INCIDENCE-BETWEENNESS GEOMETRY

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This document covers the geometry that can be developed with just the axioms related to incidence and betweenness. The full development of geometry also requires axioms related to congruence, continuity, and parallelism: these will be covered in later documents. The geometry in this document is called *incidence-betweenness geometry*. This and future documents are based on Hilbert's system (1899). I have used Greenberg as an important reference, but there are several differences between my approach and his.

This document has no diagrams. This is to emphasize that the proofs do not require any geometric intuition. However, while you are reading this, you should *make your own diagram* whenever you feel lost or confused.

1. Primitive terms, axioms, and other basics

Incidence-betweeness geometry consists of three primitive terms, eight axioms, and anything that can be defined or proved from these.

Primitive Terms. The three *primitive terms* are *point*, *line*, and *betweenness*. These terms are also called *undefined terms* since we do not give a formal, mathematical definition for them. One can only give informal, non-mathematical explanations for them.

Officially, all that we know about the primitive terms are what is expressed in the axioms, and what is proved about them in later propositions. In some sense, the meaning of these terms is not determined by definitions but instead on what is expressed in the axioms.

Remark. On the other hand, when we study models we will interpret the primitive terms, sometimes in unconventional ways. In this case, the primitive terms are in a sense defined, but only for the particular model. The fact that the primitive terms have no official definition opens the door to a variety of models.

Euclid's approach has *postulates* of geometry and other assumptions called *common notions*. In our current development postulates will be replaced by *axioms*, and the common notions will be replaced by our collection of background tools of logic. These tools include (i) propositional and predicate logic, (ii) easy set theory including basics of functions and relations, (iii) properties of equality, and (iv) from time-to-time some number systems. When we say *equality* we mean sameness, not congruence or "same size". For example, when the objects are sets (such as line segments, circles, rays), equality will be set equality. Properties of congruence and size will not be part of the background tools (although Euclid included these in his common notions), instead they will be developed in these notes.

We begin with a really basic axiom that tells you something about the primitive terms.

Axiom (Restriction on Primitive Terms)**.** *The basic type of object is the point. Lines are sets of points. Betweenness is a three place relation of points. If P, Q, R are points, then $P * Q * R$ denotes the statement that the betweenness relation holds for P, Q, R.*

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Aside from primitive terms, geometric terms must be defined. They can be defined using the primitive terms and any previously defined terms. Of course our background tools, such as sets, can be used to help define new terms.

Definition 1. If $P * Q * R$ holds, then we will say that Q *is between* P *and* R. Let P be a point, and l a line. If $P \in l$ then we say, P *is on* l, l *passes through* P, P *is incident with* l, P *is contained in* l, or P *lies on* l.

Definition 2. The set of all points is called *the plane*. We use this definition because our axioms are designed for planar geometry.[1]

There are three axioms concerning incidence of points with lines. In other words, they concern points being on or not on lines. The first axiom is inspired by Euclid's original first postulate, but is explicit about uniqueness:

Axiom (I-1)**.** *Suppose P and Q are distinct points. Then there is a unique line l such that $P \in l$ and $Q \in l$. In other words, there is a unique line passing through both P and Q.*

The next two axioms are properties of points and lines that were taken for granted by Euclid.

Axiom (I-2)**.** *If l is a line then there are at least two points on l. In other words, there are distinct points P and Q such that both P and Q are contained in l.*

In order to make our geometry into a planar geometry, we need to know that there is no line l that passes through every point in our geometry. Since this cannot be proved from the previous axioms, we need a new axiom:

Axiom (I-3)**.** *There are distinct points P, Q, R with the following property: there is no line passing through all three of the points. In other words, if l is a line, at least one of P, Q, R is not on l.*

Definition 3. The type of geometry involving only the three incidence axioms is called *incidence geometry*. It has only two primitive terms: *point* and *line*. In addition to the three incidence axioms, we also assume part of the first axiom: that lines are sets of points. A *model* for incidence geometry is an interpretation of the word *point*, and a selection of sets of points called *lines* such that Axioms I-1, I-2, I-3 all hold. If some of the axioms fail, an interpretation is called a *partial model*. Using such partial models, we can show that the above axioms are independent.[2]

There are only a few concepts that can be developed in incidence geometry alone. We give these before listing the betweenness axioms.

Definition 4. Let P and Q be distinct points, then $\overleftrightarrow{PQ}$ is defined to be the line that passes through P and Q. This definition requires the existence and uniqueness assertion from Axiom I-1 for its validity.

If $P = Q$ then $\overleftrightarrow{PQ}$ is defined to be $\{P\}$ and is sometimes called a *degenerate line*. Warning: this is *not a line* since Axiom I-2 requires that lines have at least two points.

[1]From time-to-time we will introduce models that are not planar, but the main intent of the axioms is to describe planar geometry. Axiom I-3 indicates that we want at least two dimensions, and Axiom B-4 indicates that we want only two dimensions.

[2]This uses a basic principle of models: everything you can prove from a set of assumptions (or axioms) is true of all models satisfying those assumptions. See class notes for more information.

This definition, together with Axiom I-1, immediately gives the following. Note part (iii) follows from (i) and (ii).

Proposition 1. *Let P and Q be points. Then (i) $P, Q \in \overleftrightarrow{PQ}$, (ii) if $P \neq Q$ and if l passes through P and Q then then $l = \overleftrightarrow{PQ}$, (iii) $\overleftrightarrow{PQ} = \overleftrightarrow{QP}$*

Axiom I-2 can be reinterpreted as follows:

Proposition 2. *Every line l is equal to $\overleftrightarrow{PQ}$ for some P and Q with $P \neq Q$.*

The following definition helps us restate axiom I-3.

Definition 5 (Collinear)**.** Let $P_1, P_2, \ldots, P_k$ be points. If there is a line l such that $P_i \in l$ for all i, then we say that $P_1, P_2, \ldots, P_k$ are *collinear*.

Remark. In mathematical writing, when the term 'if' is used in a definition it really means 'if and only if'. So the above definition also implies that if $P_1, P_2, \ldots, P_k$ are collinear, then there is a line l such that each $P_i \in l$ for all i. This convention only applies to definitions, and not to axioms and theorems.

