CHAPTER 9: COMPLETENESS AND CONTINUITY

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The goal of this chapter is to give a formal definition of the notion of a *complete* ordered field, and explore issues related to this concept. We will construct a complete ordered field $\mathbb{R}$ in the next chapter.

Recall that in Chapter 7 we showed that $\mathbb{Q}$ does not have a square root of 2. Informally this illustrates the incompleteness of $\mathbb{Q}$. One of the goals of this chapter is to show that any complete ordered field possesses a square root of 2, and in fact a square root of any nonnegative element. To do so we will prove a basic version of the iintermediate value theorem. Since $\mathbb{Q}$ lacks a square root for 2 we conclude that it is, as expected, incomplete in the formal sense.

The intermediate value theorem requires the notion of *continuous function*. This is one of the key concepts of analysis, and we will just scratch the surface of this concept in this course. We want just enough results about continuity to prove the intermediate value theorem, and apply it to functions such as $f(x) = x^2$ in order to obtain square roots.

1. Completeness

We begin with a formal definition of the concept of *complete*. There are several equivalent ways to define this concept. For example, the following uses the existence of suprema, but obviously one could form an alternative definition that uses the existence of infima instead.

Definition 1 (Completeness)**.** Let F be an ordered field. We say that F is *complete* if every nonempty subset $S \subseteq F$ which is bounded from above has a supremum (least upper bound).

The existence of suprema gives us the existence of infima as well:

Theorem 1. *Suppose F is a complete ordered field. Suppose $S \subseteq F$ is a nonempty subset which is bounded from below. Then S has an infimum (greatest lower bound).*

Proof. Consider the set $S' = \{-x \mid x \in S\}$ of additive inverses. Observe that S' is nonempty and bounded from above. Since F is complete, the set S' has a supremum M. Let $m = -M$. Then observe that m is the infimum of S. □

Exercise 1. Complete the above proof by giving detailed justifications for the two observations in the proof.

Theorem 2. *Suppose F is a complete ordered field containing $\mathbb{Q}$ as an ordered subfield. Then F is an Archimedean ordered field.*

Proof. From a results in Chapter 8 it is enough to show that for all $y \in F$ there is an $n \in \mathbb{N}$ such that $n \geq y$.

Suppose instead that this condition fails. Then there is a $y \in F$ such that $n < y$ for all $n \in \mathbb{N}$. This implies that $\mathbb{N}$ is bounded. Since F is complete, this means that $\mathbb{N}$ has a supremum M. Since $M - 1 < M$, the definition of supremum tells us that $M - 1$ is not an upper bound for $\mathbb{N}$, so there must be an integer $n \in \mathbb{N}$ such that $M - 1 < n$. From this we have that $M < n + 1$. Thus M is also not an upper bound of $\mathbb{N}$, a contradiction. □

Note. We noted in Chapter 8 that is is possible to think of $\mathbb{Q}$ as a subfield of any ordered field. If we do this, then the theorem can be stated more succinctly as *every complete ordered field is an Archimedean ordered field.*

2. Continuous functions

Now we explore the concept of continuity in order to set the stage for the intermediate value theorem. Due to the emphasis we place on on sequences in this course, we use the sequential definition of continuity. The next section (optional) gives another very common definition of continuity, and shows that the two definitions are equivalent.

Definition 2 (Continuity)**.** Let S be a subset of an ordered field F. Then a function $f\colon S \to F$ is said to be *continuous* on S if for all converging sequences (a_i) with terms and limit in S, the sequence $(f(a_i))$ also converges, and

$$\lim_{i\to\infty} f(a_i) \;=\; f\left(\lim_{i\to\infty} a_i\right).$$

The most basic examples of continuous functions are the identity and constant functions:

Theorem 3. *Let S be a subset of an ordered field F. Any constant function $x \mapsto c$ with $c \in F$ is a continuous function $S \to F$. The identity function $x \mapsto x$ is a continuous function $F \to F$.*

Exercise 2. Prove the above. Hint: showing this for the identity function is truly trivial. For a constant function, observe that $f(a_i)$ is a constant sequence. What do you know about limits of constant sequences?

Definition 3. Let f and g be functions $S \to F$ where F is a field. The function $f + g$ is defined to be the function $S \to F$ which sends $x \in S$ to $f(x) + g(x)$. The function $f \cdot g$ is defined to be the function $S \to F$ which sends $x \in S$ to $f(x) \cdot g(x)$.

Theorem 4 (Closure under addition)**.** *Let S be a subset of an ordered field F. If f, g are continuous on S then $f+g$ is also continuous on S. In other words, the set of continuous functions is closed under addition.*

Proof. We prove continuity for $f + g$. The proof for $f \cdot g$ is similar. By the definition of continuity (Definition 2), we take an arbitrary converging sequence (a_i) with limit a. We assume that each a_i is in S and that a is in S, and we must show that the sequence $((f + g)(a_i))$ converges with limit $(f + g)(a)$.

By Definition 3, $(f+g)(a_i) = f(a_i)+g(a_i)$, and $(f+g)(a) = f(a)+g(a)$. Since f and g are continuous on S the sequences $(f(a_i))$ and $(g(a_i))$ both converge. By the limit laws of Chapter 8, the sequence $\big(f(a_i)+g(a_i)\big)$ must also converge since it is the sum of convergent sequences. In fact,

$$\begin{aligned}
\lim_{i\to\infty}(f+g)(a_i) &= \lim_{i\to\infty}\big(f(a_i)+g(a_i)\big) && \text{(Sum of functions (Def 3))}\\
&= \lim_{i\to\infty} f(a_i) + \lim_{i\to\infty} g(a_i) && \text{(Limit law for + (Ch. 8))}\\
&= f\left(\lim_{i\to\infty} a_i\right) + g\left(\lim_{i\to\infty} a_i\right) && (f \text{ and } g \text{ continuous})\\
&= f(a)+g(a) && (a \text{ is the limit})\\
&= (f+g)(a) && \text{(Sum of functions (Def 3))}
\end{aligned}$$

□

Exercise 3. Prove the following.

