CONDITIONS

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The first goal of this handout is to show that, in Neutral Geometry, Euclid's Fifth Postulate is equivalent to the following statement: *given a line l and a point P not on l, there is a unique line containing P that is parallel to l.*

It turns out that there are many statements equivalent to Euclid's Fifth Postulate (in Neutral Geometry), and that any one of these can be used to give an axiom defining Euclidean Geometry. The second goal of this handout is to describe several of these statements (called *Euclidean conditions*).

The third goal of this handout is to describe several conditions that define Hyperbolic Geometry. These are called *Hyperbolic conditions*.

1. Euclid's Fifth Postulate

Recall Euclid's Fifth Postulate (E5P): *For any pair of distinct lines l and m, and any transversal t to these lines, if the sum of the measures of the two interior angles on a given side of t is less than 180, then l and m must intersect on that side of t.*

In other words, if (i) $l = \overleftrightarrow{PA}$ and $m = \overleftrightarrow{QB}$ are lines such that $P \neq Q$, (ii) A and B are on the same side of $t = \overleftrightarrow{PQ}$, and (iii) $|\angle APQ| + |\angle BQP| < 180$, then l and m intersect in a point C, and A and C are on the same side of t.

However, in modern texts Euclid's Fifth Postulate is not commonly used as an axiom: the following seemingly simpler postulate is often used instead.

Definition 1. The *Parallel Postulate* (PP) is the following statement: *Given any line l and any point P not on l, there is a unique line containing P that is parallel to l.*

Of course, neither Euclid's Fifth Postulate (E5P) nor the Parallel Postulate (PP) is a postulate of Neutral Geometry, and neither is a theorem of Neutral Geometry. These two statements, however, can be proved equivalent in Neutral Geometry.

Proposition 1. *In Neutral Geometry the following statements are equivalent: (i) Euclid's Fifth Postulate (E5P), (ii) The Parallel Postulate (PP).*

Proof. First assume E5P. We must prove PP. So let l be a line and P a point not on l. Drop a perpendicular from P to l, and let Q be its foot. Thus $l \perp \overleftrightarrow{PQ}$. Now let m_1 be a line perpendicular to $\overleftrightarrow{PQ}$ that contains P. By a result of Neutral Geometry, such a line exists, and by another result of Neutral Geometry $l \parallel m_1$ since l and m_1 are both perpendicular to the same line $\overleftrightarrow{PQ}$.

We must show that there are no other lines parallel to l containing P. To do so, suppose that m_2 is another such line. Then choose points X and Y on m_2 such that $X * P * Y$ (Axioms I-2 and B-2). One of the angles $\angle XPQ$ or $\angle YPQ$ must be acute since they are not

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right, and they are supplementary. Thus the sum of interior angles is less than 180 on one side of $\overleftrightarrow{PQ}$. So m_2 intersects l by E5P. This contradicts the assumption that m_2 is parallel to l. Thus m_1 is the unique parallel. We have established PP.

Now assume PP. We must prove E5P. Suppose (i) $l = \overleftrightarrow{PA}$ and $m = \overleftrightarrow{QB}$ are lines such that $P \neq Q$, (ii) A and B are on the same side of $t = \overleftrightarrow{PQ}$, and (iii) $|\angle APQ| + |\angle BQP| < 180$. We must show that l and m intersect in a point C, and A and C are on the same side of t.

Let D be a point such that $B * Q * D$ (Axiom B-2). Then $|\angle PQD| = 180 - |\angle PQB|$. This implies that $|\angle PQD| \neq |\angle APQ|$ (otherwise $|\angle APQ| + |\angle BQP| = 180$). By Axiom C-4, there is a ray $\overrightarrow{PX}$ such that $\angle XPQ \cong \angle PQD$ and such that X and A are on the same side of t (and so X and D are on opposite sides of t). Let $l' = \overleftrightarrow{PX}$. Then $l' \parallel m$ by the Alternating Interior Theorem of Neutral Geometry. Now $l \neq l'$ since $|\angle XPQ| \neq |\angle APQ|$. By PP this means that l is not parallel to m. Thus l must intersect m at a point, call it C.