Axiom I-3 can be rewritten as follows:

Proposition 3. *There exist three distinct non-collinear points.*

If you switch points and lines in the definition of collinear you get the definition of *concurrence.*

Definition 6 (Concurrent)**.** Let $l_1, l_2, \ldots, l_k$ be lines. If there is a point Q such that $Q \in l_i$ for all i, then we say that $l_1, l_2, \ldots, l_k$ are *concurrent*.

One of the most important definitions in this course is the following:

Definition 7 (Parallel)**.** Suppose that l and m are lines. If l and m do not intersect, we say that l and m are *parallel*. In this case we write $l \parallel m$.

Proposition 4. *Distinct lines intersect in at most one point. So distinct non-parallel lines intersect in a unique point.*

Proof. Let l and m be distinct lines. Suppose that l and m intersect in more than one point. Let P and Q be distinct points on the intersection. Then $l = \overleftrightarrow{PQ}$ by Proposition 1 (part *ii*). Likewise, $m = \overleftrightarrow{PQ}$. Thus $l = m$ (by basic properties of $=$), contradicting the assumption that they are distinct. $\square$

Exercise 1. Suppose we replace Axiom I-3 with the following axiom:

Axiom I-3a: *there exists a line l and a point $P \notin l$.*

Show that from Axiom I-1, I-2, I-3a, we can derive Axiom I-3. Conversely, show that from Axiom I-1, I-2, I-3, we can derive Axiom I-3a. Thus, replacing Axiom I-3 with Axiom I-3a results in an equivalent axiom system.

The betweenness axioms concern properties of betweenness that Euclid took for granted.

Axiom (B-1)**.** *Suppose P, Q, R are points such that $P * Q * R$. Then (i) $R * Q * P$, (ii) the points P, Q, and R are on a common line, and (iii) P, Q, and R are distinct.*

Axiom (B-2)**.** *Suppose B and D are distinct point. Then there are points A, C, E such that $A * B * D$ and $B * C * D$ and $B * D * E$.*

Axiom (B-3)**.** *Suppose P, Q, and R are distinct points on a line l. Then exactly one of the following occurs: (i) $Q * P * R$, (ii) $P * Q * R$, or (iii) $P * R * Q$.*

In order to express the fourth betweenness axiom, we need some definitions.

Definition 8 (Line Segment)**.** Suppose P, Q are distinct points. Then the line segment $\overline{PQ}$ is defined as follows:

$$\overline{PQ} = \{\mathrm{X} \mid P * X * Q\} \cup \{P, Q\}.$$

In other words, $\overline{PQ}$ consists of P, Q, and all the points between P and Q.

Definition 9 (Degenerate Segment)**.** Suppose P is a point, then $\overline{PP}$ is defined to be $\{P\}$. We sometimes call $\overline{PP}$ a *degenerate segment*. It is *not* considered to be a line segment.

Definition 10. Let l be a line, and let P and Q be points (distinct or equal) not on l. if $\overline{PQ}$ and l do not intersect, then we write $P \sim_l Q$. if $\overline{PQ}$ and l do intersect, we write $P \not\sim_l Q$.

Axiom (B-4)**.** *If l is a line, and if P, Q, R are points not on l, then the following hold:*

(i) If $P \sim_l Q$ and $Q \sim_l R$, then $P \sim_l R$. In other words, the relation $\sim_l$ is transitive for the points not on l.

(ii) If $P \not\sim_l Q$ and $Q \not\sim_l R$, then $P \sim_l R$.

2. Segment and ray inclusions

Proposition 5. *Let P and Q be points. Then $\overline{PQ} \subseteq \overleftrightarrow{PQ}$.*

Exercise 2. Use Axiom B-1 to prove the above. Your proof should divide into cases.

Proposition 6. *Let P and Q be points. Then $\overline{PQ} = \overline{QP}$.*

Exercise 3. Use Axiom B-1 to prove the above.

Definition 11 (Ray)**.** Suppose P, Q are distinct points. Then the ray $\overrightarrow{PQ}$ is defined as follows:

$$\overrightarrow{PQ} = \overline{PQ} \cup \{\mathrm{X} \mid P * Q * X\}.$$

The next proposition follows from the above definition (and properties of union $\cup$).

Proposition 7. *Let P and Q be distinct points. Then $\overline{PQ} \subseteq \overrightarrow{PQ}$.*

Proposition 8. *Let P and Q be distinct points. Then $\overrightarrow{PQ} \subseteq \overleftrightarrow{PQ}$.*

Exercise 4. Use Axiom B-1 to prove the above proposition.

3. Some Exercises

The following two exercises show that our geometry is not completely trivial.

Exercise 5. Show that the plane is not empty. In other words, points exist and our geometry is not trivial. You should use one, and only one axiom.

Exercise 6. Show that lines exists. You should use two, and only two axioms.

The next two exercises are relatively straightforward (based on Axiom I-1 or Proposition 1).

Exercise 7. Suppose P, Q and R are points where $P \neq Q$. Then P, Q, R are collinear if and only if R is on $\overleftrightarrow{PQ}$.

Exercise 8. Suppose P, Q and R are distinct points. Then P, Q, R are collinear if and only if $\overleftrightarrow{PQ} = \overleftrightarrow{PR} = \overleftrightarrow{QR}$.

The following are consequences of the axioms I-1 to I-3. (So they are theorems of incidence geometry).

Exercise 9. There exists three lines that are not concurrent.

Exercise 10. If l is an arbitrary line, then there is a point not on l.

Hint: you need to be a little careful: prove it for an arbitrary line, not necessarily a line formed from the three points of Axiom I-3. However, Axiom I-3 will give you the point you want.

Exercise 11. If X is a point, then there is a line not passing through it.

Hint: you need to be a careful: prove it for an arbitrary point, so do not assume that X is one of the three points mentioned in Axiom I-3. Instead of Axiom I-3, you might want to use Exercise 9 instead.

Exercise 12. If P is a point, then there are at least two (distinct) lines passing through P.

Hint: use a prior exercise to first find a line l *not* containing P. Does this line have at least two distinct points? Can you form lines with such points that go through P? Can you prove the lines are distinct?

Exercise 13. Show that if $C * A * B$ or $A * B * C$ then C is not on the segment $\overline{AB}$.

Exercise 14. Show that if $C * A * B$ then $C \notin \overrightarrow{AB}$.

Exercise 15. Show that if A and B are distinct points, then $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$.

Exercise 16. Show that if A and B are distinct points, then $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$.