Theorem 5 (Closure under multiplication)**.** *Let S be a subset of an ordered field F. If f, g are continuous on S then the function $f \cdot g$ is also continuous on S. In other words the set of continuous functions is closed under multiplication.*

Definition 4. Let S be a subset of an ordered field F. Then let $\mathcal{C}(S)$ be the set of continuous functions $S \to F$.

By Definition 4, Theorem 4, and Theorem 5, the set $C(S)$ has binary operations $+$ and $\cdot$. Is $C(S)$ a ring? Let us begin with the associative law:

Lemma 6. *The operation $+$ is associative on $\mathcal{C}(S)$.*

Proof. We must show that if $f, g, h \in \mathcal{C}(S)$ then $(f + g) + h = f + (g + h)$. To show functions are equal, it is enough to show that they have equal value for an arbitrary x in the domain. So let $x \in S$ be in the domain. Then

$$\begin{aligned}
\big((f+g)+h\big)(x) &= (f+g)(x)+h(x) && \text{(Sum of functions (Def 3))}\\
&= \big(f(x)+g(x)\big)+h(x) && \text{(Sum of functions (Def 3))}\\
&= f(x)+\big(g(x)+h(x)\big) && \text{(Addition in field } F \text{ is assoc.)}\\
&= f(x)+(g+h)(x) && \text{(Sum of functions (Def 3))}\\
&= (f+(g+h))(x) && \text{(Sum of functions (Def 3))}
\end{aligned}$$

Since $x \in S$ is arbitrary, $(f + g) + h = f + (g + h)$. □

Exercise 4. Show that $+$ for $\mathcal{C}(S)$ is also commutative. Show multiplication for $\mathcal{C}(S)$ is associative and commutative. Show that the distributive law holds for $\mathcal{C}(S)$.

Theorem 7. *Let S be a subset of an ordered field F. Then the set of continuous functions $\mathcal{C}(S)$ is a commutative ring.*

Exercise 5. Complete the proof of the above theorem. What are the 0 and 1 elements in the ring $\mathcal{C}(S)$?

Informal Exercise 6. Assume the existence of $\mathbb{R}$ (informally at this point). Informally, a continuous function on $[0, 1]$ is a function $[0, 1] \to \mathbb{R}$ whose graph is a connected curve.

Show that $\mathcal{C}([0, 1])$ is not an integral domain. Do so by sketching the graph of two continuous functions whose product is zero.

Example. Let S be a subset of an ordered field F. If $f(x) = x$, then we know that $f \in \mathcal{C}(S)$ since it is the identity function (Theorem 3). Thus $f \cdot f \in \mathcal{C}(S)$ by closure under multiplication. However $f \cdot f$ is just the function $x \mapsto x^2$. Thus $g(x) = x^2$ is a continuous function.

By induction, we can similarly show $g(x) = x^k$ is continuous for all $k \in \mathbb{N}$.

Example. The previous example shows that $g(x) = x^k$ is continuous for integers $k \in \mathbb{N}$. We can call such functions *monomial functions*. Since constant functions are continuous, and continuous functions are closed under multiplication, the function $g(x) = cx^k$ is continuous as well.

Observe that any polynomial function is the sum of functions of the form $g(x) = cx^k$. Since continuous functions are closed under sum, we conclude that any polynomial function is continuous.

3. The δ-ε definition of continuity (optional)

Definition 2 is sometimes called the definition of *sequential continuity*. There is another definition of continuity that is often used in analysis called the δ-ε definition of continuity. In this section we will show that both definitions are equivalent. This fact will not be used in this course, but it is an important fact in analysis.

Definition 5 (Second Definition of Continuity)**.** Let S be a subset of an ordered field F. Let $f : S \to F$ be a function and let $a \in S$. We say that f is *continuous at a* in the δ-ε sense if the following holds. For all $\varepsilon > 0$ in F there is a $\delta > 0$ such that for all $x \in S$ if $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

If f is continuous in this sense for all $a \in S$ then we say that $f : S \to F$ is continuous on S (in the δ-ε sense).

Notice that we defined continuity at a point. We have not yet done this for the sequential form of continuity. We do so now:

Definition 6. Let S be a subset of an ordered field F. Let $f : S \to F$ be a function and let $a \in S$. We say that f is *continuous at a* in the sequential sense if the following holds. For all sequences (a_i) converging to a with terms in S, the sequence $\big(f(a_i)\big)$ converges to $f(a)$.

By definition f is continuous in the sequential sense if and only if it is continuous at a in the sequential sense for all $a \in S$.

We will now prove the equivalence of the two notions of continuity using two lemmas, one for each direction of the equivalence.

Lemma 8. *Let S be a subset of an ordered field F. Let $f\colon S \to F$ be a function and let $a \in S$. If f is continuous at a in the δ-ε sense, then it is continuous at a in sequential sense.*

Proof. Assume (a_i) is a sequence converging to a with terms in S. Our goal according to Definition 6 is to show that the sequence $\big(f(a_i)\big)$ converges to $f(a)$. To achieve this goal, we use the definition of limit in Chapter 8. So assume $\varepsilon > 0$ is given. Our goal reduces to the following goal: find an $N \in \mathbb{N}$ such that if $i \geq N$ then $|f(a_i) - f(a)| < \varepsilon$.

By assumption, f is continuous at a in the δ-ε sense, so by Definition 5 there is a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ if $|x - a| < \delta$ and $x \in S$. Now since the sequence (a_i) converges to a, there is an $N \in \mathbb{N}$ such that if $i \geq N$ then $|a_i - a| < \delta$. (We use the definition of limit from Chapter 8, using δ for our epsilon). By the above property of δ for f, $|a_i - a| < \delta$ implies that $|f(a_i) - f(a)| < \varepsilon$.