We still need to show that A and C are on the same side of t. Suppose A and C are on opposite sides of t. Then we have a triangle $\triangle CPQ$ such that $\angle CPQ$ is supplementary to $\angle APQ$, and angle $\angle CQP$ is supplementary to $\angle BQP$. Since $|\angle APQ| + |\angle BQP| < 180$, we get $|\angle CPQ| + |\angle CQP| > 180$ which contradicts the Saccheri-Legendre Theorem (or the earlier result that any two angle measures of a triangle add up to something less than 180). Thus C and A are on the same side of t. We have established E5P. □

2. Euclidean Geometry

Euclidean Geometry consists of 5 undefined terms, 16 axioms, and anything that can be defined or proved from these.

Primitive Terms. The five primitive terms are *point*, *line*, *betweenness*, *segment congruence*, and *angle congruence*. We will adopt all the notation and definitions from Neutral geometry, so terms such as *line segment* or *triangle congruence* are ultimately defined in terms of the primitive terms.

The axioms of Euclidean Geometry include the Primitive Term Axiom of Neutral Geometry together with I-1, I-2, I-3, B-1, B-2, B-3, B-4, C-1, C-2, C-3, C-4, C-5, C-6, and the Dedekind Cut Axiom, and the Parallel Postulate below.

Axiom (Parallel Postulate (PP))**.** *Given any line l and any point P not on l, there is a unique line containing P that is parallel to l.*

Since the axioms of Neutral Geometry are a subset of the axioms of Euclidean Geometry, all the propositions of Neutral Geometry automatically hold in Euclidean Geometry.

Now by Proposition 1 we can show that E5P is a proposition of Euclidean Geometry. In this document we will investigate other important propositions of Euclidean Geometry. We will be most interested in propositions called *Euclidean Conditions* (EC). These are propositions that could replace PP as the final axioms of Euclidean Geometry. (Most of the interesting theorems that are not ECs have been proved already in earlier handouts using just Neutral Geometry).

3. Euclidean Conditions

Definition 2 (EC)**.** A *Euclidean condition*, EC for short, is a statement that is equivalent, from the point of view of Neutral Geometry, to the Parallel Postulate (PP).

PP is equivalent to itself. So trivially PP is an EC:

Condition 1 (Parallel Postulate (PP))**.** Given any line l and any point P not on l, there is a unique line containing P that is parallel to l.

Proposition 2. *Condition 1 is an EC.*

Proof. Every statement is equivalent to itself, so $PP \Leftrightarrow PP$. □

By Proposition 1, E5P is an EC.

Condition 2. [E5P] For any pair of distinct lines l and m, and any transversal t to these lines, if the sum of the measures of the two interior angles on a given side of t is less than 180, then l and m must intersect on that side of t.

Proposition 3. *Condition 2 is an EC.*

Proof. See Proposition 1. □

Since ECs are equivalent to PP, they are obviously consequences of PP. Thus they are theorems of Euclidean Geometry.

Proposition 4. *Every EC is a proposition of Euclidean Geometry.*

Furthermore, if X is an EC, and we replace PP as an axiom by X, the resulting geometry will be equivalent to Euclidean Geometry. In other words, the two geometries will have the same propositions.

Proof. Suppose X is an EC. Then $PP \Leftrightarrow X$ can be proved in Neutral Geometry. In particular $PP \Rightarrow X$ is provable in Neutral Geometry. Since PP is an axiom of Euclidean Geometry, and since every theorem of Neutral Geometry is also one of Euclidean Geometry, we have that PP and $PP \Rightarrow X$ are provable in Euclidean Geometry. By Modus Ponens (a rule of logic) we have that X is provable in Euclidean Geometry.

Now consider the geometry defined by the axioms of Neutral Geometry plus X where X is an EC. Since X is an EC we have that $PP \Leftrightarrow X$ can be proved in Neutral Geometry. So any consequence of the axioms of Neutral Geometry plus X will be a consequence of the axioms of Neutral Geometry plus PP. Thus this new geometry and Euclidean Geometry will have the same consequences. □

One of the most interesting ECs is the zero defect condition.

Condition 3. Every triangle has defect zero.

Proposition 5. *Condition 3 is an EC.*

Proof. Al-Tusi's theorem states that Condition 3 implies PP in Neutral Geometry (see the *Connections between Defect and Parallelism* handout).

