EUCLIDEAN AND HYPERBOLIC CONDITIONS

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1. Introduction

In an earlier handout we considered various *Euclidean Conditions*. These include UPP (Unique Parallel Property), E5P (Euclid's Fifth Postulate) and many others. In this handout we extend the list of Euclidean Conditions with other statements including those involving defects and similar triangles. The discussion concludes with a list of important Euclidean Conditions.

After considering Euclidean Conditions we consider Hyperbolic Geometry and various conditions called *Hyperbolic Conditions*.

Recall that a Euclidean conditions C is any statement such that

$$\text{UPP} \iff C$$

is a theorem of Neutral Geometry. All Euclidean Conditions are theorems of Euclidean Geometry, and each is a legitimate replacement for UPP as the final axiom of Euclidean Geometry.

Once Hyperbolic Geometry is accepted as consistent (through the use of models), no Euclidean Condition can be hoped to be a theorem of Neutral Geometry. They can only be theorems in Euclidean Geometry. In Neutral Geometry they are just statements that happen to be equivalent to each other, each neither provably true nor provably false.

2. Al-Tusi's Theorem

Consider the statement *all triangles have defect zero* (ZD:$\forall\triangle$). We wish to show that this is a Euclidean Condition. An important step is giving a proof (in Neutral Geometry) of

$$\text{ZD:}\forall\triangle \implies \text{UPP}.$$

A proof of this can be constructed by following the ideas of a medieval Persian mathematician, Nasir al-Din al-Tusi, who was himself very interested in questions related to Euclid's Fifth Postulate. In his honor we call this theorem *Al-Tusi's Theorem*.

We begin with a few lemmas. All proofs will in the context of Neutral Geometry.

Lemma 1. *Suppose that all triangles have defect zero. Then all regular quadrilaterals have defect zero as well.*

Proof. Let $\square ABCD$ be a regular quadrilateral. From the Quadrilateral Handout,

$$\delta ABCD = \delta ABC + \delta CDA.$$

The result follows. $\square$

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Lemma 2. *Suppose that every triangle has defect zero. Then when you double the hypothenuse of a right triangle, the legs double as well. More precisely, suppose $\triangle ABC$ is a right triangle with right angle $\angle B$. Suppose $A * B * B'$ and $A * C * C'$ are such that $\triangle AB'C'$ is a right triangle with right angle $\angle B'$. If $|\overline{AC'}| = 2|\overline{AC}|$, then $|\overline{B'C'}| = 2|\overline{BC}|$ and $|\overline{AB'}| = 2|\overline{AB}|$.*

Proof. Drop a perpendicular from C' onto the line $\overleftrightarrow{BC}$, and let D be the foot. Observe that $\square B'BDC'$ is a parallelogram with three right angles.[1] In other words, it is a Lambert quadrilateral. By the above lemma, $\delta B'BDC' = 0$. This implies that all four angles are right, so $\square B'BDC'$ is a rectangle. By a previous result (in the Quadrilateral Handout), this implies that $\overline{BD} \cong \overline{B'C'}$ and $\overline{B'B} \cong \overline{DC'}$.

Now $B * C * D$ (details are skipped[2]). By the Vertical Angle Theorem, $\angle BCA \cong DCC'$. Also, $\angle ABC \cong \angle C'DC$ since they are both right, and $\overline{AC} \cong \overline{C'C}$ by assumption. Thus $\triangle ABC \cong \triangle C'DC$ by AAS.

In particular $\overline{BC} \cong \overline{CD}$ and $\overline{AB} \cong \overline{C'D}$. The first of these tells us that $|\overline{BD}| = 2|\overline{BC}|$. However, $|\overline{BD}| = |\overline{B'C'}|$, so $|\overline{B'C'}| = 2|\overline{BC}|$.

Since $\overline{AB} \cong \overline{C'D}$, and since $\overline{B'B} \cong \overline{C'D}$, we get $\overline{AB} \cong \overline{B'B}$. Thus $|\overline{AB'}| = 2|\overline{AB}|$. □

Remark 1. What happens if you remove the assumption that all triangles have defect zero? Then all you know is that $\square B'BDC'$ is a Lambert quadrilateral with a possibly acute angle at C'. Thus $|\overline{DC'}| \geq |\overline{BB'}|$ and $|\overline{B'C'}| \geq |\overline{BD}|$. So the above proof only gives us that $|\overline{B'C'}| \geq 2|\overline{BC}|$ and $|\overline{AB'}| \leq 2|\overline{AB}|$.