In summary, for this choice of N, if $i \geq N$, then $|a_i - a| < \delta$. This implies in turn that $|f(a_i) - f(a)| < \varepsilon$. So this choice of N achieves our goal. □

Lemma 9. *Let S be a subset of an Archimedean ordered field F. Assume that $f\colon S \to F$ is a function and that $a \in S$. If f is continuous at a in the sequential sense, then it is continuous at a in the δ-ε sense.*

Proof. We prove the contrapositive. So we suppose that f is not continuous at a in the δ-ε sense. Our goal is to show that f is not continuous at a in the sequential sense.

We negate the definition of continuity at a in the δ-ε sense. This means that there exists an $\epsilon_0 > 0$ such that for all $\delta > 0$ there is an $x \in S$ such that $|x - a| < \delta$ but $|f(x) - f(a)| \geq \varepsilon$.

We use this to define a sequence (a_i). For any positive $k \in \mathbb{N}$, let $\delta = 1/k$. By the above property there is an element $a_k \in S$ such that $|a_k - a| < 1/k$ but $|f(a_k) - f(a)| \geq \varepsilon_0$.

The first claim is that (a_i) has limit a. To see this, let $\varepsilon > 0$ be given. (This ε is independent of the ε_0 above.) Since F is Archimedean, there is an $N \in \mathbb{N}$ such that $1/N < \varepsilon$ (see Chapter 8). If $i \geq N$ then

$$|a_i - a| < \frac{1}{i} \leq \frac{1}{N} < \varepsilon.$$

This (a_i) converges to a as desired.

The second claim is that $\big(f(a_i)\big)$ does not converge to $f(a)$. This follows from the fact that $|f(a_i) - f(a)| \geq \varepsilon_0$. In other words, for this particular epsilon value, there is no N' such that $i \geq N'$ implies $|f(a_i) - f(a)| < \varepsilon_0$.

If we combine the two claims, we see that f cannot be continuous at a in the sense of Definition 6. □

We now combine the above lemmas to give the following:

Theorem 10. *Let S be a subset of an Archimedean ordered field F. Assume that $f\colon S \to F$. Then f is continuous at $a \in S$ according to the sequential definition if and only if it is continuous at $a \in S$ according to the δ-ε definition. Therefore, f is continuous on S according to the sequential definition if and only if it is continuous on S according to the δ-ε definition.*

4. Intermediate value theorem

One of the key foundational results in real analysis is the intermediate value theorem. It can be proved from the completeness property.

First we remind the reader of a result proved in Chapter 8, which we restate here for the convenience of the reader.

Theorem 11. *Suppose S is a nonempty subset of an Archimedean ordered field F, and suppose that S has a supremum M. Then M is the limit of a sequence (a_i) of elements $a_i \in S$ and is the limit of a sequence (b_i) of elements $b_i \notin S$. If $a < M$ is given we can assume that each a_i is in the interval $[a, M]$. Similarly, if $b > M$ is given we can assume that each b_i is in the interval $[M, b]$.*

This gives us the main tool that we need to prove the following. (In the following, C is between A and B. This is intended to include the case where C is A or B itself.)

Theorem 12 (Intermediate Value Theorem)**.** *Let F be a complete ordered field that contains $\mathbb{Q}$ as an ordered subfield. Let $[a, b]$ be a closed interval in F where $a < b$ are elements of F. Suppose $f\colon [a, b] \to F$ is continuous. If $C \in F$ is any value between $A = f(a)$ and $B = f(b)$ then there is an element $c \in [a, b]$ such that $f(c) = C$.*

Proof. Without loss of generality we can assume $A < C < B$ (the case where $B < C < A$ is similar, and the case where C is A or B is trivial). Define the following set

$$S \stackrel{\text{def}}{=} \{u \in [a, b] \mid f(u) \le C\}.$$

Observe that S is nonempty since $a \in S$, and that b is an upper bound for S. Since F is complete, the set S has a supremum, call it c. We claim that c satisfies the conclusion of the theorem.

We begin by showing $f(c) \le C$. By Theorem 11 there is a sequence (a_i) of elements in S that converges to c. Since $a_i \in S$ we have $f(a_i) \le C$. Since f is continuous, the sequence $(f(a_i))$ converges to $f(c)$. Thus

$$f(c) = \lim_{i\to\infty} f(a_i) \le C.$$

Next we show that c is in the interval $[a, b)$. (For the sake of the theorem, we only need $c \in [a, b]$, but we need $c < b$ below in the proof.) Observe that $a \leq c$ since $a \in S$ and c is an upper bound for S. Since b is an upper bound of S, we have $c \leq b$ because c is the least upper bound of S. Finally, we have $b \neq c$ since $f(b) = B > C$ and $f(c) \leq C$ (established above). So $a \leq c < b$.

Finally we show $f(c) \geq C$, which with the above will yield the result. By Theorem 11 there is a sequence (b_i) of elements not in S converging to c, and since $c < b$ we can also assume that each $b_i \in [c, b]$. In particular, each b_i is in the interval $[a, b]$ since $[c, b]$ is a subset of $[a, b]$. Since $b_i \notin S$ we have $f(b_i) \geq C$. (Actually $f(b_i) > C$, but we just need $\geq$). Since f is continuous, the sequence $(f(b_i))$ converges to $f(c)$. Thus

$$f(c) = \lim_{i \to \infty} f(b_i) \geq C.$$

□

As an application of this theorem, we show that square roots exists in complete ordered fields.

