CHAPTER 8: SEQUENCES AND LIMITS

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As we saw in Chapter 7, the field $\mathbb{Q}$ is in some sense “incomplete”. There are numbers missing from $\mathbb{Q}$ that are essential for mathematics. Our goal is to construct a number system $\mathbb{R}$ that does not suffer from this problem. We will not construct $\mathbb{R}$ until Chapter 10; we need to do some preliminary work first. We need to understand the concept of limit, and the concepts of infimum and supremum, before we can formalize the idea of “complete” and before we have the tools necessary to carry out the construction of $\mathbb{R}$.

In this chapter we study infinite sequences and their limits. We will do so in the context of ordered fields. The most important ordered fields in mathematics are $\mathbb{Q}$ and $\mathbb{R}$, and although there are other ordered fields studied in mathematics these are in fact the only two ordered fields we will see in this course. Our approach to sequences and limits in ordered fields will thus be a unified approach that will work for both $\mathbb{Q}$ and $\mathbb{R}$. In Chapter 10 we will need sequences of rational numbers in order to construct real numbers. After we have constructed $\mathbb{R}$ we will use sequences of real numbers. Not only are such sequences important in this class, but sequences of real numbers are crucial in all of advanced mathematics, most notably in the field of real analysis.

So in this chapter we aim to prove results for a general ordered field, or sometimes a general Archimedean ordered field. Since $\mathbb{Q}$ is an Archimedean ordered field, the results will automatically be true for $\mathbb{Q}$. After we construct $\mathbb{R}$ and prove that it is too an Archimedean ordered field, the results will automatically become theorems for $\mathbb{R}$ as well.

1. Ordered fields

Informally an ordered field is a field with an order relation $<$ that satisfies the usual rules we learn in elementary algebra and arithmetic. Our formal definition does not mention $<$, but focuses on the subset of *positive* elements. The actual order relation $<$ will be formally defined in terms of these positive elements later.

Definition 1. An *ordered field* F is a field with a designated subset P such that (i) P is closed under addition and multiplication, and (ii) for any element $u \in F$ exactly one of the following occurs: $u = 0, u \in P, -u \in P$.

Remark 1. When we say that P is *closed under addition and multiplication*, we mean that if $x, y \in P$ then $x+y$ and $x \cdot y$ are in P. The second condition, that exactly one of $u = 0,\ u \in P,\ -u \in P$ holds, is called the *first form of trichotomy*.

Definition 2. Let F be an ordered field with designated subset P. The elements in P are called the *positive elements*. We sometimes write the designated subset P as $F_{>0}$.

Definition 3. Let F be an ordered field with designated subset P. If $u \in F$ is such that $-u \in P$ then u is said to be *negative*. (So the first form of trichotomy says that every element satisfies exactly one of the following properties: it is zero, positive, or negative.)

The first example of an ordered field is $\mathbb{Q}$:

Theorem 1. *The field $\mathbb{Q}$ is an ordered field.*

Exercise 1 (Easy)**.** Prove the above theorem using theorems from Chapter 7.

Exercise 2. Show that the field $\mathbb{F}_5$ cannot qualify as an ordered field. Hint: try all possible subsets for P, and show that none work.

Remark 2. This extends: no $\mathbb{F}_p$ can be an ordered field. In fact, ordered fields must be infinite. Later we will study the complex numbers $\mathbb{C}$ which is an example of an infinite field that is not an ordered field.

For most of the rest of the chapter we will consider theorems about a general ordered field F. The only ordered field we have constructed so far is $\mathbb{Q}$, so you can initially think of F as being something like $\mathbb{Q}$. You can freely use all the standard laws of algebra that are true in a general field. In Chapter 10 we will construct the real numbers $\mathbb{R}$, another ordered field, and everything we prove about ordered fields will apply to $\mathbb{R}$ as well.

As the name suggests, an ordered field is ordered by a certain order relation. This is defined as follows.

Definition 4. Let F be an ordered field, and let $x, y \in F$. If $y - x$ is positive then we write $x < y$. We also write $y > x$ in this case.

Exercise 3. Show that $y - x$ is negative if and only if $y < x$. Here x and y are in an ordered field.

Exercise 4. Prove the two following theorems.

