# INCIDENCE-BETWEENNESS GEOMETRY

MATH 410. SPRING 2006. INSTRUCTOR: PROFESSOR AITKEN

This document reviews the part of geometry that can be developed without congruence. (We will study the geometry with congruence after the first test). Let's call this geometry *incidence-betweenness geometry* since it is based on the ideas of incidence and betweeness. For the most part, I have tried to stay close to our textbook, but I have added a few extra definitions and propositions that I think clarifies the geometry. The biggest change is the addition of a defined 4-term betweenness relation.

Recall that before studying the full incidence-betweeness geometry, we studied a simple part of the geometry called *incidence geometry*.

## 1. Incidence Geometry

Incidence geometry consists of three undefined terms, three axioms, and anything that can be defined or proved from these.

**Undefined Terms.** The three undefined terms[1] are *point*, *line*, and *incidence*. Points and lines are types of objects. You should keep an open mind and allow for the possibility that these objects are not points and lines in the traditional sense. Incidence is a relation relating points and lines. In other words, given a point $P$ and a line $\ell$, either $P$ is incident to $\ell$ or it is not. We sometimes write $P \,\mathrm{I}\, \ell$ to indicate that $P$ is incident to $\ell$.[2]

In order to make things easier to read, we can express $P \,\mathrm{I}\, \ell$ informally. For example, "$P$ is on $\ell$" or "$\ell$ passes through $P$" or "$\ell$ contains $P$" are three informal ways of saying $P \,\mathrm{I}\, \ell$.

*Remark.* Note that "plane" is not an undefined term. This is because we are studying plane geometry, and so can consider the plane to be the set of *all* points. If we were studying three dimensional geometry, then we might expect "plane" to be another undefined term.

The first axiom is inspired by Euclid's original first postulate:

**Axiom** (I-1)**.** *Suppose $P$ and $Q$ are distinct points. Then there is a unique line $\ell$ such that $P \,\mathrm{I}\, \ell$ and $Q \,\mathrm{I}\, \ell$. In other words, there is a unique line passing through both $P$ and $Q$.*

**Definition 1.** Let $P$ and $Q$ be distinct points, then $\overleftrightarrow{PQ}$ is defined to be the line that passes through $P$ and $Q$. The above axiom assures us that this line exists and is unique. But if $P = Q$ then $\overleftrightarrow{PQ}$ is undefined.

This definition assures us that (i) $P$ is on $\overleftrightarrow{PQ}$, (ii) $Q$ is on $\overleftrightarrow{PQ}$, and (iii) if a line $\ell$ passes through $P$ and $Q$ then $\ell = \overleftrightarrow{PQ}$.

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[1] Actually, I prefer to call them *primitve terms*.

[2] We do not write $P \in \ell$ since, among other reasons, we haven't assumed that $\ell$ is a set. (We haven't assumed that $\ell$ isn't a set either, but we act as if we do not know one way or the other. Recall that in some models lines are sets, in others they are not.)

The following is easily proved from the above definition:

**Proposition 1.** *Let $P$ and $Q$ be distinct points. Then $\overleftrightarrow{PQ} = \overleftrightarrow{QP}$.*

We know that every two points gives us a line, but that does not automatically mean that every line has two points. (You can construct a model for Axiom I-1 where some lines have only one or zero points). The next axiom addresses this issue.

**Axiom** (I-2)**.** *If $\ell$ is a line then there are at least two points on $\ell$.*

*Remark.* This axiom implies that if $\ell$ is a line, there must be points $P$ and $Q$ such that $\ell = \overleftrightarrow{PQ}$. In other words, every line is of the form $\overleftrightarrow{PQ}$ for some $P$ and $Q$.

In order to make our geometry into a planar geometry, we need to know that there is no line $\ell$ that passes through every point in our geometry. Since this cannot be proved from the previous axioms, we need a new axiom:

**Axiom** (I-3)**.** *There are three points with the following property: there is no line passing through all three of the points.*

*Remark.* In particular, we now know that there really are points (at least three of them). So our geometry is not completely trivial. By Axiom I-1 there must be lines as well.

**Definition 2** (Collinear)**.** Let $P_1, P_2, \ldots, P_k$ be points. If there is a line $\ell$ such that $P_i \mathrm{I} \ell$ for all $i$, then we say that $P_1, P_2, \ldots, P_k$ are *collinear*.

*Remark.* Axiom I-3 tells us is that there exist three non-collinear points.

The following is easily proved (see if you can prove it in two or three sentences).

**Proposition 2.** *Suppose $P$, $Q$ and $R$ are distinct points. Then $P, Q, R$ are collinear if and only if $R$ is on $\overleftrightarrow{PQ}$.*

The following is also easy.

**Proposition 3.** *Suppose $P$, $Q$ and $R$ are distinct points. Then $P, Q, R$ are collinear if and only if $\overleftrightarrow{PQ} = \overleftrightarrow{PR} = \overleftrightarrow{QR}$.*

If you switch points and lines in the definition of collinear you get the definition of *concurrence*.

**Definition 3** (Concurrent)**.** Let $m_1, m_2, \ldots, m_k$ be lines. If there is a point $Q$ such that $Q \mathrm{I} m_i$ for all $i$, then we say that $m_1, m_2, \ldots, m_k$ are *concurrent*.

Since our textbook is about the parallel postulate, the following definition is obviously one of the most important in the course:

**Definition 4** (Parallel)**.** Suppose that $\ell$ and $m$ are lines. If there is no point common to both, then we say that $\ell$ and $m$ are *parallel*. In other words, if there is no $P$ such that $P \mathrm{I} \ell \wedge P \mathrm{I} m$ then $m$ and $\ell$ are parallel. We indicate that $m$ and $\ell$ are parallel by writing $\ell \parallel m$.