Corollary 13. *Suppose $C \in F$ where F is a complete ordered field that contains $\mathbb{Q}$ as an ordered subfield. If $C \geq 0$, then there is an element $c \in F$ such that $c^2 = C$.*

Proof. If $C < 1$, let $b = 1$, but if $C \geq 1$, let $b = C$. Consider the function $f\colon F \to F$ defined by $x \mapsto x^2$. This function is continuous since it is the product of the identity function with itself, and continuous functions are closed under products. So the restriction $f\,|_{[0,b]}$ to $[0, b]$ is also continuous.

Claim: $f(0) \leq C \leq f(b)$. To see $f(0) \leq C$, combine the facts that $C \geq 0$ and $f(0) = 0$. To show $C \leq f(b) = b^2$ we divide into two cases. First consider the case where $C < 1$, and $b = 1$. So $f(b) = 1^2 = 1$. Thus $C \leq f(b)$ as desired. Next assume that $C \geq 1$. Then

$$C = C \cdot 1 \leq C \cdot C = b \cdot b = f(b).$$

The hypotheses of the Intermediate Value Theorem are satisfied. By the Intermediate Value Theorem, there is a $c \in [0, b]$ such that $f(c) = C$. In other words, $c^2 = C$ as desired. □

Exercise 7. Suppose $C \geq 0$ in a complete ordered field F. Show that there is $c \in F$ such that $c^3 = C$. What if $C < 0$?

Corollary 14. *The field $\mathbb{Q}$ is not complete.*

Proof. In Chapter 7 we proved that there is no $r \in \mathbb{Q}$ such that $r^2 = 2$. However, if $\mathbb{Q}$ were complete, then there would be such an r by the previous corollary. □

5. Cauchy sequences

If a sequence converges, then the terms of the sequence get and stay arbitrarily close to each other. This is shown in the following theorem. Such sequences are called *Cauchy sequences*. Cauchy sequences will play a key role in our construction of $\mathbb{R}$. In addition, they play an important role in analysis quite generally. Informally, a Cauchy sequence is a sequence that seems like it "ought to converge". It might not actually converge in incomplete ordered fields, though. For example, not every Cauchy sequence in $\mathbb{Q}$ converges in $\mathbb{Q}$. However, in a complete ordered field, every Cauchy sequences will converge.

Theorem 15. *Suppose (a_i) is a convergent sequence in an ordered field F. Then for all positive ε in F there is an $N \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$*

$$i, j \geq N \quad \Rightarrow \quad |a_i - a_j| < \varepsilon.$$

Proof. Since we assume that (a_i) converges, it has a limit. Let b be the limit of the sequence (a_i).

Let $\varepsilon > 0$ be an arbitrary positive element of F. We must find a $N \in \mathbb{N}$ that satisfies the statement of the theorem.

Let $\varepsilon' = \varepsilon/2$. By the definition of limit there is a $N \in \mathbb{N}$ with $|a_i - b| < \varepsilon'$ for all $i \geq N$. So, for $i, j \geq N$ we have

$$|a_i - a_j| = |(a_i - b) + (b - a_j)| \leq |a_i - b| + |b - a_j| < \varepsilon' + \varepsilon'.$$

Here we have used the triangle inequality. Since $2\varepsilon' = \varepsilon$ we conclude that $|a_i - a_j| < \varepsilon$. Thus N has the desired property. □

The above theorem says that all convergent sequences satisfy the following definition:

Definition 7 (Cauchy sequence)**.** Suppose (a_i) is an infinite sequence in an ordered field F. We say that (a_i) is *Cauchy* if the following occurs: for all positive ε in F there is a $N \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$

$$i, j \geq N \quad \Longrightarrow \quad |a_i - a_j| < \varepsilon.$$

Remark 1. We can reinterpret Theorem 15 through its contrapositive: *if a sequence is not Cauchy, it cannot converge.*

Is the converse true? In other words, do all Cauchy sequences converge? The answer is no for $F = \mathbb{Q}$. The problem with $\mathbb{Q}$ is that it has 'holes'. For example, we saw that there is no $r \in \mathbb{Q}$ with $r^2 = 2$. Define a sequence by the rule $a_i = n_i/10^i$ where n_i is the largest integer such that $a_i^2 < 2$. This sequence will not be convergent in $\mathbb{Q}$, but can be shown to be Cauchy. Even though this Cauchy sequence does not converge in $\mathbb{Q}$, it will turn out that it is convergent in $\mathbb{R}$, and has limit $\sqrt{2}$.

Informal Exercise 8. Find the first five terms of (a_i) defined in the above remark. Assume the index set is the set of $i \geq 0$. In other words, start with $i = 0$. Hint: punch $\sqrt{2}$ into your calculator.

Our approach in the next chapter will be to assume that all Cauchy sequences in $\mathbb{Q}$ should determine a real number. Non-Cauchy sequences cannot possibly converge, so should not determine real numbers. There is a problem: different sequences can determine the same real number. For example, the sequence defined by the rule $b_i = n_i/2^i$ where n_i is the largest integer such that $b_i^2 < 2$ determines the same real number as the sequence (a_i) discussed above (in fact, they both determine $\sqrt{2}$: the sequence (a_i) is related to the decimal expansion of $\sqrt{2}$ and (b_i) is related to the base 2 expansion of $\sqrt{2}$). How do we tell if two sequences determine the same number? We can use the equivalence relation defined in Chapter 8: two Cauchy sequence determine the same real number if and only if $(a_i) \sim (b_i)$. We will make this approach more precise in Chapter 10. When we do, we will need the following.

Theorem 16. *Let F be an Archimedean ordered field. If $(a_i) \sim (b_i)$ and if (a_i) is Cauchy, then (b_i) is Cauchy.*

Exercise 9. Prove the above theorem. The proof is similar to the proof of the claim that if $(a_i) \sim (b_i)$ and if (a_i) converges, then (b_i) converges. You might wish to choose $\varepsilon' = \varepsilon/3$. The key step of the proof is

$$|b_i - b_j| = |(b_i - a_i) + (a_i - a_j) + (a_j - b_j)| < \varepsilon' + \varepsilon' + \varepsilon'.$$

We conclude with a lemma that shows that every Cauchy sequence is bounded.