Theorem 2. *Suppose $u \in F$ where F is an ordered field. Then u is positive if and only if $u > 0$. Similarly, u is negative if and only if $u < 0$.*

Theorem 3 (Transitivity)**.** *Suppose $x, y, z \in F$ where F is an ordered field. If $x < y$ and $y < z$ then $x < z$.*

Theorem 4. *Suppose $x, y, z \in F$ where F is an ordered field. If $x < y$ then $x + z < y + z$.*

Exercise 5. Prove the above theorem. Hint: simplify $(y+z)-(x+z)$.

Theorem 5. *Let $x, y, x', y' \in F$. If $x < y$ and $x' < y'$ then $x + x' < y + y'$.*

Proof. By Theorem 4 used twice, we have

$$x + x' < y + x' < y + y'.$$

(We also use the commutative law for $+$ and the transitive law for $<$). □

Theorem 6 (Trichotomy version 2)**.** *Suppose $x, y \in F$ where F is an ordered field. Then exactly one of the following occurs: (i) $x = y$, (ii) $y < x$, or (iii) $x < y$.*

Exercise 6. Prove the above. Hint: use the first version of trichotomy for $y - x$ to divide into three disjoint cases. Show for each that that being in that case is equivalent to satisfying one of (i), (ii), or (iii).

Recall the definition of an ordered set from Chapter 2. We now know that every ordered field is an ordered set. Products behave as expected:

Theorem 7. *Suppose $x, y \in F$ where F is an ordered field. If x and y are positive, then xy is positive. If x is positive, but y is negative, then xy is negative. If x and y are negative, then xy is positive.*

Proof. The first statement follows from the definition of ordered field: the positive elements are closed under multiplication.

In the second statement, $-y$ is positive by definition of negative. Thus the product $x(-y)$ is positive by closure. But $x(-y) = -(xy)$ since F is a field (this is true in any ring). Thus $-(xy)$ is positive, so xy is negative.

In the third statement, $-x$ and $-y$ are positive. So $(-x)(-y)$ is positive by closure. But, since F is a field,

$$(-x)(-y) = -\big(x(-y)\big) = -\big(-(xy)\big) = xy.$$

Thus xy is positive. □

Theorem 8. *Suppose $x, y, z \in F$ where F is an ordered field. If $x < y$, and if z is positive, then $xz < yz$. If $x < y$, and if z is negative, then $xz > yz$.*

Exercise 7. Prove the above theorem. Hint: multiply $y - x$ and z.

The follow statement is already known for $F = \mathbb{Q}$. The point of proving it here is to show that it is true of any other possible ordered field F.

Theorem 9. *The element $1 \in F$ is positive, and -1 is negative.*

Proof. Since $0 \neq 1$ in any field, we have that 1 is either positive or negative (by the first version of trichotomy). Suppose 1 is negative. Then $1 \cdot 1$ is positive by Theorem 7. But $1 \cdot 1 = 1$, so 1 is positive, a contradiction.

Since 1 is positive, -1 is negative by definition of negative. □

Theorem 10. *Suppose x is a positive element of an ordered field F. Then the inverse x^{-1} is also positive. Suppose x is a negative element of F. Then the inverse x^{-1} is also negative.*

Proof. Suppose x is positive. Observe that x^{-1} cannot be 0: otherwise

$$1 = xx^{-1} = x \cdot 0 = 0$$

which is not allowed in a field. Observe that x^{-1} cannot be negative: otherwise $xx^{-1} = 1$ must be negative (Theorem 7) contradicting Theorem 9. Thus, by trichotomy, x^{-1} is positive.

The proof of the second claim is similar. □

Theorem 11. *Suppose x, y are positive elements of an ordered field F. If $x < y$ then $y^{-1} < x^{-1}$.*

Proof. Multiply both sides of $x < y$ by $x^{-1}y^{-1}$. □

Now we consider the special case where $F = \mathbb{Q}$. Recall that $\mathbb{Z}$ is regarded as a subset of $\mathbb{Q}$. We have an order for $\mathbb{Z}$ from Chapter 4, and an order for $F = \mathbb{Q}$ defined in the current section. We now show that the new order extends the old order.