**Definition 5.** Since a line $\ell$ might not be a set, it is useful to define $\{\ell\}$ to be the set of points on $\ell$. In other words,

$$\{\ell\} = \{\text{points } P \mid P \mathrm{I} \ell\}.$$

From this definition we get that $P \mathrm{I} \ell$ if and only if $P \in \{\ell\}$. (The notation here, based on that of the textbook, is not ideal since $\{\ell\}$ might be confused with the singleton set with sole element $\ell$.[3])

We end this section with five propositions (Proposition 2.1 to 2.5) which you already proved in a homework assignment.

**Proposition 4.** *If $l$ and $m$ are distinct lines that are not parallel, then there is a unique point $P$ such that $P \mathrm{I} l$ and $P \mathrm{I} m$. In other words, non-parallel lines meet in exactly one point.*

*Proof.* Hints: existence follows from the definition of parallel. Uniqueness follows from Axiom I-1. □

**Proposition 5.** *There exists three lines that are not concurrent.*

*Proof.* Hints: let $P, Q, R$ be non-collinear (see Axiom I-3 for existence). Consider $\overleftrightarrow{PQ}$, $\overleftrightarrow{PR}$, $\overleftrightarrow{QR}$. Are they distinct? Why? Suppose they all go through a point $X$. If $X$ is not $P$ show that $\overleftrightarrow{PQ} = \overleftrightarrow{PX}$ and $\overleftrightarrow{PR} = \overleftrightarrow{PX}$. Why is that a contradiction? If $X = P$ then $X$ is on $\overleftrightarrow{QR}$, so $P, Q, R$ are collinear. Why is that a contradiction? □

**Proposition 6.** *If $l$ is a line, then there is a point not on $l$.*

*Proof.* Hint: consider the three non-collinear points from Axiom I-3. Are they all on $l$? □

**Proposition 7.** *If $P$ is a point, then there is a line not passing through it.*

*Proof.* Hint: this proof seems to be tricky for some students. It is not just a simple application of Axiom I-3. (The point $P$ might not be one of the three points discussed in the axiom). Instead, use Proposition 5. Can $P$ be on all the lines in this proposition? □

**Proposition 8.** *If $P$ is a point, then there are at least two lines passing through $P$.*

*Proof.* Hint: let $l$ be a line *not* passing through $P$. What prior proposition gives you such a line? Let $Q$ and $R$ be two distinct points on $l$. What axiom gives you these points? Can you show $\overleftrightarrow{PQ}$ and $\overleftrightarrow{PR}$ work? Why are they distinct? □

## 2. Incidence-Betweeness Geometry

Incidence-Betweeness Geometry consists of four undefined terms, seven axioms, and anything that can be defined or proved from these.

**Undefined Terms.** The four undefined terms are *point*, *line*, *incidence*, and *betweenness*. Points and lines are types of objects. Incidence is a relation between points and lines. We adopt the notation and conventions for incidence described in the previous section. Finally, betweenness is a relation: it relates triples of points.

We write $P * Q * R$ to indicate that betweenness holds for the triple $(P, Q, R)$ of points. If $P * Q * R$ we say that "$Q$ is between $P$ and $R$".

The axioms of Incidence-Betweeness Geometry include the three axioms of Incidence Geometry plus four new axioms called the *betweenness axioms*. Since we adopt the three incidence axioms, all the results (and definitions) of the previous section are still valid.

[3]Perhaps notation such as $\{\ell$ or $\Sigma\ell$ might be better.

**Axiom** (B-1)**.** *Suppose $P, Q, R$ are points such that $P * Q * R$. Then (i) $R * Q * P$, (ii) $P$, $Q$, and $R$ are collinear, and (iii) $P$, $Q$, and $R$ are distinct.*

*Remark.* In particular, if $P * Q * R$ then $\overleftrightarrow{PQ} = \overleftrightarrow{PR} = \overleftrightarrow{QR}$, and this line is the unique line containing $P, Q$, and $R$. Also, if $Q$ is between $P$ and $R$, then $Q$ is between $R$ and $P$.

**Definition 6.** (Line Segment) Suppose $A, B$ are distinct points. Then the line segment $AB$ is defined as follows:

$$AB = \{\mathrm{P} \mid A * P * B\} \cup \{A, B\}.$$

In other words, $AB$ consists of $A$, $B$, and all the points between $A$ and $B$.

**Proposition 9.** *If $A$ and $B$ are distinct points, then $AB = BA$.*

*Proof.* First we show $AB \subseteq BA$. If $P \in AB$ then either $P = A$ or $P = B$ or $A * P * B$ by the definition of line segment. If $P = A$ or $P = B$, we have $P \in BA$ by the above definition. If $A * P * B$ then $B * P * A$ by Axiom B-1. Thus $P \in BA$ by the above definition. In either case, $P \in BA$. So $AB \subseteq BA$. By a similar argument, $BA \subseteq AB$. □

*Remark.* Unlike lines, we know line segments are sets. Likewise, rays are sets.

**Definition 7.** (Ray) Suppose $A, B$ are distinct points. Then the ray $\overrightarrow{AB}$ is defined as follows:

$$\overrightarrow{AB} = AB \cup \{\mathrm{P} \mid A * B * P\}.$$

The following two propositions can be proved directly from the definition (using some set theory).

**Proposition 10.** *Suppose $A, B$ are distinct points. Then $P \in \overrightarrow{AB}$ if and only if either (i) $P = A$, (ii) $A * P * B$, (iii) $P = B$, or (iv) $A * B * P$.*

**Proposition 11.** *Suppose $A, B$ are distinct points. Then $AB \subseteq \overrightarrow{AB}$.*

The previous proposition is trivial, and can be proved without using the axioms. However, the following needs Axiom B-1 in its proof. Explain why this axiom is needed.