4. Sides of a line

In this section, let l be a fixed (but arbitrary) line. Our goal is to divide the points not on l into two sets called *sides* of l or *half-planes*.

Proposition 9. *The relation $\sim_l$ is an equivalence relation among points not on l.*

Proof. (Reflexive property). Suppose that P is a point not on l. Then $\overline{PP}$ is degenerate: it is just $\{P\}$. Since $P \notin l$, the sets $\overline{PP}$ and l do not intersect. Thus $P \sim_l P$. So the reflexive property holds.

(Symmetric property). Now suppose that $P \sim_l Q$. Then $\overline{PQ}$ does not intersect l. By Proposition 6 we have $\overline{PQ} = \overline{QP}$. Thus $\overline{QP}$ does not intersect l. Hence, $Q \sim_l P$. So the symmetric property holds.

(Transitive property). This follows from the first part of Axiom B-4. $\square$

Definition 12. If P is a point not on l, then let $[P]_l$ be the equivalence class containing P under the relation $\sim_l$. This is defined since $\sim_l$ is an equivalence relation among points not on l.

We call the equivalence class $[P]_l$ a *side* of l or a *half plane bounded by* l.

Remark. If the notion of equivalence classes is unclear to you, you should review the idea. It is covered in any good treatment of set theory. The set-theoretic definition, adopted to our case, gives us the following

$$[P]_l = \{Q \notin l \mid P \sim_l Q\}.$$

The general theory of equivalence classes gives the following (i) $Q \in [P]_l$ if and only if $P \sim_l Q$, which, in turn, is true if and only if $[P]_l = [Q]_l$, (ii) given $[P]_l$ and $[Q]_l$, either they are equal or they are disjoint. Finally, (iii) $P \in [P]_l$.

Our goal is to show that there are two sides of l.

Lemma 10. *There are at least two sides of* l.

Proof. By Exercise 10 there is a point P not on l. Consider $[P]_l$. Thus there is at least one side.

Now let Q be a point on l. Such a point exists by Axiom I-2. Let let R be a point such that $P * Q * R$. Such a point exists by Axiom B-2. Since $P \notin l$, we know $\overleftrightarrow{PQ}$ and l are distinct lines. So they intersect in at most one point (Proposition 4). Observe that that point is Q. Since $R \in \overleftrightarrow{PQ}$ and $R \neq Q$ (Axiom B-1 for both facts), it follows that $R \notin l$.

Observe that $Q \in \overline{PR}$ by definition. Thus $P \not\sim_l R$ since $Q \in l$. This means that their equivalence classes must be non-equal (and disjoint). So $[R]_l \neq [P]_l$. We have found a second side. □

So far we have only used the first part of Axiom B-4. To reach our goal, we need the second part as well.

Proposition 11. *Let* l *be a line. Then* l *has exactly two sides. In other words,* l *bounds exactly two half-planes.*

Proof. By Lemma 10 there are two distinct equivalence classes $[P]_l$ and $[Q]_l$. We must show that there is not a third distinct class.

So suppose $[R]_l$ is a third distinct side. In other words, assume $[R]_l \neq [P]_l$ and $[R]_l \neq [Q]_l$. Then, by properties of equivalence classes, $P \not\sim_l R$ and $R \not\sim_l Q$. So by the second part of Axiom B-4, we get $P \sim_l Q$, a contradiction. □

Informally, the next result shows that if a line crosses one side of a triangle, it must cross one of the other sides as well. (We have not defined triangle yet).

Theorem 12 (Pasch's Theorem)**.** *Suppose* A, B, C *are points, and* l *is a line intersecting* $\overline{AB}$. *Then (i)* l *intersects* $\overline{AC}$ *or* $\overline{CB}$, *and (ii) if* l *does not contain any of* A, B, C *then* l *intersects exactly one of* $\overline{AC}$ *and* $\overline{CB}$.

Proof. Suppose (i) fails, so l does not intersect $\overline{AC}$ and does not intersect $\overline{CB}$. This implies that A, B, C are not on l, and that $A \sim_l C$ and $C \sim_l B$. So $A \sim_l B$ by Axiom B-4. This contradicts the hypothesis that $\overline{AB}$ intersects l.

Now suppose that l does not contain any of A, B, C. Suppose (ii) fails. So, based what we just proved, l must intersect both $\overline{AC}$ and $\overline{CB}$. So $A \not\sim_l C$ and $C \not\sim_l B$. By Axiom B-4, this implies that $A \sim_l B$. This contradicts the hypothesis that $\overline{AB}$ intersects l. $\square$

Pasch was a pioneer in the study of the foundations of geometry in the late 1800s. Hilbert built on many of the ideas of Pasch and others. The above theorem gives a property of triangles that Euclid did not feel the need to prove, but which we today do prove.

Exercise 17. Suppose l is a line, and suppose P, Q, R are points not on l. If $P \not\sim_l Q$ and $Q \sim_l R$, then $P \not\sim_l R$.

Exercise 18. Suppose $A \in l$ where l is a line. Suppose $D \notin l$. Show that every point of $\overrightarrow{AD}$, except A is in $[D]_l$. In other words, show that every point on $\overrightarrow{AD} - \{A\}$ is on the same side of l.

5. Betweenness for four points

Betweenness for three points was taken as a primitive, undefined notion, and the betweenness axioms show that this type of betweenness behaves as expected. What about betweenness among four collinear points? It turns out that in may proofs, one is dealing with four points on a line and having a four term betweenness relation is very handy. Since it is not a primitive notion, we must define it in terms of earlier notions.

Definition 13. Let A, B, C, D be points. We define A-B-C-D to mean that $A * B * C$ and $A * B * D$ and $A * C * D$ and $B * C * D$.

Remark. In other words, A-B-C-D means that we can drop any one of the points, and get a valid triple betweenness. For example, if we drop B we get $A * C * D$.

Proposition 13. *If A-B-C-D then A, B, C, D are distinct and collinear.*

Proof. Since $A * B * C$, we know A, B, C are distinct and collinear by Axiom B-1. Let l be the line containing them. Since $B * C * D$, we know that B, C, D are distinct and collinear. By I-1, the line l is the only line containing B and C, thus D is on l as well. So A, B, C, D are all on l, hence are collinear.