Conversely, suppose that PP holds. We must show Condition 3 holds as well. Suppose it does not. Then there is a positive defect triangle. In the *Legendre's Defect Zero Theorem* handout we showed that all triangles must then have positive defect. In the *Connections between Defect and Parallelism* handout we showed that this implies that given a line l and a point P not on l, there are at least two lines containing P parallel to l. This gives a contradiction to the PP. Thus all triangles must have zero defect. □

Here are a few observations about ECs (they are all consequences of basic logic):

- Any statement that is equivalent to an EC (from the point of view of Neutral Geometry) is also an EC.
- In a given model of Neutral Geometry, either all ECs are true or all are false. A model of neutral geometry where the ECs are true is called a *Euclidean plane*. As we will see later, a model of Neutral Geometry where the ECs are false are models of Hyperbolic Geometry, and are called a *hyperbolic planes*.
- Any statement that is true in Euclidean Geometry and that, in Neutral Geometry, implies the Euclidean Parallel Postulate (or any other EC) is itself an EC.

Now we describe some more ECs.

Condition 4. There exists a triangle with defect zero.

Condition 5. There exists a rectangle.

Proposition 6. *Conditions 4 and 5 are ECs.*

Proof. It was shown in the *Legendre's Defect Zero Theorem* handout that Condition 4 implies Condition 3. The converse is trivially true. Thus Condition 4 and Condition 3 are equivalent (in Neutral Geometry). Since Condition 3 is an EC, Condition 4 must also be an EC.

It was shown in the *Legendre's Defect Zero Theorem* handout that Condition 5 is is implied by Condition 4, and that Condition 5 implies Condition 3. So Condition 5 must be an EC. □

4. Alternating Interior Angles

We now consider the following conditions:

Condition 6. For any pair of distinct parallel lines l and m, and any transversal t to these lines, the alternating interior angles are equal.

In other words, if $l = \overleftrightarrow{AP}$ and $m = \overleftrightarrow{QB}$ are parallel, and if A and B are on opposite sides of $t = \overleftrightarrow{PQ}$, then $\angle APQ \cong \angle BQP$.

Proposition 7. *Condition 6 is an EC. Hence it is a proposition of Euclidean Geometry.*

Proof. We will show that Condition 6 is equivalent to E5P (Condition 2). Since E5P is an EC, it follows that Condition 6 must also be an EC. □

Proof. We must show Condition 6 and PP are equivalent in Neutral Geometry.

First assume Condition 6. We must prove PP. So let l be a line and P a point not on l. Drop a perpendicular from P to l, and let Q be its foot. Thus $l \perp \overleftrightarrow{PQ}$. Now let m_1 be a line perpendicular to $\overleftrightarrow{PQ}$ that contains P. By a result of Neutral Geometry, such a line exists, and by another result of Neutral Geometry $l \parallel m_1$ since l and m_1 are both perpendicular to the same line $\overleftrightarrow{PQ}$. We must show that there are no other lines parallel to l containing P. To do so, suppose that m_2 is any such such line. Condition 6 implies that $m_2 \perp \overleftrightarrow{PQ}$. By a result of Neutral geometry there is a unique perpendicular to $\overleftrightarrow{PQ}$ containing P. Thus $m_1 = m_2$. We have established PP.

Now assume that PP holds. We must prove Condition 6. Since E5P is a consequence of PP, we can use this to establish Condition 6. Let A, B, P, Q, l, m, t be as above. Let D be such that $B * Q * D$. By E5P, $|\angle APQ| + |\angle DQP| < 180$ implies that l and m intersects. Likewise,

$|\angle APQ| + |\angle DQP| > 180$ implies that l and m intersect (look at the interior angles on the other side of t). Since l and m are parallel, we conclude that $|\angle APQ| + |\angle DQP| = 180$, in other words, $|\angle APQ| = 180 - |\angle DQP|$. But $|\angle BQP| = 180 - |\angle DQP|$ since $\angle BQP$ and $\angle DQP$ are supplementary. Thus $|\angle BQP| = |\angle APQ|$. This establishes Condition 6. $\square$

Proposition 8. *Assume Condition 6. Then for any pair of distinct parallel lines l and m, and any transversal t to these lines, corresponding angles are equal.*

In other words, suppose (i) $l = \overleftrightarrow{AP}$ and $m = \overleftrightarrow{QB}$ are parallel, (ii) A and B are on the same side of $t = \overleftrightarrow{PQ}$, and (iii) $R * P * Q$. Then $\angle APR \cong \angle BQP$.