**Proposition 12.** *Suppose $A, B$ are distinct points. Then $AB$ and $\overrightarrow{AB}$ are subsets of $\{\overleftrightarrow{AB}\}$.*

The next axiom shows that line segments and rays have more points than just the two obvious points.

**Axiom** (B-2)**.** *Suppose $B$ and $D$ are distinct point. Then there are points $A, C, E$ such that $A * B * D$ and $B * C * D$ and $B * D * E$.*

The next axiom gives one way that that betweenness behaves as expected: given three points on a line, exactly one is between the others.

**Axiom** (B-3)**.** *Suppose $A$, $B$, and $C$ are distinct points on a line $l$. Then exactly one of the following occurs: (i) $A * B * C$, (ii) $B * A * C$, or (iii) $A * C * B$.*

The following are easy now that we have Axiom B-3:

**Proposition 13.** *If $C * A * B$ then $C$ is not on the ray $\overrightarrow{AB}$. Likewise, If $C * A * B$ then $B$ is not on the ray $\overrightarrow{AC}$.*

*Proof.* Hint: use Proposition 10 and Axiom B-3. You might also use Axiom B-1 as well. $\square$

**Proposition 14.** *If $C * A * B$ or $A * B * C$ then $C$ is not on the segment $AB$.*

With the betweenness axioms we have so far, we have enough to prove a few facts about rays (for example, Proposition 3.1 in the textbook, which is divided below into two propositions):

**Proposition 15.** *If $A$ and $B$ are distinct points, then $\overrightarrow{AB} \cap \overrightarrow{BA} = AB$.*

*Proof.* Hint: (see also page 75 in the textbook) the inclusion $AB \subseteq \overrightarrow{AB} \cap \overrightarrow{BA}$ is easy. To see it, observe by Proposition 11 that $AB \subseteq \overrightarrow{AB}$. By Proposition 9, $AB = BA$, and, by Proposition 11, $BA \subseteq \overrightarrow{BA}$. Thus $AB \subseteq \overrightarrow{AB} \cap \overrightarrow{BA}$ by basic set theory.

To prove the other inclusion, suppose that $P$ is in the intersection. Use Proposition 10 and Proposition 13 to show that either $P = A$, $P = B$ or $A * P * B$. Thus $P \in AB$. $\square$

**Proposition 16.** *If $A$ and $B$ are distinct points, then $\overrightarrow{AB} \cup \overrightarrow{BA} = \{\overleftrightarrow{AB}\}$.*

*Proof.* $\overrightarrow{AB} \cup \overrightarrow{BA} \subseteq \{\overleftrightarrow{AB}\}$ follows from Proposition 12 (and the fact that $\overleftrightarrow{BA} = \overleftrightarrow{AB}$).

To show the other inclusion, suppose that $P$ is on the line $\overleftrightarrow{AB}$. If $P = A$ or $P = B$ then, by definition of ray, $P \in \overrightarrow{AB}$. Otherwise, $P, A, B$ are distinct, so by Axiom B-3 either (i) $B*A*P$ or (ii) $A*B*P$ or (iii) $A*P*B$. In case (i) we have $P \in \overrightarrow{BA}$ by Proposition 10, in case (ii) or (iii) we have $P \in \overrightarrow{AB}$ by Proposition 10. In any case, $P$ is in at least one of the two rays, so it is in the union. $\square$

To prove more advanced results concerning betweenness we need the final betweenness axiom. To explain this axiom , we need the following:

**Definition 8.** Let $l$ be a line, and let $A$ and $B$ be points not on $l$. If $A = B$ or if $AB$ does not intersect $\{l\}$ then we say that "$A$ and $B$ are on the same side of $\{l\}$". In the other case (where $A \neq B$ and $AB$ intersects $\{l\}$) we say that "$A$ and $B$ are on opposite sides of $\{l\}$".

If $A$ and $B$ are on the same side of $l$ then we write $A \sim_l B$. If $A$ and $B$ are on opposite sides, we write $A \not\sim_l B$.

Observe that the reflexive law $A \sim_l A$ follows by definition.

Also observe that, since $A$ and $B$ are not on $l$, if $A \not\sim_l B$, then $\overleftrightarrow{AB}$ intersects $l$ at a point $P$ with $A * P * B$ (and, of course, the converse holds).

**Proposition 17** (symmetric law)**.** *Let $l$ be a line, and let $A$ and $B$ be points not on $l$. Then $A \sim_l B$ implies $B \sim_l A$.*

*Proof.* Suppose $A \sim_l B$. If $A \neq B$ then the result follows from the fact (Proposition 9) that $AB = BA$. If $A = B$ the the conclusion follows from the reflexive law. $\square$

Now we are ready for the final axiom of incidence-betweenness geometry.

**Axiom** (B-4)**.** *Suppose $A, B, C$ are points not on the line $l$.*

*(i) If $A \sim_l B$ and $B \sim_l C$, then $A \sim_l C$. In other words, the relation $\sim_l$ is transitive for the points not on $l$.*

*(ii) If $A \not\sim_l B$ and $B \not\sim_l C$, then $A \sim_l C$.*

**Proposition 18.** *Suppose $l$ is a line, and suppose $A, B, C$ are points not on $l$. If $A \not\sim_l B$ and $B \sim_l C$, then $A \not\sim_l C$.*

*Proof.* Suppose otherwise that $A \sim_I C$. Observe that $C \sim_l B$ by applying the symmetric law to one of the hypotheses. By Axiom B-4 (transitive law) we get $A \sim_l B$, a contradiction. $\square$

**Proposition 19.** *If $l$ is a line, then $\sim_l$ is an equivalence relation for the set of points not on $l$.*

*Proof.* The reflexive law follows from the definition of $\sim_l$. The symmetric law was proved above, and the transitive law is part of Axiom B-3. $\square$

Since $\sim_l$ is an equivalence relation, it partitions the points not on $l$ into equivalence classes. These equivalence classes are called *half-planes bounded by $l$* or *sides of $l$*.