Here is the main theorem:

Theorem 1 (Al-Tusi's Theorem)**.** *Suppose that all triangles have defect zero. Then the following holds: given a line l and a point P not on l, there is exactly one line passing through P parallel to l. In other words,*

$$\textit{ZD:}\forall\triangle \implies \textit{UPP}.$$

Proof. Drop a perpendicular from P to l, and let Q be the foot. Thus $l \perp \overleftrightarrow{PQ}$. Now let m_1 be a line perpendicular to $\overleftrightarrow{PQ}$ that contains P. By an earlier result, such a line exists, and by another result $l \parallel m_1$ since l and m_1 are both perpendicular to the same line $\overleftrightarrow{PQ}$.

We must show that there are no other lines parallel to l containing P. To do so, suppose that m_2 is another such line. Choose a point C on m_2 that is on the same side of m_1 as Q. Drop a perpendicular from C to $\overleftrightarrow{PQ}$, and call the foot B. So $\triangle PBC$ is a right triangle with right angle at B.[3]

Now let C_1 be a point such that $P*C*C_1$ and $\overline{CC_1} \cong \overline{PC}$. In other words, $|\overline{PC_1}| = 2|\overline{PC}|$. Drop a perpendicular from C_1 to $\overleftrightarrow{PQ}$, and call the foot B_1. So $\triangle PB_1C_1$ is a right triangle

[1] This requires the routine verification of several conditions: the four points are distinct, the opposite sides are parallel.

[2] First show the three points are distinct. Next observe that C and C' are on the same side of $\overleftrightarrow{AB}$ and C' and D are on the same side of $\overleftrightarrow{AB}$, so C and D are on the same side of $\overleftrightarrow{AB}$. Similarly, B and C are on the same side of $\overleftrightarrow{DC'}$. The result follows.

[3] Also, $\overleftrightarrow{BC}$ is parallel to m_1, so B and C are on the same side of m_1. Thus B and Q are on the same side of m_1. So B is on the ray $\overrightarrow{PQ}$.

with right angle at B_1. Now P and C_1 are on opposite sides of $\overleftrightarrow{BC}$, and $\overleftrightarrow{BC} \parallel \overleftrightarrow{B_1C_1}$. It follows that P and B_1 are on opposite sides of $\overleftrightarrow{BC}$. Thus $P * B * B_1$. So we can use Lemma 2. We conclude that $\left|\overline{PB_1}\right| = 2\left|\overline{PB}\right|$.

By repeating this processes, we produce sequences $C_1, C_2, \dots$ and $B_1, B_2, \dots$ such that $\left|\overline{PB_k}\right| = 2^k\left|\overline{PB}\right|$. Furthermore, each C_i is on m_2 and each B_k is on the ray $\overrightarrow{PQ}$. For sufficiently large k, we have $\left|\overline{PB_k}\right| > \left|\overline{PQ}\right|$ by the Archimedean Principle. This implies that $P * Q * B_k$, so P and B_k are on opposite sides of l. Since $\overleftrightarrow{B_kC_k} \parallel l$, we have $B_k \sim_l C_k$. So P and C_k are also on opposite sides of l. Thus l intersects $\overline{PC_k}$. This means that $m_2 = \overleftrightarrow{PC_k}$ and l are not parallel: a contradiction. $\square$

3. Euclidean Conditions Related to Defect

From Al-Tusi's Theorem we get the following.

Proposition 3. *The statement* ZD:$\forall\triangle$ *is a Euclidean Condition.*

Proof. We must show that ZD:$\forall\triangle \iff$ UPP. One direction follows from Al-Tusi's Theorem. The other follows from that fact that ZD:$\forall\triangle$ is a theorem in Euclidean Geometry, so can be proved to follow from UPP in Neutral Geometry. $\square$

In the Legendre's Defect Zero Theorem Handout saw that several statements were equivalent to ZD:$\forall\triangle$ in Neutral Geometry. Since ZD:$\forall\triangle$ is a Euclidean Condition, the rest are as well. The following list two such conditions:

Corollary 4. *The statement "a rectangle exist"* (Rect) *is a Euclidean Condition.*

Corollary 5. *The statement "a triangle of defect zero exists"* (ZD:$\exists\triangle$) *is a Euclidean Condition.*

4. Similar Triangles

There is an important Euclidean Conditions related to similar triangles.