Lemma 12. *The order relation $<$ on $\mathbb{Q}$ extends the order relation $<$ on $\mathbb{Z}$. In other words, if $a, b \in \mathbb{Z}$, then $a < b$ (as defined in Chapter 4) if and only if $a < b$ (as defined in this section).*

Proof. Suppose that $a < b$ in the sense of Chapter 4. By the results of Chapter 4, $b - a$ must be a positive integer. By a result of Chapter 7, this means $b - a$ is a positive rational number. Thus $a < b$ in the sense of this section.

This proves one direction. The converse is similar. □

Theorem 13. *Suppose a and b are integers, and d is a positive integer. Consider $a/d, b/d \in \mathbb{Q}$. Then $a/d > b/d$ if and only if $a > b$.*

Proof. If $a/d > b/d$, then multiply both sides by d to show $a > b$. Conversely, suppose that $a > b$. Multiply both sides by d^{-1}. Now $d^{-1} > 0$ by Theorem 10. Thus $a/d > b/d$. □

2. Less than or equal

In this section let F be an ordered field.

Definition 5. If $x, y \in F$ then $x \leq y$ means $(x < y) \vee (x = y)$. We also write $y \geq x$ in this case.

Theorem 14. *Let $x, y \in F$. Then the negation of $x < y$ is $y \leq x$. The negation of $y \leq x$ is $x < y$.*

Proof. By version 2 of trichotomy (Theorem 6),

$$\neg(x < y) \iff y \leq x.$$

The contrapositive of the above gives

$$\neg(y \leq x) \iff x < y.$$

□

Theorem 15 (Mixed transitivity)**.** *Let $x, y, z \in F$. If $x < y$ and $y \leq z$ then $x < z$. Likewise, if $x \leq y$ and $y < z$ then $x < z$.*

Proof. Suppose that $x < y$ and $y \leq z$. By definition of $y \leq z$, we have either $y < z$ or $y = z$. In the first case, use transitivity of $<$ (Theorem 3). In the second case, use substitution. In either case $x < z$. This proves the first statement. The proof of the second is similar. □

Theorem 16 (Transitivity)**.** *Let $x, y, z \in F$. If $x \leq y$ and $y \leq z$ then $x \leq z$.*

Proof. Suppose that $x \leq y$ and $y \leq z$. By definition of $x \leq y$, we have either $x < y$ or $x = y$. In the first case, use mixed transitivity (Theorem 15). In the second case, use substitution. In either case $x \leq z$ as desired. □

Theorem 17. *Let $x, y, z \in F$. If $x \leq y$ then $x + z \leq y + z$.*

Proof. By definition of $x \leq y$, we have either $x < y$ or $x = y$. In the first case, $x + z < y + z$ by an earlier result (Theorem 4). In the second case, $x + z = y + z$. In either case $x + z \leq y + z$ as desired. □

Theorem 18. *Let $x, y, z \in F$ where $x \leq y$. If $z \geq 0$ then $xz \leq yz$. If $z \leq 0$ then $yz \leq xz$.*

Proof. Assume $z \geq 0$ and $x \leq y$. In the special cases where $z = 0$ or $x = y$ then $xz = yz$, so $xz \leq yz$ holds. So we can assume that $z > 0$ and $x < y$. Now use Theorem 8. This proves the first statement. The proof of the second statement is similar. □

Theorem 19. *Let $x, y, x', y' \in F$. If $x \leq y$ and $x' \leq y'$ then $x + x' \leq y + y'$.*

Proof. By Theorem 17 twice, we have

$$x + x' \leq y + x' \leq y + y'.$$

(We also use the commutative law for addition, and the transitive law for $\leq$.) □

3. Absolute values in ordered fields

Absolute values can be defined and developed in any ordered field. Throughout this section, let F be an ordered field.

Definition 6. The *absolute value* $|x|$ of $x \in F$ is defined as follows.

$$|x| \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

Theorem 20. *If $x \in F$ then $|x| \geq 0$.*

Proof. If $x \geq 0$ then $|x| = x$, so $|x| \geq 0$. If $x < 0$ then $|x| = -x$. Adding $-x$ to both sides of $x < 0$ gives $0 < -x$. Thus $|x| = -x > 0$ in this case. □

Remark 3. The above theorem shows that the absolute value defines a function $F \to F_{\geq 0}$ where $F_{\geq 0}$ is the set $\{x \in F \,|\, x \geq 0\}$.