We have mentioned that A, B, C are distinct, and that B, C, D are distinct. We still need to check $A \neq D$. This holds by Axiom B-1 and the fact that $A * B * D$. $\square$

Proposition 14. *If A-B-C-D then D-C-B-A.*

Proof. Suppose A-B-C-D. Then, by definition, $A * B * C$ and $A * B * D$ and $A * C * D$ and $B * C * D$. By Axiom B-1, $C * B * A$ and $D * B * A$ and $D * C * A$ and $D * C * B$. By definition, D-C-B-A $\square$

In real world geometry, if you sketch a line where $A * C * D$ and $A * B * C$, you will find that you are forced to make $B * C * D$ and $A * B * D$ hold as well, so A-B-C-D will be true. In other words, if you have $A * C * D$, and you "squeeze" B between the first two terms, then A-B-C-D will hold. Can we prove this intuitively obvious fact with our axioms?

Proposition 15 (Squeezing)**.** *If $A * C * D$ and $A * B * C$, then A-B-C-D.*

Proof. Let E be a point not on $\overleftrightarrow{AC}$. Such a point exists by Exercise 10. Let $l = \overleftrightarrow{CE}$. Since $A * C * D$, the line segment $\overline{AD}$ intersects l at C. Thus $A \not\sim_l D$.

By Proposition 4, the line $\overleftrightarrow{AC}$ intersects the line l only at C. Thus any line segment in $\overleftrightarrow{AC}$, intersects l at C or not at all. For example, since $A * B * C$, the point C is not in the segment $\overline{AB}$ (by Axiom B-3), and the segment $\overline{AB}$ does not intersect l. Thus $A \sim_l B$.

Since $A \sim_l B$ and $A \not\sim_l D$, it follows from Exercise 17 that $B \not\sim_l D$. Thus $\overline{BD}$ intersects l at a point between B and D, and as mentioned above that point must be at C. So $B * C * D$.

A similar argument (using $\overleftrightarrow{BE}$, the assumption $A * B * C$ and the earlier result that $B * C * D$) shows that $A * B * D$. □

The following two propositions can be proved in a similar manner.

Proposition 16 (Overlap)**.** *If $A * B * C$ and $B * C * D$, then A-B-C-D.*

Proposition 17 (Squeezing, version 2)**.** *If $A * B * D$ and $B * C * D$, then A-B-C-D.*

Proposition 18. *Suppose A, B, C, D are distinct and collinear points such that $A * B * C$. Then either (i) D-A-B-C, (ii) A-D-B-C, (iii) A-B-D-C, or (iv) A-B-C-D.*

Proof. Use Axiom B-3 to consider all the possibilities involving A, B, D. If $D * A * B$ then, since $A * B * C$, we get D-A-B-C by Proposition 16, so the result holds. If $A * D * B$ we get A-D-B-C by Proposition 15 so the result holds. So, we can assume from now on that we are in the remaining case where $A * B * D$.

Repeat with A, C, D. In other words, use Axiom B-3 again and consider all the possibilities involving A, C, D. If $D * A * C$ then, since $A * B * C$, we get D-A-B-C by Proposition 17. If $A * C * D$ we get A-B-C-D by Proposition 15. So, we can assume from now on that $A * D * C$.

We have reduced to the case where $A * B * D$ and $A * D * C$. By Proposition 15 this implies that A-B-D-C. So the result holds. □

The following generalizes Axiom B-3 to four term betweenness:

Corollary 1. *If A, B, C, D are distinct and collinear points, then there is a permutation X, Y, Z, W of these four points so that X-Y-Z-W.*

Proof. By Axiom B-3, either $A * B * C$ or $B * C * A$ or $B * A * C$. Now use the above proposition to show that in any case D can be incorporated into the betweenness. □

Four-term betweenness is very useful in proofs that involve four points. The following is an example.

Theorem 19. *Suppose $B * A * C$. Then $\overline{BA} \subseteq \overline{BC}$.*

Proof. Suppose that $D \in \overline{BA}$. We must show that $D \in \overline{BC}$. If $D = A$ then $B * D * C$ by assumption so $D \in \overline{BC}$. If $D = B$ or $D = C$ then $D \in \overline{BC}$ by definition of line segment. So from now on we will assume that D is distinct from A, B, C.

By proposition 18, either (i) D-B-A-C, (ii) B-D-A-C, (iii) B-A-D-C, or (iv) B-A-C-D.

Case (i) implies $D * B * A$ which implies that $D \notin \overline{BA}$ (Axiom B-3). So this case is out.

Case (ii) implies $B * D * C$ which implies that $D \in \overline{BC}$.

Case (iii) implies $B * D * C$ which implies that $D \in \overline{BC}$.

Case (iv) implies $B * A * D$ which implies that $D \notin \overline{BA}$ (Axiom B-3). So this case is out.

So in any allowable case, $D \in \overline{BC}$. □

Exercise 19. Suppose $B * A * C$. Use four-term betweenness to prove that $\overline{BA} \cup \overline{AC} = \overline{BC}$ and that $\overline{BA} \cap \overline{AC} = \{A\}$.

Proposition 20. *If $C * A * B$ then $\overrightarrow{AB} \cap \overrightarrow{AC} = \{A\}$.*

Proof. (sketch) The proof is similar to the proof that $\overline{BA} \cap \overline{AC} = \{A\}$. For both proofs, one need facts such as $X * A * C \Rightarrow X \notin \overrightarrow{AC}$ (which follows from Axiom B-3 and the definition of ray). □

Definition 14 (Opposite rays)**.** If $C * A * B$ then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called *opposite rays*.

The above proposition asserts that opposite rays intersect only at a point. The following gives their union.

Proposition 21. *If $C * A * B$ then $\overrightarrow{AB} \cup \overrightarrow{AC} = \overleftrightarrow{CB}$.*

Proof. (sketch) Similar to Exercise 19. □

Exercise 20. Suppose $A * B * C$. Use four-term betweenness to prove that $\overrightarrow{AB} = \overrightarrow{AC}$.

Exercise 21. Suppose $X \in \overrightarrow{AB}$. Suppose also that $X \neq A$. Show that $\overrightarrow{AB} = \overrightarrow{AX}$ and that $B \in \overrightarrow{AX}$. Hint: use the previous exercise whenever you can.

Exercise 22. Show that every line has an infinite number of points.