Definition 1 (Similar Triangles)**.** If $\triangle ABC$ and $\triangle DEF$ are triangles such that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\angle C \cong \angle F$, then we write $\triangle ABC \sim \triangle DEF$, and say that the triangles are *similar.*

Wallis proposed the following hypothesis (WallisHyp) as a postulate for geometry:

> *If* $\triangle ABC$ *is a triangle, and* x *is a positive real number, then there is a triangle* $\triangle DEF$ *such that* $\left|\overline{DE}\right| = x$ *and* $\triangle ABC \sim \triangle DEF$.

We will establish that WallisHyp is a Euclidean Condition. So we could follow Wallis' advice and use this statment as the final axiom of Euclidean Geometry if we wanted. To do this, we need the following lemma.

Lemma 6. *Suppose* $\triangle ABC \sim \triangle DEF$ *but* $\triangle ABC \ncong \triangle DEF$. *Then there exists triangle of defect zero.*

Proof. If $\overline{AB} \cong \overline{DE}$ then by SAS, $\triangle ABC \cong \triangle DEF$, contradicting our supposition. So we an assume, without loss of generality, that $\overline{AB} < \overline{DE}$.

Let X be a point on $\overrightarrow{AB}$ such that $\overline{AX} \cong \overline{DE}$, and let Y be a point on $\overrightarrow{AC}$ such that $\overline{AY} \cong \overline{DF}$ (Axiom C-2). By assumption $\angle A \cong \angle D$ so, by SAS, $\triangle AXY \cong \triangle DEF$. Observe that $\angle ABC \cong \angle DEF \cong \angle AXY$.

Since $\overline{AB} < \overline{DE}$ we have $A * B * X$. Note that $\overleftrightarrow{BX}$ is a transversal for $\overleftrightarrow{BC}$ and $\overleftrightarrow{XY}$. Since corresponding angles $\angle ABC$ and $\angle AXY$ are congruent, the alternating interior angles must also be congruent (by the Vertical Angle Theorem). By the Alternating Interior Angle Theorem of Neutral Geometry, $\overleftrightarrow{XY}$ is parallel to $\overleftrightarrow{BC}$. Thus X and Y are on the same side of $\overleftrightarrow{BC}$. This implies that A and Y are on opposite sides of $\overleftrightarrow{BC}$. Thus $A * C * Y$.

By the additivity of defects

$$\delta AXY = \delta AXC + \delta XCY = \delta ABC + \delta BCX + \delta XCY.$$

But $\delta AXY = \delta ABC$ since $\delta AXY = \delta DEF$ (congruent) and $\delta DEF = \delta ABC$ (similar). Subtracting gives $0 = \delta BCX + \delta XCY$. So $\delta BCX = \delta XCY = 0$. □

Proposition 7. *WallisHyp is a Euclidean Condition.*

Proof. First suppose that WallisHyp holds. Let $\triangle ABC$ be any fixed triangle, and let x be a real number larger than the lengths of the sides of $\triangle ABC$. Applying WallisHyp with this triangle and x gives a triangle similar to, but not congruent to, $\triangle ABC$. By the above lemma, there are triangles of defect zero. So ZD:$\exists\triangle$ holds. Since ZD:$\exists\triangle$ is equivalent to UPP (Corollary 5), we have UPP as well.

Conversely, suppose UPP holds. Let $\triangle ABC$ be given, and let x be a positive real number. Our goal is to construct a similar triangle $\triangle DEF$ such that $\left|\overline{DE}\right| = x$. To do so, let $\overline{DE}$ be a segment such that $\left|\overline{DE}\right| = x$. Such a $\overline{DE}$ exists by the Segment Measure Theorem (Neutral Geometry Handout). Let $\overrightarrow{DX}$ be a ray such that $\angle XDE \cong \angle A$. Let $\overrightarrow{EY}$ be a ray such that $\angle DEY \cong \angle B$. Also, choose X and Y on the same side of $\overleftrightarrow{DE}$. Such a X and Y exist by Axiom C-5.