Informal Exercise 8. Is the function $F \to F_{\geq 0}$ defined by $x \mapsto |x|$ injective? Is it surjective?

The following are easy consequences of the definition.

Theorem 21. *Let $x \in F$. Then*

$$|x| = 0 \iff x = 0.$$

Theorem 22. *Let $x \in F$. Then*

$$|x| > 0 \iff x \neq 0.$$

Theorem 23. *Let $x \in F$. Then $|x| = |-x|$.*

Proof. We use trichotomy to divide the proof into three cases.

If $x > 0$ then $|x| = x$ and $|-x| = -(-x) = x$ (since $-x < 0$).

If $x = 0$ then $|x| = x = 0$ and $|-x| = -x = 0$ (since $-x = 0$).

If $x < 0$ then $|x| = -x$ and $|-x| = -x$ (since $-x > 0$). □

Absolute value is compatible with multiplication.

Theorem 24 (Compatability with multiplication)**.** *Let $x, y \in F$. Then*

$$|xy| = |x| \cdot |y|.$$

Proof. We divide the proof into cases using trichotomy.

If both x and y are positive then so is xy by Theorem 7. So, by the definition of absolute value, $|xy| = xy$ and $|x||y| = xy$.

If both x and y are negative, then xy is positive (Theorem 7). So, by the definition of absolute value, $|xy| = xy$ and

$$|x||y| = (-x)(-y) = -\big(x(-y)\big) = -\big(-(xy)\big) = xy.$$

If either $x = 0$ or $y = 0$ then $|xy| = 0$ and $|x||y| = 0$.

If x is positive and y is negative, then xy is negative (Theorem 7). So, by the definition of absolute value, $|xy| = -xy$ and $|x||y| = x(-y) = -xy$.

The case where x is negative and y is positive is similar. □

Absolute value is compatible with inverses and division as well.

Theorem 25. *Let $x \in F$ be nonzero. Then*

$$\left|x^{-1}\right| = |x|^{-1}.$$

Let $x, y \in F$. Then

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}.$$

Exercise 9. Prove the above theorem. Hint: for the first equality, start with proving that $\left|xx^{-1}\right| = 1$, and solve for $\left|x^{-1}\right|$ (with the help of the previous Theorem).

Lemma 26. *If $x \in F$ then $x \leq |x|$ and $-x \leq |x|$.*

Proof. We use trichotomy to divide into cases.

If $x = 0$ then $|x| = 0$ and $-x = 0$. So obviously $x \leq |x|$ and $-x \leq |x|$.

If $x > 0$ then $x = |x|$, so $x \leq |x|$. Also $-x < 0$ and $0 < |x|$, so $-x \leq |x|$.

If $x < 0$ then $-x = |x|$, so $-x \leq |x|$. Since $x < 0$ and $0 \leq |x|$ (Theorem 20), we have $x \leq |x|$. □

Theorem 27. *Suppose $x, y \in F$ where $y \geq 0$. Then*
(i) $|x| < y$ if and only if $-y < x < y$,
(ii) $|x| > y$ if and only if $x > y$ or $x < -y$, and
(iii) $|x| = y$ if and only if $x = y$ or $x = -y$.

Proof. (ia) Suppose that $|x| < y$. Now $x \leq |x|$ (by Lemma 26). So $x < y$ by transitivity (Theorem 15). Also $-x \leq |-x|$ (Lemma 26) and $|-x| = |x|$ (Theorem 23), so $-x \leq |x|$. Thus $-x < y$ by transitivity (Theorem 15). Adding $x - y$ to both sides gives $-y < x$ (Theorem 4). We have both $x < y$ and $-y < x$, so $-y < x < y$.

(ib) Suppose $-y < x < y$. If $x \geq 0$ then $|x| = x$, so $|x| < y$. Otherwise, $x < 0$. Adding $y - x$ to both sides of $-y < x$ gives $-x < y$ (see Theorem 4). Since $|x| = -x$, we have $|x| < y$.

We leave the proofs of (ii) and (iii) to the reader. □

Exercise 10. Prove (ii) and (iii) of the above theorem.