Proof. The follows from Condition 6 together with the Vertical Angle Theorem. $\square$

We already know that in Euclidean Geometry the angle sum of any triangle is 180. This is due to the fact that Condition 3 is an EC. However, the proof that Condition 3 is an EC was a bit convoluted (since it required going through Al-Tusi's Theorem). The following gives a direct proof that is more common in books on Euclidean Geometry:

Proposition 9. *Assume Condition 6 holds. Then the sum of the angle measures of every triangle is 180.*

Proof. As mentioned above, this follows from the fact that Condition 3 and Condition 6 are equivalent (since they are both EC), but this proof is not too illuminating so we give a sketch of another classic proof.

Let $\triangle ABC$ be a triangle. Let m be a line containing A that is parallel to $\overleftrightarrow{BC}$. Let D and E be points on m such that $\angle DAB$ and $\angle B$ are alternating interior angles, and such that $\angle EAC$ and $\angle C$ are alternating interior angles. It follows that $D*A*E$ (but I skip the details that involve the Crossbar Theorem). By Condition 6, $\angle DAB \cong \angle B$ and $\angle EAC \cong \angle C$. Now $|\angle DAB| + |\angle A| + |\angle EAC| = 180$. So $|\angle B| + |\angle A| + |\angle C| = 180$.

There is yet another classic proof. Here is a sketch. Let $\triangle ABC$ be a triangle. (Think of $\angle B$ and $\angle C$ as the base angles). Let D be a point such that $\angle ACD \cong \angle A$ and such that D and B are on opposite sides of $\overleftrightarrow{AC}$. Then, by the Interior Alternating Angle Theorem of Neutral Geometry, $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$. By Proposition 8, $\angle DCE \cong \angle B$ where E is a point such that $B * C * E$. Now $|\angle BCA| + |\angle ACD| + |\angle DCE| = 180$ and $\angle BCA = \angle C$, $\angle ACD \cong \angle A$, and $\angle DCE \cong \angle B$. The result follows. $\square$

5. Similar Triangles

Definition 3 (Similar Triangles)**.** If $\triangle ABC$ and $\triangle DEF$ are triangles such that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$, then we write $\triangle ABC \sim \triangle DEF$, and say that the triangles are *similar.*

Wallis proposed the following as a postulate for geometry. We will show that it is an EC, so is suitable as a replacement for E5P or PP.

Condition 7 (Wallis's Hypothesis)**.** If $\triangle ABC$ is a triangle, and x is a positive real number, then there is a triangle $\triangle DEF$ such that $\left|\overline{DE}\right| = x$ and $\triangle ABC \sim \triangle DEF$.

Before we prove that this is an EC, we will prove the following lemma.

Lemma 10. *Suppose $\triangle ABC \sim \triangle DEF$ but $\triangle ABC \ncong \triangle DEF$. Then there exists triangle of defect zero.*

Proof. If $\overline{AB} \cong \overline{DE}$ then by SAS, $\triangle ABC \cong \triangle DEF$, contradicting our supposition. So we an assume, without loss of generality, that $\overline{AB} < \overline{DE}$.

Let X be a point on $\overrightarrow{AB}$ such that $\overline{AX} \cong \overline{DE}$, and let Y be a point on $\overrightarrow{AC}$ such that $\overline{AY} \cong \overline{DF}$ (Axiom C-1). By assumption $\angle A \cong \angle D$ so, by SAS, $\triangle AXY \cong \triangle DEF$. Observe that $\angle ABC \cong \angle DEF \cong \angle AXY$.

Since $\overline{AB} < \overline{DE}$ we have $A * B * X$. Note that $\overleftrightarrow{BX}$ is a transversal for $\overleftrightarrow{BC}$ and $\overleftrightarrow{XY}$. Since corresponding angles $\angle ABC$ and $\angle AXY$ are congruent, the alternating interior angles must also be congruent (by the Vertical Angle Theorem). By the Alternating Interior Angle Theorem of Neutral Geometry, $\overleftrightarrow{XY}$ is parallel to $\overleftrightarrow{BC}$. Thus X and Y are on the same side of $\overleftrightarrow{BC}$. This implies that $A * C * Y$.[1]

By the additivity of defects

$$\delta AXY = \delta AXC + \delta XCY = \delta ABC + \delta BCX + \delta XCY.$$

But $\delta AXY = \delta ABC$ since $\delta AXY = \delta DEF$ (congruent) and $\delta DEF = \delta ABC$ (similar). So $\delta BCX = \delta XCY = 0$. □

Proposition 11. *Condition 7 is an EC.*

Proof. We will show that Condition 7 is equivalent to Condition 4. Since Condition 4 is an EC, the result follows.