To complete the proof, we can use any established Euclidean Condition (since they are equivalent to UPP). We know $|\angle A| + |\angle B| < 180$ by a result of Neutral Geometry. Thus $\overrightarrow{DX}$ and $\overrightarrow{EY}$ must intersect at a point F by E5P (we can use E5P since it is a Euclidean Condition). Note that $\delta ABC = \delta DEF = 0$ by Proposition 3. Since $\angle A \cong \angle D$, and $\angle B \cong \angle E$, it follows that $\angle C \cong \angle F$. Thus $\triangle ABC \sim \triangle DEF$. □

5. Equidistant Lines

Here is another important Euclidean Condition:

If l and m are parallel, then every point of m is equidistant from l. (EqD)

Actually, you only need to show three points are equidistant to prove UPP. Here is the definition of distance from a point to a line:

Definition 2. Let P be a point not on the line l. Let m be the unique line perpendicular to l containing P. Let Q be the foot (the intersection of l and m). Then the *distance from P to l* is defined to be $\left|\overline{PQ}\right|$.

Remark 2. Let P, Q, l be as above. If X is another other point on l then $\overline{PA} < \overline{PX}$, so Q is actually is the closest point on l to P. To see this, consider the right triangle $\triangle PQX$ and observe that $\angle Q < \angle A$.

Proposition 8. *if l and m are parallel, and three distinct points P, Q, R on m have the same distance from l, then rectangles exist.*

Proof. Without loss of generality, we can suppose that $P * Q * R$. Drop a perpendicular from P to l, and let A be the foot. Drop a perpendicular from Q to l, and let B be the foot. Drop a perpendicular from R to l, and let C be the foot. Now A, B, C must be distinct (since perpendiculars through a given point are unique, and two distinct lines intersect m in at most one point).

By hypothesis $\overline{PA} \cong \overline{QB} \cong \overline{RC}$. So $\square PABQ$ is a Saccheri quadrilateral. By an earlier result (from the Quadrilateral Handout), $|\angle APQ| = |\angle BQP| \leq 90$. Also, $\square QBCR$ is a Saccheri quadrilateral. So $|\angle BQR| = |\angle CRQ| \leq 90$. But $\angle BQP$ and $\angle BQR$ are supplementary. Since neither is obtuse, they must both be right. So $|\angle APQ| = |\angle BQP| = 90$. Thus $\square PABQ$ is a Rectangle. $\square$

Proposition 9. *EqD is a Euclidean Condition.*

Proof. If EqD holds, then Rect holds by the above proposition. But Rect is equivalent to UPP by Corollary 4. So UPP holds.

Conversely, if UPP holds then let A and B be two points on m where $l \parallel m$. Drop perpendiculars from A and B to l, and let F and G be the respective feet. Thus $\angle F$ and $\angle G$ are right. Now if we draw a perpendicular to $\overleftrightarrow{AF}$ containing A, we get a parallel to l by a theorem of Neutral Geometry. By our uniqueness assumption, that line must be m. Thus $\angle A$ is right. A similar argument shows $\angle B$ is right. Thus $\square ABGF$ is a rectangle. So, by a previous result, $\overline{AF} \cong \overline{BG}$. Since A and B were arbitrary points on m, we conclude that all points on m have the same distance to l. $\square$

6. Areas of Triangles

We have not defined area in this course, so we cannot prove theorems concerning area. It turns out, however, that in Hyperbolic Geometry area is proportional to defect. Since defect is bounded by 180, we conclude that areas of triangles too must be bounded in Hyperbolic Geometry. Because of this, the following turns can be shown to be a Euclidean Condition.

> *There are triangles of arbitrarily large area. In other words, given a positive real number x, one can find a triangle of area at least x.* (Big△)

Remark 3. The modern definition of area and volume are part of a branch of mathematics called *measure theory.*

7. Legendre's Crossbar Hypothesis

One of Legendre's proofs of the Euclidean Parallel Postulate uses the following.[4]

> *If D is an interior point in the angle $\angle BAC$, then there is a line passing through D that intersects both rays, $\overrightarrow{AB}$ and $\overrightarrow{AC}$, of the angle.* (CrossbarHyp)

Proposition 10. *CrossbarHyp is a Euclidean Condition.*

[4]Another of his proof uses the assumption that if D is interior to an angle $\angle BAC$, then every line containing D must intersect the angle. This gives yet another Euclidean Condition. See *Greenberg* for more details.

Roughly speaking Legendre proved UPP as follows. Given $\triangle ABC$, one forms a triangle that contains two congruent copies of $\triangle ABC$ (using CrossbarHyp). By repeating this process, one gets unbounded area. Since Big$\triangle$ is a Euclidean Condition, one gets UPP as well. See *Greenberg* for the details.

8. Other ECs

The following was proposed as an axiom by Farkas Bolyai.