Hint: Start with A_1 and A_2 distinct points on the line. How do you get these two points? Next describe how, once you have produced points up to A_n, you can find a new point A_{n+1}. In order to guarantee that A_{n+1} is not equal to any previous points, you will need to produce A_{n+1} carefully. For example, can you find an A_{n+1} so that it is not in $A_1 * A_n * A_{n+1}$? Can you prove (by induction on n) that every A_i with $i < n+1$ is in $\overline{A_1 A_{n+1}}$, but not equal to A_{n+1}. Finally, can you use this result to show that $A_i \neq A_j$ if $i \neq j$?

Remark. The suggestions for solving the previous exercise and the following use what are known as a *recursive sequences.*[3]

Exercise 23. Show that every line segment has an infinite number of points.

Hint: let A_1 and A_2 be the endpoints. The strategy is opposite that described in the hint of the previous exercise.[4] If you choose A_{n+1} in a certain way, it will follow that, if $i < n$, the point A_i is not in $\overline{A_n A_{n+1}}$. Prove this by induction. Finally, use this fact to show that $A_i \neq A_j$ if $i \neq j$.

6. Vertices and endpoints

Definition 15. If $\overrightarrow{AB}$ is a ray, then A is called the *vertex*.

Remark. In order to show that the definition of a vertex is a well defined property of a ray, we must show that it is a property of the ray itself (considered as a set), and not dependent on the points A and B used to specify the ray. In other words, if $\overrightarrow{AB} = \overrightarrow{A'B'}$ then we must show $A = A'$. In contrast, B is not necessarily equal to B'. The following lemma helps to establish the definition of vertex by characterizing the vertex in terms of betweenness.

[3]They also use a form of the axiom of choice. The axiom of choice can be avoided once the congruence axioms are brought into play.

[4]Alternately, if you want to be efficient you can start with this exercise, and then conclude the previous exercise as a corollary.

Lemma 22. *The point A of a ray $\overrightarrow{AB}$ is the unique point on the ray that is not between two other points on the ray.*

Proof. First we show that A is not between two points on the ray $\overrightarrow{AB}$. Suppose $X * A * Y$ where $X, Y \in \overrightarrow{AB}$. Then $Y \neq A$ by Axiom B-1. Thus $\overrightarrow{AB} = \overrightarrow{AY}$ by Exercise 21. Finally $X \notin \overrightarrow{AY}$ by Exercise 14, contradicting the assumption that $X \in \overrightarrow{AB}$.

Now let $C \in \overrightarrow{AB}$ be distinct from A. We will show that C is between two points on the ray. Since $C \neq A$, we have $\overrightarrow{AB} = \overrightarrow{AC}$ (Exercise 21). By Axiom B-2, there is a point D such that $A * C * D$, and $D \in \overrightarrow{AC}$ by the definition of ray. Thus C is between $A, D \in \overrightarrow{AC}$. □

Corollary 2. *If $\overrightarrow{AB}$ and $\overrightarrow{A'B'}$ are equal as sets, then $A = A'$.*

Proof. By the above lemma, A is the unique point of the set $\overrightarrow{AB} = \overrightarrow{A'B'}$ that is not between two other points of the set. Likewise, A' is the the unique point of the set $\overrightarrow{AB} = \overrightarrow{A'B'}$ that is not between two other points of the set. By uniqueness $A = A'$. □

Definition 16. The points A and B are called the *endpoints* of the segment $\overline{AB}$.

Remark. We are in a similar situation in the definition of endpoints as we were in the definition of vertex. In order to show that the definition of endpoints is well defined we must show that the endpoints do not depend on how the segment is specified, but only on the segment itself (considered as as set). In other words, if $\overline{AB} = \overline{CD}$ we must show that $\{A, B\} = \{C, D\}$, that is either $A = C$ and $B = D$, or $A = D$ and $B = C$.

This can be accomplished by showing that A and B are the only points on $\overline{AB}$ not between two points of $\overline{AB}$. The key is the following exercise.

Exercise 24. Let $\overline{AB}$ be a line segment. Show that A cannot be between two points E and F of $\overline{AB}$. Of course, a similar argument shows that B cannot be between two points.

Hint: suppose $E * A * F$ and get a contradiction. The case where $E = B$ or $F = B$ is not so bad (Exercise 13). For the case where A, B, E, F are distinct, use four point betweenness.

7. Angles and their interiors

In our system an angle is *not* defined to be the space between two rays, but is defined to be the union of two rays. However, we can define the space between the rays, called the *interior of the angle*, as the intersection of two half planes.

Definition 17 (Angle)**.** Suppose A, B, C are non-collinear points. Then the *angle* $\angle BAC$ is defined to be $\overrightarrow{AB} \cup \overrightarrow{AC}$.

You can move the points on the rays, and the angle will not change.

Proposition 23. *Suppose A, B, C are non-collinear points. If $B' \in \overrightarrow{AB}$ and $C' \in \overrightarrow{AC}$ with $B' \neq A$ and $C' \neq A$, then A, B', C' are also non-collinear and $\angle BAC = \angle B'AC'$.*

Proof. Suppose l is a line containing A, B', C'. Since A and B' are on $\overrightarrow{AB}$ and $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$, we have that $\overleftrightarrow{AB}$ is a line containing A and B'. By Axiom I-1 we have $l = \overleftrightarrow{AB}$. A similar argument gives, $l = \overleftrightarrow{AC}$. Together, these conclusions contradict the hypothesis that A, B, C are non-collinear. We conclude that A, B', C' are not collinear.

By Exercise 21, $\overrightarrow{AB'} = \overrightarrow{AB}$ and $\overrightarrow{AC'} = \overrightarrow{AC}$. Thus

$$\angle B'AC' = \overrightarrow{AB'} \cup \overrightarrow{AC'} = \overrightarrow{AB} \cup \overrightarrow{AC} = \angle BAC.$$

□

Definition 18. The point A in $\angle BAC$ is called the *vertex* of $\angle BAC$.

Remark. With the tools we have developed, the vertex of an angle can be shown to be the unique point on the angle that is not between two other points on the angle. This fact can be used to argue that the definition of vertex is well defined.

Definition 19. Let $\angle BAC$ be an angle. Then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are called the *sides* of the angle.

Remark. With the tools we have developed, the sides of an angle can be shown to be independent of the points B and C used to specify the angle. This fact can be used to argue that the definition is well defined.