First suppose that Condition 7 holds. Then Condition 4 holds by the above lemma (using $x = 2$ say).

Conversely, suppose Condition 4 holds. Since we know already that Condition 4 is an EC, if it holds then all known ECs must also hold. Let $\triangle ABC$ be given, and let x be a positive real number. Our goal is to construct a similar triangle $\triangle DEF$ such that $|\overline{DE}| = x$. To do so, let $\overline{DE}$ be a segment such that $|\overline{DE}| = x$. Such a $\overline{DE}$ exists by the Segment Measure Theorem (*Neutral Geometry* handout). Let $\overrightarrow{DX}$ be a ray such that $\angle EDX \cong \angle A$. Let $\overrightarrow{EY}$ be a ray such that $\angle DEX \cong \angle B$. Also, choose X and Y on the same side of $\overleftrightarrow{DE}$. Such a X and Y exist by Axiom C-4. Since $|\angle A| + |\angle B| < 180$, the rays $\overrightarrow{DX}$ and $\overrightarrow{EY}$ intersect in a point F by E5F (Condition 2). Note that $\delta ABC = \delta DEF = 0$ by Condition 3. Since $\angle A \cong \angle D$, and $\angle B \cong \angle E$, it follows that $\angle C \cong \angle F$. Thus $\triangle ABC \sim \triangle DEF$. □

6. Equidistant Lines

Here is another famous EC.

Condition 8. If l and m are parallel, then every point of m is equidistant from l.

Actually, you only need to show three points are equidistant. Recall that the distance from a point P to a line l is defined as follows: drop a perpendicular from P to l and let A be the foot. Then the distance from P to l is defined to be $|\overline{PA}|$.[2]

Proposition 12. *if l and m are parallel, and three points P, Q, R on m are equidistant from l, then rectangles exist.*

[1] Observe that $A * Y * C$ leads to a contradiction: it makes A and Y on the same side $\overleftrightarrow{BC}$, but we know that A and X are on opposite sides.

[2] It is not too hard to show that if X is another other point on l then $\overline{PA} < \overline{PX}$, so A really is the closest point on l to P. To see this, consider the right triangle $\triangle PAX$ and observe that $\angle X < \angle A$.

Proof. Without loss of generality, we can suppose that $P * Q * R$. Drop a perpendicular from P to l, and let A be the foot. Drop a perpendicular from Q to l, and let B be the foot. Drop a perpendicular from R to l, and let C be the foot. Now A, B, C must be distinct (since perpendiculars through a given point are unique, and two distinct lines intersect m in at most one point).

By hypothesis $\overline{PA} \cong \overline{QB} \cong \overline{RC}$. So $\square PQBA$ is a Saccheri quadrilateral. By an earlier result (from the Quadrilateral Handout), $|\angle APQ| = |\angle BQP| \leq 90$. Also, $\square QRCB$ is a Saccheri quadrilateral. So $|\angle BQR| = |\angle CRQ| \leq 90$. But $\angle BQP$ and $\angle BQR$ are supplementary. Since neither is obtuse, they must both be right. So $|\angle APQ| = |\angle BQP| = 90$. Thus $\square PQBA$ is a Rectangle. $\square$

Proposition 13. *Condition 8 is an EC.*

Proof. If Condition 8 holds then rectangles exist by the above proposition. So the PP holds since the existence of rectangles is an EC (Condition 5).

Conversely, if PP holds then let A and B be two points on m where $l \parallel m$. Drop perpendiculars from A and B to l, and let F and G be the respective feet. Thus $\angle F$ and $\angle G$ are right. Now if we draw a perpendicular to $\overleftrightarrow{AF}$ containing A, we get a parallel to l by a theorem of Neutral Geometry. By our uniqueness assumption, that line must be m. Thus $\angle A$ is right. A similar argument shows $\angle B$ is right. Thus $\square ABGF$ is a rectangle. So, by a previous result, $\overline{AF} \cong \overline{BG}$. Since A and B were arbitrary points on m, we conclude that all points on m have the same distance to l. $\square$

7. Areas of Triangles

We have not defined area in this course, so we cannot prove theorems concerning area. But it turns out that in Hyperbolic Geometry area is proportional to defect. Since defect is bounded by 180, we conclude that areas of triangles too must be bounded. Because of this, the following turns out to be an EC.