> *If P, Q, R are three non-collinear points, then there is a circle containing P, Q, and R.* (3ptCircle)

The following assumption was made by al-Tusi.

> *Suppose that A, B, C are three points on the same side of a line l such that $A * B * C$. Let D, F, G be points on l such that $\overleftrightarrow{AD}$, $\overleftrightarrow{BF}$, and $\overleftrightarrow{CG}$ are each perpendicular to l. If $\angle ABF$ is acute then $\overline{AD} < \overline{BF} < \overline{CG}$.* (AlTusiHyp)

Proposition 11. 3ptCircle *and* AlTusiHyp *are Euclidean Conditions.*

We skip the proof.

9. List of Euclidean Conditions

Here is a list of Euclidean Conditions in one place for convenient reference.

(1) UPP. Unique Parallel Property.
(2) E5P. Euclid's Fifth Postulate.
(3) ConAIA. The converse to the Alternating Interior Angle Theorem.
(4) Proclus. The Proclus Property concerning lines intersecting parallels.
(5) TPP. Transitivity of Parallel Property.
(6) (ZD:$\forall\triangle$). All triangle have zero defect.
(7) Rect. Rectangles exist. All triangle have zero defect.
(8) (ZD:$\exists\triangle$). Triangles with zero defect exist.
(9) WallisHyp. Similar triangles exist of arbitrary size.
(10) EqD. Equal distances for points on a parallel line.
(11) Big$\triangle$. Triangles of arbitrarily large areas exist.
(12) CrossbarHyp. Crossbars exist through arbitrary points.
(13) 3ptCircle. A circle exists containing three given non-collinear points.
(14) AlTusiHyp. Al-Tusi Hypothesis (for decreasing length segments).

10. Hyperbolic Geometry

Hyperbolic Geometry consists of 5 undefined terms, 16 axioms, and anything that can be defined or proved from these.

Primitive Terms. The five primitive terms are *point*, *line*, *betweenness*, *segment congruence*, and *angle congruence*. We will adopt all the notation and definitions from Neutral Geometry.

The Primitive Term Axiom for Hyperbolic Geometry is a preliminary axiom telling us what type of objects all the primitive terms are supposed to represent. The Primitive Term Axiom in Hyperbolic Geometry is exactly the same as for IBC Geometry.

Axiom (Primitive Terms)**.** *The basic type of object is the point. Lines are sets of points. Betweenness is a three place relation of points. If P, Q, R are points, then $P * Q * R$ denotes the statement that the betweenness relation holds for (P, Q, R). Segment congruence is a two place relation of line segments, and angle congruence is a two place relation of angles. If $\overline{AB}$ and $\overline{CD}$ are line segments, then $\overline{AB} \cong \overline{CD}$ denotes the statement that the segment congruence relation holds between $\overline{AB}$ and $\overline{CD}$. If α and β are angles, then $\alpha \cong \beta$ denotes the statement that the angle congruence relation holds between α and β.*

The axioms of Hyperbolic Geometry include the above Primitive Term Axiom together with the axioms I-1, I-2, I-3, B-1, B-2, B-3, B-4, C-1, C-2, C-3, C-4, C-5, C-6, Dedekind's Axiom, and the following Axiom.

Axiom (¬UPP)**.** *There is a line l, a point P not on l, and two lines m_1 and m_2 containing P such that $l \parallel m_1$ and $l \parallel m_2$.*

Remark 4. This axiom is called ¬UPP since it is essentially the negation of UPP. To see this, recall that in Neutral Geometry given $P \notin l$ there is at least one parallel m to l containing P. So UPP essentially says that for all l and $P \notin l$ there is at most one such parallel. The negation of this is that there is an l and $P \notin l$ with more than one such parallel. Note the change in quantifiers from $\forall$ to $\exists$.

Remark 5. Since the axioms of Neutral Geometry are a subset of the axioms of Hyperbolic Geometry, all the propositions of Neutral Geometry automatically hold in Hyperbolic Geometry. Thus every proposition of Neutral Geometry is a common proposition for Euclidean Geometry and Hyperbolic Geometry.

Here is are two meta-propositions concerning the two types of propositions of Hyperbolic Geometry.