**Definition 9.** Let $l$ be a line, and let $A$ be a point not on $l$. Then the equivalence class $[A] = \{B \text{ not on } l \mid B \sim_l A\}$ is called a *side of $l$* or a *half plane bounded by $l$*.

Since $\sim_l$ is an equivalence relation, we have that the sides of $l$ are disjoint. How many sides are there?

**Proposition 20.** *Let $l$ be a line. Then $l$ has exactly two sides. In other words, $l$ bounds exactly two half-planes.*

*Proof.* Let $A$ be a point not on $l$. Such a point exists by Proposition 6. Let $P$ be a point on $l$. Such a point exists by Axiom I-2. Finally, let $B$ be a point such that $A * P * B$. Such a point exists by Axiom B-2.

Since distinct lines intersect in at most one point (Proposition 4), we know that $B$ is also not on $l$. We know $A \not\sim_l B$ since $AB$ intersects $l$ at $P$.

Consider the sides $[A]_l$ and $[B]_l$. They are distinct since $A \not\sim_l B$. So we have at least two sides.

Now suppose $[C]_l$ is a third side. In other words, assume $[C]_l \neq [A]_l$ and $[C]_l \neq [B]_l$. Then, by definition of equivalence class, $A \not\sim_l C$ and $C \not\sim_l B$. So by Axiom B-4, we get $A \sim_l B$, a contradiction. $\square$

Informally, the next result shows that if a line crosses one side of a triangle, it must cross one of the other sides as well. (The book assumes that $A, B, C$ are non-collinear, but I do not see why that is necessary, so I state it in more generality.)

**Theorem 1** (Pasch's Theorem)**.** *Suppose $A, B, C$ are points with $A \neq B$, and $l$ is a line intersecting $AB$. Then (i) $l$ intersects $AC$ or $CB$, and (ii) if $l$ does not contain any of $A, B, C$ then $l$ intersects exactly one of $AC$ and $CB$.*

*Proof.* Suppose (i) fails, so $l$ does not intersect $AC$ and does not intersect $CB$. This implies that $A, B, C$ are not on $l$, and that $A \sim_l C$ and $C \sim_l B$. So $A \sim_l B$ by Axiom B-3. This contradicts the hypothesis that $AB$ intersects $l$.

Now suppose that $l$ does not pass through any of $A, B, C$. Suppose (ii) fails. So, using what we just proved, $l$ intersects both $AC$ and $CB$. So $A \not\sim_l C$ and $C \not\sim_l B$. By Axiom B-3, this implies that $A \sim_l B$. This contradicts the hypothesis that $AB$ intersects $l$. $\square$

## 3. Betweenness for four points

Betweenness for three points was taken as an undefined notion, and the above axioms show that betweenness for three points behaves as expected. What about betweenness among four collinear points? We first define the notion:

**Definition 10.** Let $A, B, C, D$ be points. We define $A$-$B$-$C$-$D$ to mean that $A * B * C$ and $A * B * D$ and $A * C * D$ and $B * C * D$. (In other words, $A$-$B$-$C$-$D$ means that we can drop any one of the points, and get a valid triple betweenness.)

**Proposition 21.** *If $A$-$B$-$C$-$D$ then $A, B, C, D$ are distinct and collinear.*

*Proof.* Hint: use the definition and Axiom B-1. □

**Proposition 22.** *If $A$-$B$-$C$-$D$ then $D$-$C$-$B$-$A$.*

*Proof.* Suppose $A$-$B$-$C$-$D$. Then, by definition, $A * B * C$ and $A * B * D$ and $A * C * D$ and $B * C * D$. By Axiom B-1, $C * B * A$ and $D * B * A$ and $D * C * A$ and $D * C * B$. By definition, $D$-$C$-$B$-$A$ □

If you try to draw a picture where $A * B * C$ and $A * C * D$, you will find that you are forced to make $B * C * D$ and $A * B * D$ hold as well, so $A$-$B$-$C$-$D$ appears to hold. Can we prove this intuitively obvious fact with the axioms so far? The following is essentially Proposition 3.3 in the textbook.

**Proposition 23.** *If $A * B * C$ and $A * C * D$, then $A$-$B$-$C$-$D$.*

*Proof.* Let $E$ be a point not on $\overleftrightarrow{AB}$. Such a point exists by Proposition 6. By Axiom B-1, $C$ is on $\overleftrightarrow{AB}$, so $C$ and $E$ are distinct. Thus $l = \overleftrightarrow{CE}$ exists (by Axiom I-1). Since $A * C * D$, the line segment $AD$ intersects $l$ at $C$. Thus $A \not\sim_l D$.

By Proposition 4, the line $\overleftrightarrow{AB}$ intersects the line $l$ only at $C$. Thus any line segment in $\{\overleftrightarrow{AB}\}$, intersects $l$ at $C$ or not at all. For example, since $A * B * C$, the point $C$ is not in the segment $AB$ (by Axiom B-3), and the segment $AB$ does not intersect $l$. Thus $A \sim_l B$.

Since $A \sim_l B$ and $A \not\sim_l D$, it follows from Proposition 18 that $B \not\sim_l D$. Thus $BD$ intersects $l$ at a point between $B$ and $D$, and as mentioned above, that point must be at $C$. So $B * C * D$.

A similar argument (using $\overleftrightarrow{BE}$, the assumption $A * B * C$ and the earlier result that $B * C * D$) shows that $A * B * D$. □

The following two propositions can be proved in a similar manner.