Definition 20 (Interior and Exterior of an Angle)**.** Suppose A, B, C are non-collinear points. Then the *interior of* $\angle BAC$ is defined to be the following intersection of half-planes:

$$[B]_{\overleftrightarrow{AC}} \cap [C]_{\overleftrightarrow{AB}}.$$

In other words, D is in the interior of $\angle BAC$ if and only if (i) B and D are on the same side of $\overleftrightarrow{AC}$ and (ii) C and D are on the same side of $\overleftrightarrow{AB}$.

A point is said to be *exterior* to $\angle BAC$ if it is not on $\angle BAC$ and is not in the interior of $\angle BAC$. The set of exterior points to $\angle BAC$ is called the *exterior* of $\angle BAC$.

Remark. These definitions do not depend on the choice of B and C used to specify the angle $\angle BAC$. The key to showing this is Exercise 18. We then conclude that the definition of interior and exterior are well defined.

The following gives a way to produce points in the interior (and exterior) of an angle.

Proposition 24 (Crossbar-Betweenness)**.** *Suppose A, B, C are non-collinear points, and suppose D is on the line $\overleftrightarrow{BC}$. Then D is in the interior of $\angle BAC$ if and only if $B * D * C$.*

Remark. The line $\overleftrightarrow{BC}$ is sometimes called the *crossbar line*, and the segment $\overline{BC}$ the *crossbar segment*.

Proof. First suppose that D is in the interior of $\angle BAC$. Then, by definition, (i) B and D are on the same side of $\overleftrightarrow{AC}$ and (ii) C and D are on the same side of $\overleftrightarrow{AB}$. In particular, D is not on $\overleftrightarrow{AC}$ or $\overleftrightarrow{AB}$, so $D \neq C$ and $D \neq B$. By Axiom B-3, either (i) $D * B * C$, (ii) $B * D * C$ or (iii) $B * C * D$. We show $B * D * C$ by eliminating the other cases. In case (i) we have that D and C are on opposite sides of $\overleftrightarrow{AB}$, a contradiction. In case (iii) we have that B and D are on opposite sides of $\overleftrightarrow{AC}$, a contradiction. Thus only case (ii) holds: $B * D * C$.

Now suppose $B * D * C$. Then $B \notin \overline{DC}$ by Exercise 13. But B is the unique point of intersection between the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ (Proposition 4), and of course $\overleftrightarrow{BC} = \overleftrightarrow{DC}$ (by Axiom B-1 and the fact that $B * D * C$). So the subset $\overline{DC}$ cannot intersect $\overleftrightarrow{AB}$. Thus D and C are on the same side of $\overleftrightarrow{AB}$. Similarly, D and B are on the same side of $\overleftrightarrow{AC}$. Thus D is in the interior of $\angle BAC$. □

8. Betweenness of rays

We now develop the concept of betweenness of rays. Since this type of betweenness is not one of our primitive terms, we must define it. We will use the concept of the interior of angle in its definition.

Proposition 25. *Suppose that A, B, C are non-collinear points, and that D is in the interior of $\angle BAC$. Then every point on the ray $\overrightarrow{AD}$, except the vertex A, is in the interior of $\angle BAC$.*

Proof. The interior of the angle is defined to be the intersection of two half-planes:

$$[B]_{\overleftrightarrow{AC}} \cap [C]_{\overleftrightarrow{AB}}.$$

Since $D \in [B]_{\overleftrightarrow{AC}}$ we have $[B]_{\overleftrightarrow{AC}} = [D]_{\overleftrightarrow{AC}}$ (basic property of equivalence classes). By Exercise 18, every point of the ray $\overrightarrow{AD}$ except A is in $[B]_{\overleftrightarrow{AC}} = [D]_{\overleftrightarrow{AC}}$. Likewise, every point of the ray $\overrightarrow{AD}$ except A is in $[C]_{\overleftrightarrow{AB}}$. Thus every point of the ray $\overrightarrow{AD}$ except A is in the intersection. □

Definition 21 (Betweenness of rays)**.** Suppose that A, B, C, D are distinct points such that the rays $\overrightarrow{AB}, \overrightarrow{AC}$, and $\overrightarrow{AD}$ are distinct, and such that A, B, C are not collinear (in other words, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays). If D is in the interior of $\angle BAC$ then we say that $\overrightarrow{AD}$ is *between* $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and we write

$$\overrightarrow{AB} \text{ -- } \overrightarrow{AD} \text{ -- } \overrightarrow{AC}.$$

From this definition, it is clear that $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$ is equivalent to $\overrightarrow{AC}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AB}$.

Remark. Proposition 23 shows that the angle $\angle BAC$ depends on the rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$, and not on the particular points B and C used to specify the rays. Proposition 25 implies that if $\overrightarrow{AC}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AB}$ then the whole ray $\overrightarrow{AD}$, except A, is in the interior of $\angle BAC$. This shows that the property of betweenness of rays is a well defined property of the rays themselves, and not how they are written. In particular, the property is independent of the choice of B, C, D on the respective rays.

9. The crossbar theorem and consequences

The main goal of this section is to prove the crossbar theorem. To do so we need the following lemma.

Lemma 26. *If A, B are two distinct points on a line l and C and D are on opposite sides of l, then $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are disjoint.*

Proof. Exercise 18 shows that every point on $\overrightarrow{AC}$ except the vertex A is on the same side of l as C. Likewise, every point on $\overrightarrow{BD}$ except the vertex B is on the same side of l as D. Since opposite sides of l are disjoint (a basic property of equivalence classes), it follows that the only possible intersection between the two rays occurs at the vertices. But $A \neq B$. □

The following is a key step toward proving the crossbar theorem. Although its truth intuitively obvious, its proof is a bit more involved than the proofs we have seen so far.

Proposition 27 (Separation property)**.** *If $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$ then $B \not\sim_l C$ where $l = \overleftrightarrow{AD}$.*

Proof. Let E be such that $C * A * E$ (Axiom B-2). Thus C and E are on opposite sides of $\overleftrightarrow{AB}$. By definition of betweenness of rays, the point D is in the interior of $\angle BAC$. By definition of the interior of an angle, D and C are on the same side of $\overleftrightarrow{AB}$. So D and E are on opposite sides of $\overleftrightarrow{AB}$. By Lemma 26, $\overrightarrow{AD}$ and $\overrightarrow{BE}$ cannot intersect. In particular, the segment $\overline{BE}$ cannot intersect the ray $\overrightarrow{AD}$.