Proposition 12. *Every proposition of Neutral Geometry is a proposition of Hyperbolic Geometry.*

Proof. See the above remark. □

Proposition 13. *The negation of any Euclidean Condition is a proposition of Hyperbolic Geometry.*

Proof. Let C be a Euclidean Condition. By definition $C \iff UPP$ is provable in Neutral Geometry. Thus the contrapositive $\neg UPP \iff \neg C$ is provable in Neutral Geometry. By the previous proposition, $\neg UPP \iff \neg C$ is provable in Hyperbolic Geometry. Since $\neg UPP$ is an axiom, we can prove $\neg C$ in Hyperbolic Geometry □

For example, the existence of a triangle of defect zero is a Euclidean Condition, thus the negation is a proposition of Hyperbolic Geometry. Since all triangles have non-negative defect in Neutral Geometry, we get the following:

Proposition 14. *The defect of any triangle is postive. So the angle sum of any triangle is strictly less than 180.*

11. The Hyperbolic Parallel Property (HPP)

From ¬UPP, which applies to only one P and l, we can actually prove something much stronger.

Definition 3 (HPP)**.** The *Hyperbolic Parallel Property (HPP)* is the following statement:

> *For all lines l and points $P \notin l$ there are at least two lines parallel to l that contain P.*

Proposition 15. *The statement*

$$\neg UPP \iff HPP$$

is provable in Neutral Geometry.

Proof. Proving HPP from ¬UPP is easy, so we focus on proving HPP from ¬ UPP. Since we assume ¬ UPP we get all the propositions of Hyperbolic Geometry. In particular, all triangles have positive defect.

Let l be a line, and $P \notin l$. Our goal is to find two parallel to l containing P. Drop a perpendicular from P to l, and let Q be the foot. Thus $l \perp \overleftrightarrow{PQ}$. Now let m_1 be a line perpendicular to $\overleftrightarrow{PQ}$ that contains P. By an earlier result, such a line exists, and by another result $l \parallel m_1$ since l and m_1 are both perpendicular to the same line $\overleftrightarrow{PQ}$.

We must find another, distinct parallel. Let R be a point of l not equal to Q. Let $\alpha = |\angle QPR| + |\angle QRP|$. Since $\delta PQR > 0$, we have $\alpha < 90$.

By the Angle Measure Theorem, there is a ray $\overrightarrow{PA}$ such that $\angle QPA$ has measure α, and such that A and R are on the same side of $\overleftrightarrow{PQ}$. Let $m_2 = \overleftrightarrow{AP}$. Since $\angle QPA$ is not right, we know that m_1 and m_2 are distinct lines containing P.

Now $|\angle QPA| = |\angle QPR| + |\angle QRP|$ by definition of α, and $|\angle QPA| = |\angle QPR| + |\angle RPA|$ since $\overrightarrow{PR}$ is interior to $\angle QPA$. Thus $|\angle RPA| = |\angle QRP|$. By the Alternating Interior Angle Theorem, $m_2 \parallel l$. So m_1 and m_2 are distinct parallels.[5] □

Corollary 16. HPP *holds in Hyperbolic Geometry*

12. Hyperbolic Conditions

Definition 4 (Hyperbolic Condition)**.** A *Hyperbolic Condition* is a statement that is provably equivalent, in Neutral Geometry, to ¬UPP. In other words, it is a statement C such that

$$C \iff \neg\text{UPP}$$

is provable in Neutral Geometry.

Proposition 17. ¬UPP *and* HPP *are each Hyperbolic Conditions.*

Proof. Since ¬UPP $\iff$ ¬UPP is trivially provable, we have that ¬UPP is a Hyperbolic Condition. By Proposition 15, HPP is also a Hyperbolic Condition. □

[5]This proof uses the following: $\overrightarrow{PR}$ is interior to $\angle QPA$, and A and Q are on opposite sides of $\overleftrightarrow{PR}$. The first follows from $\alpha > |\angle QPR|$. The second is a consequence of the first: since since R is interior to $\angle QPA$, the ray $\overrightarrow{PR}$ intersects $\overline{AQ}$ by the crossbar theorem, showing that A and Q are on opposite sides of $\overleftrightarrow{PR}$.

Every Hyperbolic Condition is in fact a theorem of Hyperbolic Geometry. In fact, each Hyperbolic Condition can be used as the final axiom of Hyperbolic Geometry.