**Proposition 24.** *If $A * B * C$ and $B * C * D$, then $A$-$B$-$C$-$D$.*

**Proposition 25.** *If $A * B * D$ and $B * C * D$, then $A$-$B$-$C$-$D$.*

**Proposition 26.** *Suppose $A, B, C, D$ are distinct and collinear points such that $A * B * C$. Then either (i) $D$-$A$-$B$-$C$, (ii) $A$-$D$-$B$-$C$, (iii) $A$-$B$-$D$-$C$, or (iv) $A$-$B$-$C$-$D$.*

*Proof.* Use Axiom B-3 to consider all the possibilities involving $A, B, D$. If $D * A * B$ then, since $A * B * C$, we get $D$-$A$-$B$-$C$ by Proposition 24, so the result holds. If $A * D * B$ we get $A$-$D$-$B$-$C$ by Proposition 23 so the result holds. So, we can assume from now on that we are in the remaining case where $A * B * D$.

Repeat with $A, C, D$. In other words, use Axiom B-3 again and consider all the possibilities involving $A, C, D$. If $D * A * C$ then, since $A * B * C$, we get $D$-$A$-$B$-$C$ by Proposition 25. If $A * C * D$ we get $A$-$B$-$C$-$D$ by Proposition 23. So, we can assume from now on that $A * D * C$.

We have reduced to the case where $A * B * D$ and $A * D * C$. By Proposition 23 this implies that $A$-$B$-$D$-$C$. So the result holds. $\square$

The following generalizes Axiom B-3 to four term betweenness:

**Corollary 1.** *If $A, B, C, D$ are distinct and collinear points, then there is a permutation $X, Y, Z, W$ of these four points so that $X$-$Y$-$Z$-$W$.*

*Proof.* By Axiom B-3, either $A * B * C$ or $B * C * A$ or $B * A * C$. Now use the above proposition to show that in any case $D$ can be incorporated into the betweenness. $\square$

Having these results about four term betweenness makes it easier to prove some of the harder results involving rays:

**Proposition 27.** *If $C * A * B$ then $\overrightarrow{AB} \cap \overrightarrow{AC} = \{A\}$.*

*Proof.* Clearly, $A$ is on both rays, so $\{A\} \subseteq \overrightarrow{AB} \cap \overrightarrow{AC}$. So we only need to establish the other inclusion.

Suppose that $P \in \overrightarrow{AB} \cap \overrightarrow{AC}$. We will give an indirect proof: we assume that $P \neq A$, and derive a contradiction.

Observe that $P \neq B$ (otherwise $P$ would fail to be on $\overrightarrow{AC}$ by Proposition 13). Likewise, $P \neq C$. So we can assume that $A, B, C, D$ are distinct. They are collinear. (Do you see why?)

By Proposition 26, and the fact that $C * A * B$, we know that either (i) $P$-$C$-$A$-$B$, (ii) $C$-$P$-$A$-$B$, (iii) $C$-$A$-$P$-$B$, or (iv) $C$-$A$-$B$-$P$. In cases (i) and (ii) we have $P * A * B$, so $P \notin \overrightarrow{AB}$ (by Proposition 13). Likewise, in cases (iii) and (iv) we have $C * A * P$ so $P \notin \overrightarrow{AC}$. Thus each case gives a contradiction. $\square$

**Definition 11.** (opposite rays) If $C * A * B$ then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called *opposite rays*. The above proposition asserts that opposite rays intersect only at a point. What is their union?

**Proposition 28.** *If $C * A * B$ then $\overrightarrow{AB} \cup \overrightarrow{AC} = \{\overleftrightarrow{BC}\}$.*

*Proof.* By Proposition 11, $\overrightarrow{AB} \subseteq \{\overleftrightarrow{AB}\}$ and $\overrightarrow{AC} \subseteq \{\overleftrightarrow{AC}\}$. By Axiom B-1, the points $A, B, C$ are collinear, so $\overleftrightarrow{AB} = \overleftrightarrow{BC}$ and $\overleftrightarrow{AC} = \overleftrightarrow{BC}$. Thus both rays are subsets of $\{\overleftrightarrow{BC}\}$. It follows from basic set theory that the union is a subset of $\{\overleftrightarrow{BC}\}$ as well.

For the other inclusion, let $P$ be a point on $\overleftrightarrow{BC}$. This implies that $A, B, C, P$ are collinear. (Can you tell me why?). If $P$ is $A$, $B$, or $C$ then the result is obvious, so we can assume that $A, B, C, P$ are distinct. (Can you fill in the details?)

By Proposition 26, and the fact that $C * A * B$, we know that either (i) $P$-$C$-$A$-$B$, (ii) $C$-$P$-$A$-$B$, (iii) $C$-$A$-$P$-$B$, or (iv) $C$-$A$-$B$-$P$. In case (i) we get $P * C * A$, so $P \in \overrightarrow{AC}$. In case (ii) we get $C * P * A$ so $P \in AC \subseteq \overrightarrow{AC}$. In case (iii) we get $A * P * B$ so $P \in AB \subseteq \overrightarrow{AB}$. In case (iv) we get $A * B * P$ so $P \in \overrightarrow{AB}$. In any case, the point $P$ is in the union of the two rays. $\square$

The following was done in a homework assignment (before you knew about Proposition 26). Observe how Proposition 26 can be used to produce a short proof.

**Proposition 29.** *Suppose $A * B * C$. Then $AC = AB \cup BC$.*

*Proof.* Suppose $P \in AC$. If $P = A$ or $P = B$ or $P = C$ then $P$ is in the union. So we can assume that $P, A, B, C$ are distinct. Since $P \in AC$ we have that $A * P * C$. By Proposition 26, and the fact that both $A * B * C$ and $A * P * C$, we know that either (i) $A$-$B$-$P$-$C$ or (ii) $A$-$P$-$B$-$C$. In the first case, $B * P * C$ so $P \in BC$. In the second case $A * P * B$ so $P \in AB$. In either case, $P$ is in the union. This establishes one inclusion.