Now let F be such that $F * A * D$ (Axiom B-2). Thus F and D are on opposite sides of $\overleftrightarrow{AC}$. However, B and D are on the same sides of $\overleftrightarrow{AC}$ since D is in the interior of $\angle CAB$. So F and B are on opposite sides of $\overleftrightarrow{AC}$. By Lemma 26, $\overrightarrow{AF}$ and $\overrightarrow{EB}$ cannot intersect. In particular, the segment $\overline{BE}$ cannot intersect the ray $\overrightarrow{AF}$.

Now the line $l = \overleftrightarrow{AD}$ is the union of $\overrightarrow{AD}$ and $\overrightarrow{AF}$ (Proposition 21). So $\overline{BE}$ cannot intersect l at all. So $B \sim_l E$. However, $E \not\sim_l C$ since $C * A * E$. Thus $B \not\sim_l C$. □

The above proposition implies that $\overleftrightarrow{AD}$ must intersect the segment $\overline{BC}$. The following shows that in fact the ray $\overrightarrow{AD}$ intersects $\overline{BC}$.

Theorem 28 (Crossbar Theorem)**.** *Suppose $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$. In other words, suppose A, B, C are non-collinear points, and that D is in the interior of $\angle CAB$. Then the ray $\overrightarrow{AD}$ intersects $\overline{BC}$. The intersection point is between B and C.*

Proof. By the previous proposition, the line $\overleftrightarrow{AD}$ intersects $\overline{BC}$ at some point P such that $B * P * C$. Now suppose P is not on the ray $\overrightarrow{AD}$. In other words, suppose $P * A * D$.

Since $P * A * D$, it follows that P and D are on opposite sides of $\overleftrightarrow{AB}$. But D is interior to $\angle CAB$, so D and C are on the same side of $\overleftrightarrow{AB}$. Thus P and C are on opposite sides of $\overleftrightarrow{AB}$. However, $B * P * C$ implies that P and C are on the same side of $\overleftrightarrow{AB}$, a contradiction. □

The Crossbar-Betweenness Proposition (Proposition 24) gives a converse which we restate below for convenience.

Proposition 29. *Suppose A, B, C are non-collinear. If $B * D * C$ then $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$.*

We now apply the Crossbar Theorem and prove some basic facts about betweenness of rays. We could also develop a whole theory about betweenness of four rays in a manner similar to what we did for points, but this seems like overkill for our needs. However, the following proposition concerning four rays is similar to a result for the betweenness of four points.

Proposition 30 (Squeezing)**.** *Suppose $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$ and $\overrightarrow{AB}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AD}$. Then $\overrightarrow{AB}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AC}$ and $\overrightarrow{AE}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$.*

Proof. By the Crossbar Theorem, the ray $\overrightarrow{AD}$ intersects $\overline{BC}$ in a point D', and $B * D' * C$. Note that $\overrightarrow{AD} = \overrightarrow{AD'}$, so $\overrightarrow{AB}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AD'}$ by hypothesis. By the Crossbar Theorem again, the ray $\overrightarrow{AE}$ intersects the segment $\overline{BD'}$ in a point E', and $B * E' * D'$. By Proposition 15, we have B-E'-D'-C.

By definition of four point betweenness, $B * E' * C$. Thus, $\overrightarrow{AB}$ -- $\overrightarrow{AE'}$ -- $\overrightarrow{AC}$ by Proposition 29. A similar argument gives $\overrightarrow{AE'}$ -- $\overrightarrow{AD'}$ -- $\overrightarrow{AC}$. The desired result follows from the fact that $\overrightarrow{AD} = \overrightarrow{AD'}$ and $\overrightarrow{AE} = \overrightarrow{AE'}$ (Exercise 21). □

Here is an application of Proposition 30.

Proposition 31 (Interior Inclusion)**.** *Suppose D is in the interior to $\angle BAC$. Then the interior of $\angle BAD$ is a subset of the interior of $\angle BAC$.*

Proof. Let E be in the interior of $\angle BAD$. We must show that E is in the interior to $\angle BAC$.

Since E is in the interior of $\angle BAD$, we have $\overrightarrow{AB}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AD}$. Since D is in the interior of $\angle BAC$, we have $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$. By Proposition 30, we have $\overrightarrow{AB}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AC}$. By definition of betweenness for angles, this implies E is in the interior of $\angle BAC$. □

Here is another property of betweenness of rays.

Proposition 32. *Given three distinct rays with vertex at the same point A. At most one ray can be between the other two.*

Proof. Suppose $\overrightarrow{AB}$ -- $\overrightarrow{AC}$ -- $\overrightarrow{AD}$ and $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$. Then $\overrightarrow{AC}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AC}$ by Proposition 30, contradicting the definition of betweenness of rays. □

Exercise 25. (Informal: no formal proofs required.) Draw three rays on a piece of paper, all starting from the same point A, such that none of the three is between the other two. Thus Axiom B-3 does not hold for rays. (It does hold if two of the rays are on the same side of the line defined by the third ray.)

10. Supplementary Angles

Definition 22 (Supplementary Angles)**.** Let $B * A * C$, and let D be a point not on $\overleftrightarrow{BC}$. Then $\angle BAD$ and $\angle DAC$ are called *supplementary angles.*

Proposition 33 (Supplementary Existence)**.** *Every angle α has a supplementary angle β.*

Proof. Write $\alpha = \angle BAD$. By Axiom B-2 there is a point C such that $B * A * C$. So $\alpha = \angle BAD$ and $\beta = \angle DAC$ are supplementary. □

Proposition 34 (Supplementary Interiors: Part 1)**.** *The interiors of supplementary angles are disjoint.*

Proof. Let $B * A * C$, and let D be a point not on $\overleftrightarrow{BC}$. We must show that the interiors of $\angle BAD$ and $\angle DAC$ are disjoint.