Proposition 18. *Every HC is a proposition of Hyperbolic Geometry. Furthermore, if C is an Hyperbolic Condition, and we replace* $\neg$UPP *as an axiom by C, the resulting geometry will be equivalent to Hyperbolic Geometry. In other words, the two geometries will have the same propositions.*

Proof. Follows from the definition of Hyperbolic Condition and Hyperbolic Geometry. □

Our earlier study of Euclidean Conditions was also a study of Hyperbolic Conditions due to the following:

Proposition 19. *A statement is a Hyperbolic Condition if and only if its negation is a Euclidean Condition.*

Proof. $\neg$UPP $\iff C$ is equivalent to its contrapositive UPP $\iff \neg C$. The first defines a Hyperbolic Condition, the second defines a Euclidean Condition (for $\neg C$). □

The above proposition gives us many Hyperbolic Conditions: just negate the Euclidean Conditions discussed above. Examples: no rectangles exist,there are lines l_1, l_2, l_3 such that $l_1 \parallel l_2$ and $l_2 \parallel l_3$ but l_1 and l_3 intersect, there are points on a parallel line of different distances from a given line, etc.

An interesting Hyperbolic Condition is the following:

Definition 5 (AAA)**.** The *Angle-Angle-Angle property* (AAA) is the following statement: *Similar triangles are congruent.*

Proposition 20. *AAA is a Hyperbolic Condition. Hence it is a theorem of Hyperbolic Geometry.*

Proof. First assume $\neg$UPP, so we can use theorems from Hyperbolic Geometry. Suppose $\triangle ABC \sim \triangle DEF$, but $\triangle ABC \not\cong \triangle DEF$. By Lemma 6 there exists triangles of defect zero. But in Hyperbolic Geometry all triangles have positive defect, a contradiction. So similar triangles must be congruence.

Now suppose AAA. Then Wallis' Hypothesis (WallisHyp) must be false: otherwise we could take x larger than any side of the given triangle $\triangle ABC$ and derive a contradiction to AAA. Thus the negation of Wallis' Hypothesis holds. The negation of any Euclidean Condition is a Hyperbolic Condition, and every Hyperbolic Condition implies $\neg$UPP. So $\neg$UPP holds. □

Here is a list of some of the more interesting Hyperbolic Conditions:

1. $\neg$UPP concerning one line l, one point $P \notin l$, and two parallels.
2. HPP. Hyperbolic Parallel Property: a property of all l and $P \notin l$.
3. AAA. Similar triangles are congruent.
4. There are parallel lines, and a third line that only intersects one of the two parallels (negation of Proclus).
5. There are distinct lines $l_1 \parallel l_2$ and $l_2 \parallel l_3$ such that l_1 and l_3 intersect. (negation of TPP)
6. There is a triangle of non-zero defect (negation of (ZD:$\forall\triangle$)).
7. Rectangles do not exist (negation of Rect).

(8) All triangles have positive defect (negation of (ZD:$\exists\triangle$)).
(9) If $l \parallel m$, then there are points A and B on m whose distances to l differ (negation of EqD).
(10) There is a bound on the areas of triangles (negation of Big$\triangle$).
(11) There is an angle and a point P in the interior of the angle such that there are no crossbars going through P. (negation of the CrossbarHyp)

All the above conditions hold in Hyperbolic Geometry, but fail in Euclidean Geometry. They truly illustrates that Hyperbolic Geometry is, in the words of János Bolyai, a "strange new universe"!

13. Appendix: Historical comments

Nasir al-Din al-Tusi (1201–1274) (or Nasir Eddin al Tusi or Nasiraddin Tusi) was a medieval Persian mathematician who was famous for his work in mathematics, astronomy, philosophy, and logic. In mathematics he is known for his innovations in planar and spherical trigonometry. Some consider him to be the best astronomer between Ptolomy and Copernicus.

He was very familiar with Greek mathematics. In fact, he wrote several commentaries concerning important Greek mathematicians. Like other geometers, he was not content to accept Euclid's Fifth Postulate as an axiom, but wished to prove it instead. In doing so he made an assumption concerning distances: suppose m and l are lines such that a transversal t is perpendicular to l but not to m. Roughly speaking he assumed that on the side of the acute angle, points of m get closer to l and on the obtuse side they get farther away. From this assumption he was actually able to prove that the points do not just get closer on the side of the acute angle, but they actually intersect. This gives, without too much trouble, the Euclidean Fifth Postulate. His work made use of Saccheri quadrilaterals long before Saccheri was born.[6] Here is a slight restatement of his assumption.