Lemma 17. *If $(a_i)_{i \geq n_0}$ is a Cauchy sequence in an ordered field F, then there is a bound $B \in F$ such that $|a_i| \leq B$ for all $i \geq n_0$.*

Proof. First we show that $a_i \leq B_1$ for some positive upper bound B_1.

Since (a_i) is Cauchy, there is a $N \in \mathbb{N}$ such that $|a_i - a_j| < 1$ for all $i, j \geq N$ (choose $\varepsilon = 1$). Let A be the maximum of $0, a_{n_0}, \ldots, a_N$, and let $B_1 = A+1$. Since $A \geq 0$ we have B_1 positive. We will show that B_1 is in fact an upper bound for (a_i).

First consider the case where $i \leq N$. In this case

$$a_i \leq A < A + 1.$$

Since $B_1 = A + 1$, we have $a_i \leq B_1$ as desired.

Next consider the case where $i > N$. Since $i, N \geq N$, we have the inequality $|a_i - a_N| < 1$. Thus $-1 < a_i - a_N < 1$. So

$$a_i < a_N + 1 \leq A + 1 = B_1,$$

and we get $a_i \leq B_1$ in this case as well.

The proof of the existence of a negative lower bound is similar. (Subtract one from a mininum). Write the lower bound as $-B_2$ where B_2 is positive. So we get

$$-B_2 \leq a_i \leq B_1$$

for all $i \geq n_0$. Let B be the maximum of B_1 and B_2. Then

$$-B \leq a_i \leq B.$$

So $|a_i| \leq B$ as desired. □

6. Cauchy criterion for completeness

Our goal is to prove the following theorem. We will need this later to show that our construction of $\mathbb{R}$ gives a complete ordered field. The converse is true as well, and will be proved in a later section.

Theorem 18 (Cauchy criterion)**.** *Let F be an Archimedean ordered field. If every Cauchy sequence in F converges in F then F is complete.*

The proof of this uses the notion of an *ε-almost-supremum* We first build a Cauchy sequence out of such "almost-Sups", and the limit can be shown to be the actual supremum.

Recall the definition (from Chapter 8): let S be a nonempty subset of an ordered field F, and let $\varepsilon > 0$ be in F. An *ε-almost-supremum* A of S is an upper bound of S such that there is an $x \in S$ in the interval $(A - \epsilon, A]$. In Chapter 8 we proved the following (which we restate for convenience):

Theorem 19. *Let S be a nonempty subset of an Archimedean ordered field F, and let $\varepsilon > 0$ be in F. If S is bounded from above, then S has an ε-almost-supremum.*

Using this theorem we can prove the Cauchy criterion:

Proof of Theorem 18. Let S be a nonempty subset of F with an upper bound. Our goal is to show that S has a supremum. This will show F is complete as desired.

For each positive integer $n \in \mathbb{N}$, let A_n be an $1/n$-almost-supremum of S. This exists by Theorem 19 (proved in Chapter 8).

Claim: (A_i) is a Cauchy sequence. Let $\varepsilon > 0$ be given. To prove the claim we need to find an N such that if $i, j \geq N$ then $|A_i - A_j| < \varepsilon$. Since F is an Archimedean ordered field there is an N such that $1/N \leq \varepsilon$. Now suppose $i, j \geq N$. Without loss of generality, suppose $A_i \geq A_j$. Since A_i is an $1/i$-almost-supremum, $A_i - 1/i$ is not an upper bound of S. Since A_j is an upper bound of S, we have $A_i - 1/i < A_j$. Hence

$$|A_i - A_j| = A_i - A_j < 1/i \leq 1/N \leq \varepsilon.$$

Thus (A_i) is Cauchy. By the assumption of the theorem, (A_i) has a limit, call it A.

Claim: A is an upper bound of S. To see this we must show $x \leq A$ for all $x \in S$. Suppose otherwise, that $x > A$ for some $x \in S$. Let $\varepsilon = x - A$. Since A is the limit of (A_i), there is an N such that

$$i \geq N \implies |A_i - A| < \varepsilon.$$

In particular, $|A_N - A| < \varepsilon$. Observe that $A < x \leq A_N$ since A_N is an upper bound of S. In particular, $A < A_N$ so

$$A_N - A = |A_N - A| < \varepsilon = x - A.$$

This implies that $A_N < x$, a contradiction since A_N is an upper bound of S.

Claim: for all $\varepsilon > 0$ there is an $x \in S$ such that $|A - x| < \varepsilon$. To show this, fix $\varepsilon > 0$. Since F is an Archimedean ordered field, there is a positive integer N such that $1/N \leq \varepsilon/2$ (Chapter 8). Since A is the limit of (A_i) there is a positive integer N' such that $|A_i - A| < \varepsilon/2$ for all $i \geq N'$. Let n be the maximum of N and N'. Since A_n is an $1/n$-almost supremum, there is an $x \in S$ with $|A_n - x| < 1/n$. So

$$|A - x| \leq |A - A_n| + |A_n - x| < \frac{\varepsilon}{2} + \frac{1}{n} \leq \frac{\varepsilon}{2} + \frac{1}{N} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We know that A is an upper bound, but is it the least upper bound? We show that A is indeed the supremum of S showing that if $A' < A$ then A' is not an upper bound. To do so, let $\varepsilon = A - A'$. By the above claim, there is an $x \in S$ such that $|A - x| < \varepsilon$. (Note $A - x = |A - x|$ since A is an upper bound of S). So

$$A - x = |A - x| < \varepsilon = A - A'.$$

This implies $A' < x$, so A' is not an upper bound. □

7. Bounded monotonic sequences converge

Monotonic sequences are commonly used in mathematics and are often easier to deal with than arbitrary sequences. One of the most useful basic facts about $\mathbb{R}$ is that every bounded monotonic sequences converges. We will show that this follows from the completeness property.