Corollary 28. *Suppose $x, y \in F$ where $y \geq 0$. Then*
(i) $|x| \leq y$ if and only if $-y \leq x \leq y$,
(ii) $|x| \geq y$ if and only if $x \geq y$ or $x \leq -y$, and

The following is sometimes called the "triangle inequality" since the analogous vector version says that the third side of a triangle can be no larger than the sum of the lengths of the other two sides.

Theorem 29 (Triangle inequality)**.** *If $x, y \in F$ then*

$$|x + y| \leq |x| + |y|.$$

Proof. We have $x \leq |x|$ and $y \leq |y|$ (Lemma 26). Thus $x + y \leq |x| + |y|$ by Theorem 19.

We have $-x \leq |x|$ and $-y \leq |y|$ (Lemma 26). Adding $x - |x|$ to both sides of $-x \leq |x|$ gives $-|x| \leq x$ (see Theorem 17). Likewise, $-|y| \leq y$. So $-(|x| + |y|) \leq x + y$ (Theorem 19). Thus

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

By Corollary 28 (i),

$$|x + y| \leq |x| + |y|.$$

□

4. Intervals and density in ordered fields

We begin with the definition of finite intervals:

Definition 7 (Intervals)**.** Let F be an ordered field, and let $x, y \in F$ such that $x < y$. Then we define four intervals with endpoints x, y. The associated *open interval* is defined as follows:

$$(x, y) \stackrel{\text{def}}{=} \{z \in F \mid x < z < y\}.$$

The associated *closed interval* is defined as follows:

$$[x, y] \stackrel{\text{def}}{=} \{z \in F \mid x \le z \le y\}.$$

The associated *half-closed intervals* are defined as follows

$$[x, y) \stackrel{\text{def}}{=} \{z \in F \mid x \le z < y\} \qquad (x, y] \stackrel{\text{def}}{=} \{z \in F \mid x < z \le y\}.$$

If $x > y$ then all of these intervals are defined to be the empty set.

It is common also to define *infinite intervals*. For example, (x, ∞) is defined as $\{z \in F \mid z > x\}$.

Exercise 11. Define four other types of infinite intervals (including the interval $(-\infty, \infty)$). Show that all five types of infinite intervals are nonempty.

It is obvious that all the closed and half-closed intervals are nonempty (if $x < y$ as in the above definition). However, is it obvious that (x, y) is nonempty? We will end this section with a proof that (x, y) is indeed nonempty. In other words, for all $x, y \in F$ with $x < y$ there is a $z \in F$ with $x < z < y$. So you can always find new elements between two given elements. Because of this we say that F is *dense*.

The proof is simple: show that the average $(x + y)/2$ is between x and y. One question: is $2 \in F$? Of course, in the case of $\mathbb{Q}$ the answer is clearly yes: $\mathbb{Q}$ contains $\mathbb{Z}$. In general, it can be shown that $\mathbb{Z}$ can be embedded into any ordered field. Instead of showing this now, we make an *ad hoc* definition of the number two. This makes 2 a member of any ordered field. Later we will need 3 as well, so we define 3 along with 2.

Definition 8. Let 1 be the multiplicative identity of an ordered field F. Define 2 to be $1 + 1$. Define 3 to be $2 + 1$.

Remark 4. Observe that 2 is positive in any ordered field F since $1 \in F$ is positive and the set of positive elements is closed under addition. As a consequence, the multiplicative inverse $2^{-1} = 1/2$ exists and is positive in any ordered field. Similarly 3 and $1/3$ are positive in any ordered field.

Remark 5. If we needed to, we could define 4 to be $3 + 1$, and so on. This trick works not just for ordered fields, but for any ring whatsoever. However, in some rings, as in the ring $\mathbb{Z}_4$, we would have $4 = 0$. In ordered fields, positive elements are closed under addition, so we never get 0 by this process. So there is an injection from $\mathbb{N}$ into any ordered field.

Theorem 30 (Density)**.** *Let F be an ordered field. Let $x, y \in F$ be such that $x < y$. Then we can find an element $z \in F$ with $x < z < y$. In other words, then interval (x, y) is nonempty, and the field F is dense.*

Proof. Let $z = (x + y)/2$.