Hypothesis (Al-Tusi)**.** *Suppose that A, B, C are three collinear points all on the same side of a line l such that $A * B * C$. Let D, F, G be points on l such that $\overleftrightarrow{AD}$, $\overleftrightarrow{BF}$, and $\overleftrightarrow{CG}$ are each perpendicular to l. If $\angle ABF$ is acute then $\overline{AD} < \overline{BF} < \overline{CG}$.*

From this al-Tusi proved the following:

Proposition 21. *If the above hypothesis holds then every Saccheri quadrilateral is a rectangle, so rectangles exist.*

Proof. Let $\square ABCD$ be a Saccheri quadrilateral with $\angle B$ and $\angle C$ right angle, and with $\overline{AB} \cong \overline{CD}$. Then we know from the Quadrilateral Handout that the summit angles $\angle A$ and $\angle D$ are congruent. Let X be such that $X * A * D$ and such that X, A, D are all on the same side of $\overleftrightarrow{BC}$. Drop a perpendicular from X to $\overleftrightarrow{BC}$, and let Y be the foot.

If $\angle BAD$ is acute then by the above hypothesis $\overline{DC} < \overline{AB} < \overline{XY}$. But this contradicts the assumption that $\overline{AB} \cong \overline{CD}$. Thus $\angle BAD$ is not acute. If $\angle BAD$ is obtuse, then $\angle XAB$ is acute. So, by the above hypothesis, $\overline{XY} < \overline{AB} < \overline{DC}$. But this also contradicts

[6] He wasn't the first. For example, the famous Persian Poet and Scientist Omar Khayyam used them earlier. But al-Tusi used them to great effect.

the assumption that $\overline{AB} \cong \overline{CD}$. Therefore $\angle BAD$ is a right angle. Similarly, $\angle ADC$ is right. So $\square ABCD$ is a rectangle.

Thus every Saccheri quadrilateral is a rectangle. But you can form Saccheri quadrilaterals $\square ABCD$ with sides $\overline{AB}$ and $\overline{BC}$ of any length you want. Also $\triangle ABC$ is right. Since $\delta ABCD = 0$, it follows that $\delta ABC = 0$. Thus all right triangles have defect zero. Then, as in the handout on Lagendre's Defect Zero Theorem, every triangle must have defect zero.

This shows all Saccheri quadrilateral is a rectangle. Since Saccheri quadrilaterals exist (in Neutral Geometry), so do rectangles. □

14. Appendix: Aristotle's Hypothesis

Aristotle assumed that the distance between two rays of an acute angle tends to infinity as you go out along a ray. This turns out to be true and provable in both Euclidean and Hyperbolic Geometry. In fact, it is a theorem of Neutral Geometry. Apparently, Sacchari was the first to prove this hypothesis of Aristotle in Neutral Geometry (it is easy to prove in Euclidean Geometry; the challenge is to prove it without the parallel postulate).

Proposition 22 (Aristotle's Hypothesis)**.** *Let $\angle CAD$ be an acute angle. For every number M, we can find a point C' of $\overrightarrow{AC}$ such that the distance from C' to the line $\overleftrightarrow{AD}$ is greater than M.*

Proof. Drop a perpendicular from C to the line $\overleftrightarrow{AD}$. Let B be the foot. So $\triangle ABC$ is a right triangle with right angle $\angle B$. Now let C_1 be a point such that $A * C * C_1$ and $\overline{CC_1} \cong \overline{AC}$. Drop a perpendicular from C_1 to the line $\overleftrightarrow{AD}$. Let B_1 be the foot. So $\triangle AB_1C_1$ is a right triangle with right angle $\angle B_1$.

Observe that $\left|\overline{AC_1}\right| = 2\left|\overline{AC}\right|$. If we are in Euclidean Geometry, then we have defect zero for all triangles and so, by Lemma 2, $\left|\overline{B_1C_1}\right| = 2\left|\overline{BC}\right|$. Now since $\left|\overline{B_1C_1}\right|$ is the distance from C_1 to $\overleftrightarrow{AD}$, we have shown how to double the distance. What if we are not in Euclidean Geometry? In this case, the remark after Lemma 2 shows us that, because of the properties of Lambert quadrilateral, we have $\left|\overline{B_1C_1}\right| > 2\left|\overline{BC}\right|$. So we do even better: we more than double the distance.[7]

By repeating this process we get a sequence $C_1, C_2, C_3, \ldots$. Each term doubles, or more than doubles, the distance to $\overleftrightarrow{AD}$. So eventually the distance will be larger than any given M. □

[7] In Elliptic Geometry or Spherical Geometry, this does not happen.