For the opposite inclusions. Suppose that $P \in AB \cup BC$. If $P = A$ or $P = B$ or $P = C$ then $P$ is clearly in $AC$. So we can restrict ourselves to the case where $A, B, C, P$ are distinct. If $P$ is in $AB$ then $A * P * B$. Since, in addition, $A * B * C$, we have by Proposition 23 that $A$-$P$-$B$-$C$. So $A * P * C$ and $P \in AC$. A similar argument show that if $P \in BC$ then $P \in AC$. In either case, $P \in AC$. □

**Proposition 30.** *Suppose $A * B * C$. Then $\overrightarrow{AC} = \overrightarrow{AB}$.*

*Proof.* In order to establish one inclusion, suppose that $P \in \overrightarrow{AC}$. In other words, assume that $P * A * C$ is false. If $P = A$, $P = B$, or $P = C$ the inclusion follows from the fact that $A * B * C$. So we can assume that $P, A, B, C$ are distinct, that $A * B * C$ is true, and that $P * A * C$ is false. By Proposition 26, and using the above assumptions, we restrict to the cases whare (i) $A$-$P$-$B$-$C$, (ii) $A$-$B$-$P$-$C$ or (iii) $A$-$B$-$C$-$P$. In the first case, $A * P * B$, in the second $A * B * P$, and in the third $A * B * P$. In either case $P \in \overrightarrow{AB}$. This establishes one inclusion. The other inclusion is established by a similar argument. □

## 4. The interior of angles and triangles

An angle is *not* defined to be space between two rays, but is defined to be the union of two rays. However, we can define the space between two angles, called the *interior* as the intersection of two half planes.

**Definition 12** (Angle)**.** Suppose $A, B, C$ are non-collinear points. Then the angle $\angle BAC$ is defined to be $\overrightarrow{AB} \cup \overrightarrow{AC}$.

**Definition 13** (Interior and Exterior of an Angle)**.** Suppose $A, B, C$ are non-collinear points. Then the interior of $\angle BAC$ is defined to be the following intersection of half-planes:

$$[B]_{\overleftrightarrow{AC}} \cap [C]_{\overleftrightarrow{AB}}.$$

In other words, $D$ is in the interior of $\angle BAC$ if and only if (i) $B$ and $D$ are on the same side of $\overleftrightarrow{AC}$ and (ii) $C$ and $D$ are on the same side of $\overleftrightarrow{AB}$.

A point is said to be *exterior* to $\angle BAC$ if (and only if) it is not on $\angle BAC$ and not in interior of $\angle BAC$. The set of exterior points to $\angle BAC$ is called the *exterior* of $\angle BAC$.

The following is easy. (Can you prove it?)

**Proposition 31.** *If $A, B, C$ are not collinear, then they are distinct.*

The following gives a way to produce points in the interior (and exterior) of an angle.

**Proposition 32** (Pre-Crossbar)**.** *Suppose $A, B, C$ are non-collinear points, and suppose $D$ is on the line $\overleftrightarrow{BC}$. Then $D$ is in the interior of $\angle BAC$ if and only if $B * D * C$.*

*Proof.* Suppose that $D$ is in the interior. Then , by definition, (i) $B$ and $D$ are on the same side of $\overleftrightarrow{AC}$ and (ii) $C$ and $D$ are on the same side of $\overleftrightarrow{AB}$. In particular, $D$ is not on $\overleftrightarrow{AC}$ or $\overleftrightarrow{AB}$, so $D \neq C$ and $D \neq B$. By Axiom B-3, either (i) $D * B * C$ or (ii) $B * D * C$ or (iii) $B * C * D$. We show $B * D * C$ by eliminating the other cases. In case (i) we have that $D$ and $C$ are on opposite sides of $\overleftrightarrow{AB}$, a contradiction. In case (iii) we have that $B$ and $D$ are on opposite sides of $\overleftrightarrow{AC}$, a contradiction. Thus case (ii) holds: $B * D * C$.

Now suppose $B * D * C$. Then $B \notin \overline{DC}$ by Proposition 14. But $B$ is the unique point of intersection between the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ (see Proposition 4), and of course $\overleftrightarrow{BC} = \overleftrightarrow{DC}$ (by Axiom B-1 and the fact that $B * D * C$). So $DC$ cannot intersect $\overleftrightarrow{AB}$. Thus $D$ and $C$ are on the same side of $\overleftrightarrow{AB}$. Similarly, $D$ and $B$ are on the same side of $\overleftrightarrow{AC}$. Thus $D$ is in the interior of $\angle BAC$. ☐

**Proposition 33.** *Let $l$ be a line, $A$ a point on $l$ and $B$ a point not on $l$. Then every point on the ray $\overrightarrow{AB}$ except $A$ itself is on the same same half plane bounded by $l$. In other words, every point $P$, except $A$, on $\overrightarrow{AB}$ has the property that $P \sim_l B$.*

*Furthermore, if $C * A * B$, then every point $Q$, except $A$, on the opposite ray $\overrightarrow{AC}$ has the property that $Q \not\sim_l B$.*

**Corollary 2.** *Suppose $A$ and $A'$ are points on a line $l$ (possibly the same point), and suppose $B$ and $C$ are two points not on $l$. If $B \not\sim_l C$ then the rays $\overrightarrow{AB}$ and $\overrightarrow{A'C}$ do not intersect unless $A = A'$. If $A = A'$, then $A$ is the unique point of intersection.*

**Definition 14.** Let $l$ be a line, $A$ a point on $l$ and $B$ a point not on $l$. We now know that $\overrightarrow{AB}$ is almost a subset of the half-plane $[B]_l$. More precisely,

$$\overrightarrow{AB} - \{A\} \subseteq [B]_l, \quad \text{and} \quad \overrightarrow{AB} \subseteq [B]_l \cup \{l\}$$

where $\overrightarrow{AB} - \{A\}$ is the set formed by removing $A$ from the ray. Because of this, we say that $\overrightarrow{AB}$ *is on the side* $[B]_l$ *of* $l$.