Since $x < y$ we have $x+x < x+y$. Now $x+x = 1\cdot x+1\cdot x = (1+1)x = 2x$. So $2x < x + y$. Since 2 is positive, 2^{-1} is positive. Thus $x < (x + y)/2$.

Since $x < y$ we have $x+y < y+y$. Now $y+y = 1\cdot y+1\cdot y = (1+1)y = 2y$. So $x + y < 2y$. Since 2 is positive, 2^{-1} is positive. Thus $(x + y)/2 < y$. □

5. The Archimedean property

There are two ordered fields that we will study in this course: $\mathbb{Q}$, and later $\mathbb{R}$. Both of these have an important property: they are *Archimedean*. Before explaining what this means, we need the following preliminary definition:

Definition 9 (Subfield)**.** Suppose that K and F are fields, and that K is a subset of F. Suppose also that the ring operations $+$ and $\times$ of K are compatible with the ring operations of F in the sense that the ring operations of F restricted to K gives the ring operations on K. Then we say that K is a *subfield* of F.

Suppose further that K and F are ordered fields, and that the order relation $<$ on K is compatible with the order relation on F in the sense that for $x, y \in K$ we have $x <_K y$ if and only if $x <_F y$. Then we say that K is an *ordered subfield* of F.

We are most interested in ordered fields that contain $\mathbb{Q}$ as an ordered subfield. Such an ordered field F must then contain $\mathbb{N}$ as a subset, and every positive integer is in the set P of positive elements of F.

Definition 10 (Archimedean ordered field)**.** Let F be an ordered field. We say that F is an *Archimedean ordered field* if (i) it contains $\mathbb{Q}$ as an ordered subfield, and (ii) for all $x > 0$ and y in F, there is an $n \in \mathbb{N}$ such that $nx \geq y$.

Remark 6. In the appendix we will discuss the fact that (i) is in some sense true of any ordered field, so we can actually remove this from the definition. The condition (ii) is the main condition. Informally it says that if you have an x which is possibly very small, the integer multiples nx become arbitrarily large. This means that F cannot have so called *infinitesimal* elements. A version of this property was used in geometry by the famous ancient Greek mathematician Archimedes, hence the name.

Note. There is a version of calculus and analysis, called *nonstandard analysis*, where infinitesimals are allowed. This uses a version of the real numbers that is not Archimedean. Advocates of nonstandard analysis claim that some of the simplicity of the original form of calculus of Newton and Leibniz can be preserved if we keep infinitesimals. In addition, infinitesimals dx, dy and so on are often used informally in applied mathematics and in the sciences. However, the majority of current mathematicians use the version of $\mathbb{R}$ with no infinitesimals. We follow this usage: our version of $\mathbb{R}$ will be Archimedean with no infinitesimals.

In an Archimedean field, the integers are unbounded:

Theorem 31. *Let F be an ordered field with ordered subfield $\mathbb{Q}$. Then F is Archimedean if and only if the following holds: for all $y \in F$ there is an $n \in \mathbb{N}$ such that $n \geq y$.*

Exercise 12. Prove the above theorem.

Corollary 32. *Let $y \in F$ where F is an Archimedean ordered field. Then there are integers m, n such that*

$$m \leq y \leq n.$$

Proof. The existence of a suitable n follows from the previous theorem. To find m, apply the above theorem to $-y$. By Theorem 31 there is an integer n' such that $-y \leq n'$. Thus $-n' \leq y$. So if $m = -n'$, then $m \leq y$. □

Exercise 13. Let F be an Archimedean field. Suppose $u > 0$ in F. Show that there is a positive integer n such that $1/n \leq u$. Hint: use Theorem 31.

Theorem 33. *Let F be an ordered field with $\mathbb{Q}$ as an ordered subfield. Then F is Archimedean if and only if the following holds: if $u > 0$ is a positive element of F then there is a positive $n \in \mathbb{N}$ such that $1/n \leq u$.*

Proof. One direction is an exercise (see above). For the other direction, we assume that for all $u > 0$ there is a positive $n \in \mathbb{N}$ such that $1/n \leq u$.