Condition 9. There are triangles of arbitrarily large area. In other words, given a positive real number x, one can find a triangle of area at least x.

Proposition 14. *The above is an EC.*

8. Legendre's Crossbar Hypothesis

One of Legendre's proofs of the Euclidean Parallel Postulate uses the following.[3]

Condition 10. If $\angle BAC$ is an acute angle, and if D is an interior point in this angle, then there is a line passing through D that intersects both rays, $\overrightarrow{AB}$ and $\overrightarrow{AC}$, of the angle.

Proposition 15. *The above is an EC.*

Actually, Legendre only needed to assume only for angles of measure less than or equal to 60. See the textbook for the proof that this an EC (pages 157–159). It is closely related to Condition 9 above. The basic ideas is that given $\triangle ABC$, one can form a triangle that contains two congruent copies of $\triangle ABC$. By repeating this process, one gets unbounded area. However, this process requires the use of Condition 10.

[3]Another of his proof uses the assumption that if D is interior to an angle $\angle BAC$, then every line containing D must intersect the angle. This gives yet another EC. See Page 21–22 of the textbook for more.

9. Other ECs

The following turn out to be ECs. The first dates back to Proclus.

Condition 11 (Proclus)**.** Let l, m, t be distinct lines. If $l \parallel m$ and if t intersects m, then t must intersect l.

Condition 12 (Transitivity of Parallels)**.** Let l, m, n be distinct lines. If $l \parallel m$ and $m \parallel n$ then $l \parallel n$.

The following was proposed as an axiom by Farkas Bolyai.

Condition 13. If P, Q, R are three non-collinear points, then there is a circle containing P, Q, and R.

The following assumption was made by al-Tusi (and is discussed in more detail in an earlier Handout).

Condition 14. Suppose that A, B, C are three collinear points all on the same side of a line l such that $A * B * C$. Let D, F, G be points on l such that $\overleftrightarrow{AD}$, $\overleftrightarrow{BF}$, and $\overleftrightarrow{CG}$ are each perpendicular to l. If $\angle ABF$ is acute then $\overline{AD} < \overline{BF} < \overline{CG}$.

We now show Conditions 11 and 12 are ECs (we skip showing the last two are).

Proposition 16. *Condition 11 is an EC.*

Proof. We first show Condition 11 implies PP. So let l be a line and P a point not on l. We know from Neutral Geometry that there is at least one parallel m to l containing P. Suppose m' is another. Observe that m' intersects m at P. So by Condition 11, m' must also intersect l, a contradiction. Therefore, m is the unique parallel to l containing P.

Now we show that PP implies Condition 11. So suppose that $l \parallel m$ and that a third line t intersects m. Let P be the point of intersection. Our goal is to show that t also intersects l. Suppose it didn't, then t and m would both be parallel to l and would both contain P, a contradiction. □

The next lemma is very easy. It implies that Condition 12 is also an EC since Condition 11 is known to be an EC.

Lemma 17. *In Neutral Geometry, Condition 11 is equivalent to Condition 12.*

Proof. First we show Condition 11 implies Condition 12. So let l, m, n be distinct lines with $l \parallel m$ and $m \parallel n$. Our goals is to show $l \parallel n$. Suppose otherwise, that l and n intersect. By Condition 11 applied to $m \parallel n$, the line l must intersect m: a contradiction.

Now we show Condition 12 implies Condition 11. So let l, m, t be distinct lines such that $l \parallel m$ and such that t intersects m. We must show t intersects l. Suppose not, then $t \parallel l$. So by Condition 12, $t \parallel m$: a contradiction. □

Corollary 18. *Condition 12 is also an EC.*

We skip the proofs of the following.

Proposition 19. *Condition 13 and 14 are ECs.*

10. Hyperbolic Geometry

Hyperbolic Geometry consists of 5 undefined terms, 16 axioms, and anything that can be defined or proved from these.

Primitive Terms. The five primitive terms are *point*, *line*, *betweenness*, *segment congruence*, and *angle congruence*. We will adopt all the notation and definitions from Neutral geometry, so terms such as *line segment* or *triangle congruence* are ultimately defined in terms of the primitive terms.