**Corollary 3.** *Suppose that $A, B, C$ are non-collinear points, and that $D$ is in the interior of $\angle BAC$. Then every point on the ray $\overrightarrow{AD}$, except $A$ itself, is in the interior of $\angle BAC$.*

**Definition 15** (Interior Ray)**.** Suppose that $A, B, C$ are non-collinear points. If, as in the above corollary, every point of a ray $\overrightarrow{AD}$, except $A$ itself, is in the interior of $\angle BAC$, then we say that $\overrightarrow{AD}$ is an *interior ray*, or that $\overrightarrow{AD}$ is *between* $\overrightarrow{AB}$ and $\overrightarrow{AC}$.[4]

The following shows that the two rays of an angle are separated by the line formed from an interior ray.

**Proposition 34.** *Suppose $A, B, C$ are non-collinear points, and suppose $D$ is in the interior of $\angle CAB$. Then $B \not\sim_l C$ where $l = \overleftrightarrow{AD}$.*

[4] I prefer "interior", but the textbook uses "between".

*Proof.* Let $E$ be such that $C * A * E$ (Axiom B-2). Thus $C$ and $E$ are on opposite sides of $\overleftrightarrow{AB}$. However, by definition of the interior of an angle, $D$ and $C$ are on the same side of $\overleftrightarrow{AB}$. So $D$ and $E$ are on opposite sides of $\overleftrightarrow{AB}$. By Corollary 2, $\overrightarrow{AD}$ and $\overrightarrow{BE}$ cannot intersect. In particular, the segment $BE$ cannot intersect the ray $\overrightarrow{AD}$.

Now let $F$ be such that $F * A * D$ (Axiom B-2). Thus $F$ and $D$ are on opposite sides of $\overleftrightarrow{AC}$. However, $B$ and $D$ are on the same sides of $\overleftrightarrow{AC}$ since $D$ is in the interior of $\angle CAB$. So $F$ and $B$ are on opposite sides of $\overleftrightarrow{AC}$. By Corollary 2, $\overrightarrow{AF}$ and $\overrightarrow{EB}$ cannot intersect. In particular, the segment $EB = BE$ cannot intersect the ray $\overrightarrow{AF}$.

Now the line $l = \overleftrightarrow{AD}$ is the union of $\overrightarrow{AD}$ and $\overrightarrow{AF}$. So $BE$ cannot intersect $l$ at all. So $B \sim_l E$. However, $E \not\sim_l C$ since $C * A * E$. Thus $B \not\sim_l C$. □

In the above proposition, it follows that $\overleftrightarrow{AD}$ must intersect the segment $BC$. The following shows that in fact the ray $\overrightarrow{AD}$ intersects $BC$.

**Theorem 2** (Crossbar Theorem)**.** *Suppose $A, B, C$ are non-collinear points, and suppose $D$ is in the interior of $\angle CAB$. Then the ray $\overrightarrow{AD}$ intersects $BC$.*

*Proof.* By the previous proposition, the line $\overleftrightarrow{AD}$ intersects $BC$ at some point $P$ such that $B * P * C$. Now suppose $P$ is not on the ray $\overrightarrow{AD}$. In other words, suppose $P * A * D$.

Since $P * A * D$, it follows that $P$ and $D$ are on opposite sides of $\overleftrightarrow{AB}$. But $D$ is interior to $\angle CAB$, so $D$ and $C$ are on the same side of $\overleftrightarrow{AB}$. Thus $P$ and $C$ are on opposite sides of $\overleftrightarrow{AB}$. However, $B * P * C$ implies that $P$ and $C$ are on the same side of $\overleftrightarrow{AB}$, a contradiction. □

**Definition 16** (Supplementary Angles)**.** Let $B * A * C$, and let $D$ be a point not on $\overleftrightarrow{BC}$. Then $\angle BAD$ and $\angle DAC$ are called *supplementary angles.*

**Proposition 35** (Supplementary Existence)**.** *Every angle $\alpha$ has a supplementary angle $\beta$.*

*Proof.* Write $\alpha = \angle BAD$. By Axiom B-2 there is a point $C$ such that $B * A * C$. So $\alpha = \angle BAD$ and $\beta = \angle DAC$ are supplementary. □

**Proposition 36** (Supplementary Interiors: Part 1)**.** *The interiors of supplementary angles are disjoint.*

*Proof.* Let $B * A * C$, and let $D$ be a point not on $\overleftrightarrow{BC}$. We must show that the interiors of $\angle BAD$ and $\angle DAC$ are disjoint.

Suppose on the contrary that $E$ is in both interiors. Let $l = \overleftrightarrow{AD}$ Then $E \sim_l B$ since $E$ is in the interior of $\angle BAD$. Likewise, $E \sim_l C$ since $E$ is in the interior of $\angle DAC$. So $B \sim_l C$ which contradicts $B * A * C$. □

**Proposition 37** (Supplementary Interiors: Part 2)**.** *Let $B * A * C$, and let $D$ a point not on $\overleftrightarrow{BC}$. If $E$ is on the same side of $\overleftrightarrow{BC}$ as $D$, then exactly one of the following must hold: (i) $E$ is in the interior of $\angle BAD$, (ii) $E$ is on the ray $\overrightarrow{AD}$, (iii) $E$ is in the interior of the supplementary angle $\angle DAC$.*

*Proof.* If $E \in \overrightarrow{AD}$ then it is obviously not in either interior (why?). So we can consider the case where $E \notin \overrightarrow{AD}$. We must show that (i) or (iii) holds (both cannot hold by the previous proposition).