Let $x > 0$ and y be in F. We must show there is an $n \in \mathbb{N}$ with $nx \geq y$. If $y \leq 0$ then $n = 0$ works, so assume that $y > 0$. Let $u = x/y$, and let $n \in \mathbb{N}$ be such that $1/n \leq u$. Then $nx \geq y$ as desired. □

Remark 7. In the above we can also find an n such that $1/n < u$ (strict inequality). This follows from the above and the fact that $1/(k+1) < 1/k$.

Theorem 34. *The field $\mathbb{Q}$ is an Archimedean ordered field.*

Proof. By Theorem 33 we just need to show that if $r \in \mathbb{Q}$ is positive then we can find a positive integer n with $1/n \leq r$. So assume $r \in \mathbb{Q}$ is positive. Hence we can write $r = a/b$ with a, b both positive integers. Since $1 \leq a$ we have $1/b \leq a/b = r$. □

We can strengthen Corollary 32:

Theorem 35. *Let F be an Archimedean ordered field. Suppose $x \in F$. Then there is a unique integer $n \in \mathbb{Z}$ such that $n \leq x < n + 1$.*

Remark 8. In other words, $x = n + y$ where $0 \leq y < 1$. The number y is sometimes called the *fractional part* of x (although it is not always a fraction in the sense of being in $\mathbb{Q}$). We call n the *floor* of x.[1]

[1]In contrast, the smallest integer greater than or equal to x is called the *ceiling of x*. If $n < x < n+1$ then $n+1$ is the ceiling of x, but if $n = x$ then n is both the floor and ceiling of x. The floor of x is written $\lfloor x \rfloor$ and the ceiling is written $\lceil x \rceil$.

Proof. By Corollary 32 there are integers a and b such that $a \leq x \leq b$. In particular the set of integers less than or equal to x has upper bound $b \in \mathbb{Z}$. By a result of Chapter 4, this implies that there is a largest integer n such that $n \leq x$. In particular, $n + 1 > x$.

To prove uniqueness, suppose $n_1, n_2 \in \mathbb{N}$ satisfy both $n_1 \leq x < n_1 + 1$ and $n_2 \leq x < n_2 + 1$. So $n_1 < n_2 + 1$ and $n_2 < n_1 + 1$. This implies that $n_1 - 1 < n_2 < n_1 + 1$. Thus $n_2 = n_1$ since n_1 is the only integer between $n_1 - 1$ and $n_1 + 1$. □

The following will be important to us in the case where $F = \mathbb{R}$.

Theorem 36. *Let F be an Archimedean ordered field. Then $\mathbb{Q}$ is dense in F. In other words, if $x < y$ with $x, y \in F$, then there exists an $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. Let n be a positive integer such that $1/n < (y - x)$. Such an n exists by Theorem 33. In particular $1 < ny - nx$.

By Theorem 35 there is an integer m such that $m \leq nx < m + 1$. Since $m \leq nx$ and $1 < ny - nx$ we have $m + 1 < nx + (ny - nx) = ny$. So $nx < m + 1 < ny$. Hence $x < (m + 1)/n < y$. So $r = (m + 1)/n$ is a rational number between x and y. □

Remark 9. The converse is true as well: if $\mathbb{Q}$ is a dense ordered subfield of and ordered field F then F is an Archimedean ordered field. This follows from Theorem 33.

6. Infinite sequences and limits

In this section we consider infinite sequences in an ordered field F. Recall the definition of infinite sequences from Section 9 of Chapter 5. An infinite sequence with values in F is a function whose domain is a set (called the *index set*) of the form $\{i \in \mathbb{Z} \mid i \geq n_0\}$, and whose codomain is F. We use notation such as $(a_i)_{i \geq n_0}$ to denote such a sequence. We simply write (a_i) when the domain is not important to the discussion.

An important concept associated to sequences is that of a *limit*. What do we mean by the limit of a sequence? Informally, a sequence (a_i) has limit b if the terms of the sequence eventually get and stay arbitrarily close to b. This informal description is a bit ambiguous and is unsuitable to use in a proof, so we give a more precise definition.