The axioms of Hyperbolic Geometry include the Primitive Term Axiom of Neutral Geometry together with I-1, I-2, I-3, B-1, B-2, B-3, B-4, C-1, C-2, C-3, C-4, C-5, C-6, and the Dedekind Cut Axiom, and the Hyperbolic Parallel Postulate below.

Axiom (Hyperbolic Parallel Postulate (HPP))**.** *Given any line l and any point P not on l, there are at least two lines containing P that are parallel to l.*

Since the axioms of Neutral Geometry are a subset of the axioms of Hyperbolic Geometry, all the propositions of Neutral Geometry automatically hold in Hyperbolic Geometry.

Thus every proposition of Neutral Geometry is a common proposition for Euclidean Geometry and Hyperbolic Geometry. However, not every proposition of Euclidean Geometry is a proposition of Hyperbolic Geometry. In fact, the negation of every EC is a proposition of Hyperbolic Geometry, so no EC can be a proposition of Hyperbolic Geometry (assuming the consistency of Hyperbolic Geometry). For example, there is an EC that asserts that there is a triangle with zero defect (Condition 4). So this statement must be false in Hyperbolic Geometry. This leads to the following:

Proposition 20. *Every triangle has positive defect.*

Proof. Suppose $\triangle ABC$ does not have positive defect. By the Saccheri-Legendre Theorem (from the *Neutral Geometry* handout), we conclude that $\delta ABC = 0$. However, the existence of such a triangle is an EC (Condition 4). By definition of EC, this implies that PP holds. But PP clearly contradictions the HPP axiom of Hyperbolic Geometry. Thus no such $\triangle ABC$ exists. □

The technique employed in the above proof generalizes:

Proposition 21. *The negation of every EC is a proposition of Hyperbolic Geometry.*

Proof. Let X be an EC. We wish to prove $\neg X$. We do so by contradiction. We suppose X and get a contradiction.

So suppose X. By the definition of EC, we know that we can prove $X \Leftrightarrow PP$. In particular, we can prove $X \Rightarrow PP$. Using Modus Ponens, we get PP. However, PP clearly contradictions the HPP axiom of Hyperbolic Geometry. □

To summarize, we have two big sources of proposition for Hyperbolic Geometry: (i) all the theorems of Neutral Geometry, and (ii) all the negations of ECs. However, many interesting theorems of Hyperbolic Geometry are obtained without explicit appeal to these tactics.

11. Hyperbolic Conditions

Definition 4 (HC)**.** A *Hyperbolic condition*, HC for short, is a statement that is equivalent, from the point of view of Neutral Geometry, to the Hyperbolic Parallel Postulate (HPP).

HPP is equivalent to itself. So trivially HPP is an HC:

H-Condition 1 (Hyperbolic Parallel Postulate (HPP))**.** Given any line l and any point P not on l, there are at least two lines containing P that are parallel to l.

Proposition 22. *H-Condition 1 is an HC.*

Proof. Every statement is equivalent to itself, so $HPP \Leftrightarrow HPP$. □

Here is an interesting HC.

H-Condition 2. Every triangle has positive defect.

Proposition 23. *H-Condition 2 is an HC.*

Proof. We must show H-Condition 2 $\iff$ HPP. In the *Connections between Defect and Parallelism* handout it was shown that if Condition 2 then HPP holds.

Conversely, suppose that HPP holds. All the propositions of Hyperbolic Geometry follow from HPP (and the other axioms of Neutral Geometry). Thus Proposition 20 holds. This gives us H-Condition 2 as desired. □

Every HC is in fact a theorem of Hyperbolic Geometry. In fact, each HC can be used to define Hyperbolic Geometry.

Proposition 24. *Every HC is a proposition of Hyperbolic Geometry.*

Furthermore, if X is an HC, and we replace HPP as an axiom by X, the resulting geometry will be equivalent to Hyperbolic Geometry. In other words, the two geometries will have the same propositions.

Proof. Similar to the corresponding proof for ECs. □

Thus, in Hyperbolic Geometry, every triangle has positive defect. We can use this to prove an interesting theorem that obviously fails in Euclidean Geometry.