First observe that $E$ is not on the opposite ray to $\overrightarrow{AD}$ since that ray is on the other side of the line $\overleftrightarrow{BC}$. So $E$ is not on the line $\overleftrightarrow{AD}$. It must be on one side or the other. If $E$ and $B$ are on the same side of $\overleftrightarrow{AD}$, then (since $E$ and $D$ are on the same side of $\overleftrightarrow{BC}$) we must have $E$ is in the interior of $\angle BAD$. If $E$ and $B$ are on opposite sides of $\overleftrightarrow{AD}$ then $E$ and $C$ are on the same side of $\overleftrightarrow{AD}$ (since $B$ and $C$ are on opposite sides of $\overleftrightarrow{AD}$). This, together with the assumption that $E$ and $D$ are on the same side of $\overleftrightarrow{BC}$, implies that $E$ is in the interior of the supplementary angle $\angle DAC$. □

**Proposition 38** (Interior Inclusion)**.** *Suppose $D$ is interior to $\angle BAC$. Then the interior to $\angle BAD$ is a subset of the interior to $\angle BAC$.*

*Proof.* Let $E$ be interior to $\angle BAD$. We must show that $E$ is interior to $\angle BAC$.

By the crossbar theorem, the ray $\overrightarrow{AD}$ intersects $\overleftrightarrow{BC}$ at a point $D'$ with $C * D' * B$. Now $E$ is interior to $\angle BAD'$, so by the crossbar theorem again the ray $\overrightarrow{AE}$ intersects $\overleftrightarrow{BD'}$ at a point $E'$ with $D' * E' * B$. Thus $C$-$D'$-$E'$-$B$. In particular, $C * E' * B$. By the pre-crossbar theorem, $E'$ is interior to $\angle BAC$. This implies that $E$ must also be interior to $\angle BAC$ (why?). □

The following will be important in the proof that any two right angles are congruent (in IBC geometry).

**Proposition 39** (Overlapping Angles)**.** *Suppose that $A, B, C, D, E$ are distinct points such that $B * A * C$, and such that $D$ and $E$ are on the same side of $\overleftrightarrow{BC}$. If $D$ is in the interior of $\angle EAC$, then $E$ is in the interior of $\angle BAD$.*

*Proof.* By Proposition 37, either (i) $E$ is interior to $\angle BAD$, (ii) $E$ is on $\overrightarrow{AD}$ or (iii) $E$ is interior to $\angle DAC$. We just need to show (ii) and (iii) are impossible.

Suppose (ii). Then $A, D, E$ are collinear. So $D$ cannot be in the interior of $\angle EAC$, a contradiction.

Suppose (iii). By the previous proposition, the interior of $\angle DAC$ is contained in the interior of $\angle EAC$. So $E$ is in the interior of $\angle EAC$, a contradiction (why?). □

**Definition 17** (Triangle)**.** Let $A, B, C$ be non-collinear points. The *triangle* $\triangle ABC$ is defined to be $AB \cup BC \cup CA$. The points $A, B, C$ are called the *vertices* of $\triangle ABC$. The segments $AB, BC, CA$ are called the *sides* of $\triangle ABC$. The angles $\angle ABC$, $\angle BCA$, and $\angle CAB$ are called the angles of $\triangle ABC$.

The *interior* of $\triangle ABC$ is defined to be the intersection of the interiors of the three angles. The *exterior* of $\triangle ABC$ is the set of points both not on the triangle and not in the interior.

**Proposition 40.** *Let $A, B, C$ be non-collinear points. The interior of $\triangle ABC$ is the intersection of three half planes. In fact, it is just*

$$[A]_{\overleftrightarrow{BC}} \cap [B]_{\overleftrightarrow{CA}} \cap [C]_{\overleftrightarrow{AB}}.$$

## 5. Convexity

**Definition 18.** A set $S$ of points is called *convex* if the following condition holds: if $A$ and $B$ are distinct points in $S$ then $AB \subseteq S$. If $S$ has no points, or just one point, then $S$ is automatically convex since the condition is vacuously true.

We expect segments, rays, lines, half-planes, interiors of angles, and interiors of triangles to be convex. We expect exteriors of angles, angles themselves, and triangles themselves not to be convex.

**Proposition 41.** *The intersection of convex sets is convex.*

THIS SECTION IS INCOMPLETE. As the course progresses, I will add more information to this section whenever we need facts about convexity.

## 6. Questions for further research

**Question 1.** Are the endpoints of a line segment determined by the segment itself? In other words, suppose $AB = CD$. Does it follow that $A = C$ or $A = D$, and that $B = C$ or $B = D$. Another way to phrase this is as follows: does $AB = CD$ imply $\{A, B\} = \{C, D\}$ ?

**Question 2.** Are the vertex of a ray determined by the ray itself? In other words, suppose as sets $\overrightarrow{AB} = \overrightarrow{CD}$. Does it follow that $A = C$ ?

**Question 3.** Are the rays of an angle determined by the angle itself?

**Question 4.** Is the vertex of an angle determined by the angle itself? In other words, suppose as sets $\angle ABC = \angle DEF$. Does it follow that $A = E$ ?

**Question 5.** Is the vertices of an triangle determined by the triangle itself? In other words, suppose as sets $\triangle ABC = \triangle DEF$. Does it follow that $\{A, B, C\} = \{D, E, F\}$?

**Question 6.** Is the interior of an angle determined by the angle itself? In other words, suppose as sets $\angle ABC = \angle DEF$. Does it follow that the interior of $\angle ABC$ is the same set as the interior of $\angle DEF$? A similar question can be ask about the interior of triangles.

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