Definition 8 (Monotonic)**.** Let $(a_i)_{i \geq n_0}$ be a sequence in an ordered field F. The sequence is said to be *increasing* if $a_{i+1} \geq a_i$ for all $i \geq n_0$. The sequence is *decreasing* if $a_{i+1} \leq a_i$ for all $i \geq n_0$. In either case (a_i) is said to be *monotonic*. (Observe that a constant sequence is considered both increasing and decreasing).

The sequence $(a_i)_{i \geq n_0}$ is said to be *strictly increasing* if $a_{i+1} > a_i$ for all $i \geq n_0$. The sequence $(a_i)_{i \geq n_0}$ is *strictly decreasing* if $a_{i+1} < a_i$ for all $i \geq n_0$. In either case (a_i) is said to be *strictly monotonic*. (Observe that a constant sequence is monotonic, but not strictly monotonic)

The following is a simple consequences of the definition. It is stated for increasing sequences, but the statement holds, with the obvious modifications, for decreasing sequence. There are obvious versions for strictly monotonic sequences as well.

Lemma 20. *Suppose that $(a_k)_{k \geq n_0}$ is a monotonically increasing sequence in an ordered field F. If $j \geq i \geq n_0$ then $a_j \geq a_i$.*

Proof. Fix $i \geq n_0$, and consider the set $S_i = \{u \in \mathbb{Z} \mid u \geq n_0 \text{ and } a_u \geq a_i\}$ we will show by induction that all $j \geq i$ are in S_i.

For the base case, observe that $a_i \geq a_i$ (reflexive). Thus $i \in S_i$.

Now suppose $k \in S_i$. This implies $a_{k+1} \geq a_k \geq a_i$ (the first inequality by Definition 8, the second since $k \in S_i$). So $k + 1 \in S_i$.

By induction, S_i contains all $j \geq i$. In particular $a_j \geq a_i$ if $j \geq i$. □

An increasing sequence is automatically bounded from below: if a_{n_0} is the first term of such a sequence then the above lemma shows a_{n_0} is a lower bound. So to say that such a sequence is *bounded* really means that it is also bounded from above. Obviously the same idea, but reversed, applies to decreasing sequences.

Theorem 21 (Convergence of bounded monotonic sequences)**.** *Let (a_i) be a bounded monotonic sequence in a complete field F. Then (a_i) converges.*

Proof. We assume that $(a_i)_{i \geq n_0}$ is an increasing sequence. The decreasing case is similar. Since F is complete, and the set $\{a_i \mid i \geq n_0\}$ is bounded, this set has a supremum B. We will show that B is in fact the limit.

Let $\varepsilon > 0$ be given. By a result in Chapter 8, since B is the supremum, the interval $(B - \varepsilon, B]$ must contain an element of $\{a_i \mid i \geq n_0\}$. In other words, there is an $N \in \mathbb{N}$ such that $B - \varepsilon < a_N \leq B$. We will show that N has the desired property (as in the definition of limit). So suppose $i \geq N$. Then $a_N \leq a_i$ since the sequence is increasing. But B is an an upper bound for the sequence. So $a_i \leq B$. Thus

$$B - \varepsilon < a_N \leq a_i \leq B.$$

Observe $|B - a_i| = (B - a_i)$ since $a_i \leq B$. Observe also that $B - a_i < \varepsilon$ since $B - \varepsilon < a_i$. Therefore, $|B - a_i| < \varepsilon$ as desired. □

Exercise 10 (Optional)**.** Use ε-almost suprema (and infima) from Chapter 8 to show that if F is an archimedean ordered field, then every bounded monotonic sequence is Cauchy. (Even if F is not complete). Hint: it is enough to do the increasing case. The proof is similar to that of the previous theorem. For each $n \in \mathbb{N}$, let B_n be an $1/n$-almost supremum. Now given an arbitrary $\varepsilon > 0$, choose $1/n < \varepsilon$. Show that you can choose an $N \in \mathbb{N}$ such that $a_N \in (B_n - 1/n, B_n]$. Show that $i, j \geq N$ implies $|a_i - a_j| < 1/n < \varepsilon$.

8. Accumulation points (optional)

Consider the sequence defined by the equation $a_k = (-1)^k$. It is obviously not a Cauchy sequences, so it cannot converge. (Recall that all convergent sequences must be Cauchy). However, it does have an infinite number of terms equal to 1 and an infinite number of terms equal to -1. So 1 an -1 are in some sense limits in a more general sense. A similar phenomenon occurs for the sequence defined by $b_k = 1/k + (-1)^k$; this sequence seems to have values that "accumulate" near both 1 and -1. This leads to an important concept which we now define:

Definition 9 (Accumulation point)**.** Let (a_i) be a sequence in an ordered field F. We say that the element $b \in F$ is an *accumulation point* of (a_i) if the following occurs: for all $\varepsilon > 0$ and $n \in \mathbb{N}$ there is an integer $i \geq n$ such that $|a_i - b| < \varepsilon$.

Remark 2. We can rephrase the above condition to an equivalent condition: *for all* $\varepsilon > 0$ *there are an infinite number of* $i \in \mathbb{N}$ *such that* $|a_i - b| < \varepsilon$. Note that there are two ways for a value b to be an accumulation point of the sequence: either the value b itself occurs an infinite number of times in the sequence, or there are terms of the sequence that get *arbitrarily close* to b.

Exercise 11. Let F be an Archimedean ordered field. Show that 1 is an accumulation point of the sequence defined by $a_k = (-1)^k$ and of the sequence defined by $b_k = 1/k + (-1)^k$.