Proposition 25 (AAA)**.** *If $\triangle ABC \sim \triangle DEF$ then $\triangle ABC \cong \triangle DEF$.*

Proof. Suppose $\triangle ABC \ncong \triangle DEF$. By Lemma 10 there exists triangles of defect zero. But in Hyperbolic Geometry all triangles have positive defect, a contradiction. □

Warning: HPP is not the simple negation of PP. In fact, the simple negation of PP would say that there exists a P and l with a certain property (see H-Condition 4). HPP is much stronger: it makes a claim for *all* suitable P and l. Despite these observations, the following lemma shows that HPP is equivalent, in Neutral Geometry, to PP.

Lemma 26. *In Neutral Geometry we have the following:*

$$HPP \iff \neg PP$$

Proof. Suppose HPP. Then every triangle must have positive defect by Proposition 20. Thus Condition 5 is false: there are no triangles of defect zero. Since Condition 5 is an EC, this implies that PP must be false.

Now suppose $\neg PP$ holds. Then every EC must be false. Thus there are no triangles of defect zero (see Condition 5). By the Saccheri-Legendre Theorem (in the *Neutral Geometry* handout), all triangles must have positive defect. This is H-Condition 2, which we showed was an HC. Thus HPP holds. □

The following proposition gives very a powerful tool for generating HCs.

Proposition 27. *The negation of any EC is an HC. Likewise, the negation of any HC is an EC.*

Proof. PART 1. Let S be an EC. In other words, S is a statement such that $S \iff PP$ can be proved in Neutral Geometry. So $\neg S \iff \neg PP$ can be proved in Neutral Geometry. We can then use Lemma 26 to prove that $\neg S \iff HPP$. So $\neg S$ is an HC.

PART 2. Let S be an HC. In other words, S is a statement such that $S \iff HPP$ can be proved in Neutral Geometry. We can then use Lemma 26 to prove that $S \iff \neg PP$. Thus $\neg S \iff PP$ can be proved in Neutral Geometry. So $\neg S$ is an EC. □

Now we can show that AAA is not just a proposition of Hyperbolic Geometry, but that it is actually a HC.

H-Condition 3. Similar triangles are congruent.

Proposition 28. *H-Condition 3 is an HC.*

Proof. We must show H-Condition 3 $\iff$ HPP.

First suppose H-Condition 3 holds. Then Wallis's Hypothesis (Condition 7) must be false: just take x larger than any side of the given triangle $\triangle ABC$ and derive a contradiction to AAA. Thus the negation of Wallis' Hypothesis holds. By Proposition 27, this negation is an HC. By definition, any HC gives HPP.

Conversely, suppose that HPP holds. All the propositions of Hyperbolic Geometry follow from HPP (and the other axioms of Neutral Geometry). Thus Proposition 25 (AAA) holds. This gives us H-Condition 3 as desired. □

Since PP is trivially an EC, Proposition 27 implies that the negation of PP is an HC. Thus the following is an HC:

H-Condition 4. There is a line l and a point P not on l for which there is not exactly one line containing P that is parallel to l.

The following is the negation of Condition 3. By Proposition 27, it is an HC.

H-Condition 5. There exists a triangle with non-zero defect.

The following is the negation of Condition 8. By Proposition 27, it is an HC.

H-Condition 6. There are parallel lines $l \parallel m$ such that m contains points A and B that are not equidistant from l.

The following is the negation of Condition 10. By Proposition 27, it is an HC.

H-Condition 7. There is an acute angle $\angle BAC$ and a point D interior to $\angle BAC$ such that there are no lines containing D that hit both $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

The following is the negation of Condition 11. By Proposition 27, it is an HC.

H-Condition 8. There are three lines l, m, t such that $l \parallel m$, and such that t that intersects l but fails to intersect m.

The following is the negation of Condition 12. By Proposition 27, it is an HC.

H-Condition 9. There are three lines l, m, n such that $l \parallel m$ and $m \parallel n$ but where l is not parallel to n.

The following is the negation of Condition 13. By Proposition 27, it is an HC.

H-Condition 10. There are three non-collinear points P, Q, R such that there is no circle containing P, Q, and R. In other words, there is a triangle that cannot be inscribed in a circle.

The following is the negation of Condition 5. By Proposition 27, it is an HC.

H-Condition 11. There are no rectangles.

All the above HC-Conditions are true in Hyperbolic Geometry. They truly illustrates that Hyperbolic Geometry is, in the words of János Bolyai, a "strange new universe"!

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