Suppose on the contrary that E is in both interiors. Let $l = \overleftrightarrow{AD}$ Then $E \sim_l B$ since E is in the interior of $\angle BAD$. Likewise, $E \sim_l C$ since E is in the interior of $\angle DAC$. So $B \sim_l C$ which contradicts $B * A * C$. □

Proposition 35 (Supplementary Interiors: Part 2)**.** *Let $B * A * C$, and let D a point not on $\overleftrightarrow{BC}$. If E is on the same side of $\overleftrightarrow{BC}$ as D, then exactly one of the following must hold: (i) E is in the interior of $\angle BAD$, (ii) E is on the ray $\overrightarrow{AD}$, (iii) E is in the interior of the supplementary angle $\angle DAC$.*

Proof. If $E \in \overrightarrow{AD}$ then it is not in either interior (why?). So we can consider the case where $E \notin \overrightarrow{AD}$. We must show that (i) or (iii) holds in this case (both cannot hold by the previous proposition).

First observe that E is not on the opposite ray to $\overrightarrow{AD}$ since that ray (minus A) can be shown to be on the other side of the line $\overleftrightarrow{BC}$. So E is not on the line $\overleftrightarrow{AD}$. Thus it must be on one side or the other. If E and B are on the same side of $\overleftrightarrow{AD}$, then (since E and D are on the same side of $\overleftrightarrow{BC}$, and $\overleftrightarrow{AB} = \overleftrightarrow{BC}$) we have that E is in the interior of $\angle BAD$. If E and B are on opposite sides of $\overleftrightarrow{AD}$ then E and C are on the same side of $\overleftrightarrow{AD}$ (since B and C are on opposite sides of $\overleftrightarrow{AD}$). This, together with the assumption that E and D are on the same side of $\overleftrightarrow{BC}$, implies that E is in the interior of the supplementary angle $\angle DAC$. □

Proposition 36. *Suppose $B * A * C$. If $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AE}$ then $\overrightarrow{AD}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AC}$.*

Proof. Since $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AE}$, the point D is in the interior of $\angle BAE$. Thus D and E are on the same side of $\overleftrightarrow{BA}$. By Proposition 35, either (i) E is interior to $\angle BAD$, (ii) E is on $\overrightarrow{AD}$ or (iii) E is interior to $\angle DAC$. We start by showing (i) and (ii) are impossible.

Suppose (i). Then $\overrightarrow{AB}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AD}$. But $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AE}$ by hypothesis. This contradicts Proposition 32.

Suppose (ii). Then $\overrightarrow{AD} = \overrightarrow{AE}$ (Exercise 21). So D cannot be interior to $\angle BAE$, contradicting $\overrightarrow{AB}$ -- $\overrightarrow{AD}$ -- $\overrightarrow{AE}$.

Thus (iii) holds. So $\overrightarrow{AD}$ -- $\overrightarrow{AE}$ -- $\overrightarrow{AC}$ by definition of ray betweenness. □

11. Triangles

Definition 23 (Triangle)**.** Let A, B, C be non-collinear points. The *triangle* $\triangle ABC$ is defined to be $\overline{AB} \cup \overline{BC} \cup \overline{CA}$. The points A, B, C are called the *vertices* of $\triangle ABC$. The segments $\overline{AB}, \overline{BC}, \overline{CA}$ are called the *sides* of $\triangle ABC$. The angles $\angle ABC$, $\angle BCA$, and $\angle CAB$ are called the angles of $\triangle ABC$.

The *interior* of $\triangle ABC$ is defined to be the intersection of the interiors of the three angles. The *exterior* of $\triangle ABC$ is the set of points both not on the triangle and not in the interior.

Remark. The vertices can be shown to be the only points on the triangle not between two other points of the triangle. Thus the set of vertices is well defined: it doesn't depend on how the triangle was specified, but only on the triangle itself (considered as a set).

From the fact that the vertices are well defined, it follows that the sides, angles, interior, and exterior are also well defined.

Proposition 37. *Let A, B, C be non-collinear points. The interior of $\triangle ABC$ is the intersection of three half planes. In fact, it is just*

$$[A]_{\overleftrightarrow{BC}} \cap [B]_{\overleftrightarrow{CA}} \cap [C]_{\overleftrightarrow{AB}}.$$

Proof. This follows from the definition of the interior of angles. □

12. Convexity

Definition 24. A set S of points is called *convex* if the following condition holds: if $A, B \in S$ then $\overline{AB} \subseteq S$. If S has no points then S is automatically convex since the condition is vacuously true.

Our experience with geometry leads us to expect that segments, rays, lines, half-planes, interiors of angles, and interiors of triangles are all convex. We expect exteriors of angles, angles themselves, and triangles themselves not to be convex.

Exercise 26. (Easy) Show that if a set has a single point, then it is convex. Hint: what is $\overline{AA}$?

Exercise 27. (Easy) Show that lines are convex.

Exercise 28. Show that rays are convex. Hint: given two points A and B on a ray $\overrightarrow{CX}$, you must show that $\overline{AB}$ is a subset of $\overrightarrow{CX}$. If $A = B = C$ then this is trivial, so you can assume that either A or B is not the vertex. In other words, you can suppose $A \neq C$, say. This means $\overrightarrow{CX} = \overrightarrow{CA}$ (why?). So you can replace X with A. The case where $B = C$ is easy (why?), and the case where $B = A$ is easy (why?). So you can assume A, B, C are distinct points on $\overrightarrow{CA}$. For all $D \in \overline{AB}$, show that $D \in \overrightarrow{CA}$. In the case that A, B, C, D are distinct, use four point betweenness.

Exercise 29. Show that half-planes are convex. Hint: what is the definition of $A \sim_l B$?

Proposition 38. *The intersection of convex sets is convex.*

Proof. Suppose S and T are convex. We must show that if $A, B \in S \cap T$ then $\overline{AB} \subseteq S \cap T$.

Suppose that $A, B \in S \cap T$. Then $A, B \in S$ by the definition of intersection. So $\overline{AB} \subseteq S$ since S is convex. A similar argument shows $\overline{AB} \subseteq T$. By basic set theory, $\overline{AB} \subseteq S \cap T$. $\square$

Exercise 30. Use the above to show that the interior of angles and triangles are convex.

Exercise 31. Show that line segments are convex. Hint: This is not as trivial as it sounds. It is easy, however, once you recall that $\overline{AB} = \overrightarrow{AB} \cap \overrightarrow{BA}$.

Exercise 32. Show that angles are not convex. Hint: Proposition 24.

Exercise 33. Show that triangles are not convex. Hint: Let $\triangle ABC$ be given. Consider a point D with $B * D * C$. Show that $\overline{AD}$ is not a subset of the triangle.

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