Exercise 12. If a sequence has a limit, show that its limit is the unique accumulation point.

Exercise 13. From the previous exercise we see that a convergent sequence has exactly one accumulation point. Is the converse true? In other words, if a sequence has exactly one accumulation point, does it follow that the sequence converges?

The following will be useful later in showing the converse of Theorem 18.

Theorem 22. *Let* (a_i) *be a Cauchy sequence in an ordered field* F*. If* a *is an accumulation point of* (a_i) *then* a *is the limit of* (a_i)*.*

Proof. By the definition of limit, for any given $\varepsilon > 0$ we must show that there is an $N \in \mathbb{N}$ such that $|a_i - a| < \varepsilon$ for all $i \geq N$.

So suppose $\varepsilon > 0$ is given. Let $\varepsilon' = \varepsilon/2$. Since (a_i) is a Cauchy sequence, there is an N such that $|a_i - a_j| < \varepsilon'$ for all $i, j \geq N$. By the definition of accumulation point, there is an $j_0 \geq N$ such that $|a_{j_0} - a| < \varepsilon'$. Now assume that $i \geq N$. Then

$$|a_i - a| \leq |a_i - a_{j_0}| + |a_{j_0} - a| < \varepsilon' + \varepsilon' = \varepsilon.$$

□

Exercise 14. Show that if $(a_i) \sim (b_i)$ are equivalent sequences, then they have the same accumulation points. Is the converse true?

9. Lim infs and lim sups (optional)

We will see that in any complete ordered field, bounded sequence (bounded in both direction) must have accumulation points. In this case there is a greatest accumulation point call the *superior limit* and a least accumulation point called the *inferior limit*.

The superior limit, often called $\limsup$, is formed by seeing how the bounds to the sequence change as a larger and larger number of terms are removed from the *beginning* of the sequence. These bounds will tend to go down, or stay the same, as more terms are removed. What happens in the long term, as measured by the infimum of these bounds, is the *superior limit*. The *inferior limit*, often called lim inf, is formed in a similar way except with lower bounds. The following definition makes this idea precise.

Definition 10 (Lim sup and lim inf)**.** Suppose $(a_i)_{i\geq n_0}$ is a bounded sequence in a complete ordered field F. In other words, suppose there is a bound B such that $|a_i| \leq B$ for all $i \geq n_0$. For each $k \geq n_0$ consider the following set

$$S_k = \{a_i \mid i \geq k\}.$$

Observe that S_k is a nonempty set with upper bound B and lower bound $-B$. Let M_k be the supremum of S_k and let m_k be the infimum of S_k. These exist since F is complete. Observe that $-B \leq m_k \leq M_k \leq B$, so the sets $\{m_k \mid i \geq k\}$ and $\{M_k \mid i \geq k\}$ are themselves bounded by $-B$ and B. The superior limit is defined as follows:

$$\limsup_{i\to\infty} a_i \stackrel{\text{def}}{=} \inf\{M_k \mid k \geq n_0\}.$$

The inferior limit is defined as follows:

$$\liminf_{i\to\infty} \stackrel{\text{def}}{=} \sup\{m_k \mid k \geq n_0\}.$$

These both exist since F is complete.

Remark 3. Often lim sups and lim infs are defined even for unbounded sequences. For example, if a sequence (a_i) has no upper bound then the lim sup of (a_i) is sometimes said to be ∞. Similarly, if (a_i) has no lower bound then the lim inf of (a_i) would be $-\infty$. In this course, however, we will stick to bounded sequences where lim sup and lim inf are elements of F (assuming F is complete).

Lemma 23. *Let (a_i) be a bounded sequence in a complete ordered field F. Suppose $M \in F$ is such that*

$$M > \limsup_{i\to\infty} a_i.$$

Then all but a finite number of terms of (a_i) are strictly smaller than M. In other words, there is an $N \in \mathbb{N}$ such that if $i \geq N$ then $a_i < M$.

Proof. Let $X = \limsup_{i\to\infty} a_i$. By definition X is the greatest lower bound of the M_k in the above definition. Since $M > X$, observe that M is not a lower bound of $\{M_k\}$. Thus there is an $N \in \mathbb{N}$ such that $M_N < M$. Since M_N is the least upper bound of $S_N = \{a_i \mid i \geq N\}$, we have $a_i \leq M_N$ for all $i \geq N$. Since $M_N < M$ we have $a_i < M$ for all $i \geq N$. □

We can give an alternate characterization of lim sups that is sometimes easier to work with than Definition 10.

Theorem 24. *Let* (a_i) *be a bounded sequence in a complete ordered field* F*. Then* $\limsup_{i\to\infty} a_i$ *is the minimal element* $X \in F$ *with the following property: For all* $M > X$ *there is an* $N \in \mathbb{N}$ *such that if* $i \geq N$ *then* $a_i < M$*. In other words,* $\limsup_{i\to\infty} a_i$ *is the smallest element of* F *such that any larger element is a strict upper bound for all but a finite number of terms of* (a_i)*.*

Proof. The above lemma shows that $X = \limsup_{i\to\infty} a_i$ has the desired property. Now we need to show that no smaller element Y has the property. Suppose $Y < X$ has the property. Let Z be chosen so that $Y < Z < X$. By the property assumed for Y there is an $N \in \mathbb{N}$ such that if $i \geq N$ then $a_i < Z$. This implies that Z is an upper bound of the set $S_N = \{a_i \mid i \geq N\}$. So $M_N \leq Z$ where M_N is the supremum of S_N. Since X is defined to be the infimum of the set of such M_k we have $X \leq M_N$. Now we get the following contradiction:

$$X \leq M_N \leq Z < X.$$

□

For lim infs we have the following:

Theorem 25. *Let* (a_i) *be a bounded sequence in a complete ordered field* F*. Then* $\liminf_{i\to\infty} a_i$ *is the maximal element* $x \in F$ *with the following property: For all* $m < x$ *there is an* $N \in \mathbb{N}$ *such that if* $i \geq N$ *then* $a_i > m$*. In other words,* $\liminf_{i\to\infty} a_i$ *is the largest element of* F *such that any smaller element is a strict lower bound for all but a finite number of terms of* (a_i)*.*

Exercise 15. How would you change the proof of Lemma 23 and Theorem 24 in order to prove Theorem 25?

We give a third characterization of lim sups and lim infs in terms of accumulation points. This is perhaps the simplest way of describing them.

Theorem 26. *Suppose* (a_i) *is a bounded sequence in a complete ordered field* F*. Then* $\limsup_{i\to\infty} a_i$ *is an accumulation point of* (a_i)*. In fact, it is the greatest accumulation point of* (a_i)*.*

Proof. Let $X = \limsup_{i\to\infty} a_i$. We will first show that X is an accumulation point. We assume we are given $\varepsilon > 0$ and $n \in \mathbb{N}$. According to the definition of accumulation point, our goal is to show that there is an $i \geq n$ such that $|a_i - X| < \varepsilon$.

We have $X + \varepsilon > X$. So by Theorem 24 there is an $N \in \mathbb{N}$ such that $i \geq N$ implies $a_i < X + \varepsilon$. Let k be the maximum of n and N. As in the definition of lim sup, let $S_k = \{a_i \mid i \geq k\}$ and let M_k be the supremum of S_k.

By definition of lim sup, we have $X \leq M_k$, so $X - \varepsilon < M_k$. Observe that $X - \epsilon$ is not an upper bound of S_k (since M_k is the least upper bound). So there is an $a_i \in S_k$ with $X - \varepsilon < a_i$. Since $i \geq k \geq N$, we have $a_i < X + \varepsilon$. So

$$X - \varepsilon < a_i < X + \varepsilon.$$

This implies $|a_i - X| < \varepsilon$. Observe also that $i \geq k \geq n$. This concludes the proof that $X = \limsup_{i\to\infty} a_i$ is an accumulation point.

Now suppose that $Y > X$. We must show that Y is not an accumulation point. Let $Z \in F$ be such that $X < Z < Y$. By Theorem 24, there is an $N \in \mathbb{N}$ such that if $i \geq N$ then $a_i < Z$. For $i \geq N$ we use $a_i < Z < Y$ to conclude

$$|a_i - Y| = Y - a_i = (Y - Z) + (Z - a_i) \geq (Y - Z).$$

So if $\varepsilon = Y - Z$ then $|a_i - Y| \geq \varepsilon$ for all $i \geq N$. This shows that Y is not an accumulation point of (a_i). □

The proof of the following is similar to the proof of the above theorem, so we omit it.

Theorem 27. *Suppose* (a_i) *is a bounded sequence in a complete ordered field* F*. Then* $\liminf_{i\to\infty} a_i$ *is an accumulation point of* (a_i)*. In fact, it is the least accumulation point of* (a_i)*.*

Corollary 28. *Every bounded sequence in a complete ordered field* F*. has an accumulation point.*

Corollary 29. *Suppose* (a_i) *is a bounded sequence in a complete ordered field* F*. Then*

$$\liminf_{i\to\infty} a_i \leq \limsup_{i\to\infty} a_i.$$

Furthermore (a_i) *converges if and only if equality hold. In this case,*

$$\lim_{i\to\infty} a_i = \liminf_{i\to\infty} a_i = \limsup_{i\to\infty} a_i.$$

Proof. Since $\lim_{i\to\infty} a_i$ is the least accumulation point and $\limsup_{i\to\infty} a_i$ is the greatest accumulation point, we have

$$\liminf_{i\to\infty} a_i \leq \limsup_{i\to\infty} a_i.$$

By Exercise 12, if the sequence (a_i) has a limit, that limit is the unique accumulation point, so

$$\lim_{i\to\infty} a_i = \liminf_{i\to\infty} a_i = \limsup_{i\to\infty} a_i.$$

Finally, suppose equality holds, and let X be its value:

$$\liminf_{i\to\infty} a_i = \limsup_{i\to\infty} a_i = X.$$

We need to show that X is the limit of (a_i). Let $\varepsilon > 0$ be given. We will find an $N \in \mathbb{N}$ such that $|a_i - X| < \varepsilon$ for all $i \geq N$. First use Theorem 24 to obtain an $N_1 \in \mathbb{N}$ such that if $i \geq N_1$ then $a_i < X + \varepsilon$. Use Theorem 25 to get an $N_2 \in \mathbb{N}$ such that if $i \geq N_2$ then $a_i > X - \varepsilon$. Thus if $i \geq N$ where N is the maximum of N_1 and N_2, then

$$X - \varepsilon < a_i < X + \varepsilon.$$

In particular, $|a_i - X| < \varepsilon$ as desired.

□

Remark 4. The above shows that for *bounded* sequences, convergence is equivalent to the existence of exactly one accumulation point.

10. Cauchy sequences converge (optional)

Earlier we showed that if F is an Archimedean ordered field such that every Cauchy sequences converges, then F is complete. Now we show the converse.

Theorem 30. *If F is a complete ordered field, then every Cauchy sequence converges.*

Proof. Let (a_i) be a Cauchy sequence in a complete ordered field F. By Lemma 17 the sequence (a_i) is bounded. By Theorem 26 the $\limsup$ yields an accumulation point for (a_i). So, by Theorem 22, (a_i) converges. □

This gives another characterization of *completeness* for Archimedean ordered fields.

Corollary 31. *Let F be an Archimedean ordered field. Then F is complete if and only if every Cauchy sequence converges.*