Definition 11 (Limit)**.** Suppose F is an ordered field, that (a_i) is a sequence in F, and that $b \in F$. We say that b is the *limit* of (a_i) if the following holds: for all positive $\varepsilon \in F$ there is an $N \in \mathbb{N}$ such that if $i \geq N$ then $|a_i - b| < \varepsilon$. We can write this with three quantifiers as follows:

$$(\forall \varepsilon \in F_{>0})(\exists N \in \mathbb{N})(\forall i \in \mathbb{N})\Big(i \geq N \implies |a_i - b| < \varepsilon\Big).$$

Here $F_{>0}$ denotes the positive elements of F.

Not all sequences have limits. If a limit exists the we say the sequence *converges*. This is captured in the following definition:

Definition 12 (Convergence)**.** A sequence that has a limit is said to *converge*. We can write this with *four* quantifiers as follows:

$$(\exists b \in F)(\forall \varepsilon \in F_{>0})(\exists N \in \mathbb{N})(\forall i \in \mathbb{N})\Big(i \geq N \implies |a_i - b| < \varepsilon\Big).$$

A sequence that does not have a limit is said to *diverge*.

Exercise 14. Use the rules of quantifiers in logic to negate the definition of limit. Complete the following sentence: *the sequence* (a_i) *does not have limit* b *means that there is a positive* $\varepsilon \in F$ *such that for all* $N \in \mathbb{N}$ *...*

In a similar manner, use the rules of basic logic to negate the definition of converges. In other words, complete the following sentence: *the sequence* (a_i) *diverges means that for all* $b \in F$ *there exists a positive ...*

Exercise 15. Suppose F is an Archimedean ordered field, and consider the sequence $(i)_{i\in\mathbb{N}}$. In other words, consider the sequence given by the identity function. Show that this sequence diverges. Hint: work with $\varepsilon = 1$, and either use the previous exercise or give a proof by contradiction.

Notice that in the above definition we used the term *the* limit. It sounds like we are treating limits as if they are unique. This is justified by the following theorem.

Theorem 37. *A convergent sequence in an ordered field has a unique limit.*

Proof. Suppose otherwise that (a_i) is a sequence in F with two distinct limits b and c. Let $\varepsilon = |b - c|/2$. Since $b \neq c$, we have that $b - c \neq 0$. Hence $|b-c| > 0$. Since $2 > 0$ we have $2^{-1} > 0$. Thus the product $|b-c|2^{-1}$ is positive. In other words $\varepsilon > 0$.

By definition of limit, there is a $N_1 \in \mathbb{N}$ such that $|a_i - b| < \varepsilon$ for all $i \geq N_1$. Likewise there is a $N_2 \in \mathbb{N}$ such that $|a_i - c| < \varepsilon$ for all $i \geq N_2$. Let N be the maximum of N_1 and N_2. Then, for $i \geq N$ we have

$$|b - c| = |(b - a_i) + (a_i - c)| \leq |b - a_i| + |a_i - c| < \varepsilon + \varepsilon.$$

Here we used the triangle inequality. Since $2\varepsilon = |b - c|$ we have that

$$|b - c| < |b - c|,$$

a contradiction. □

Remark 10. We often write

$$\lim_{i\to\infty} a_i = b$$

when we wish to assert (1) that the sequence (a_i) converges, and (2) that the unique limit of the sequence (a_i) is b.

Informal Exercise 16. Draw a picture of a number line representing F. Draw b and c in the above proof, and indicate the sets defined by $|x - b| < \varepsilon$ and $|x - c| < \varepsilon$ where ε is as in the above proof. Observe that the sets do not intersect so there can be no a_i simultaneously in both. This explains why we chose $\varepsilon = |b-c|/2$. Note, we could have chosen $\varepsilon = |b-c|/4$, for instance, and obtained a contradiction. However, $\varepsilon = 2|b-c|$ does not work. Why not?

Remark 11. In the above theorem we get to choose ε to be whatever we want since we are *assuming* a limit exists. If instead you are trying to *prove* a limit exists, you cannot choose ε, but must allow ε to be an arbitrary positive element of F.

Exercise 17. Let F be an Archimedean ordered field. Show that

$$\lim_{j\to\infty} 1/j = 0.$$

(Assume that the domain of the sequence is the positive integers, since this sequence is not defined when $j = 0$). Hint: by Theorem 33 (and the following remark) we know that if $\varepsilon > 0$ then $1/n < \varepsilon$ for some positive integer n.

Exercise 18. Show that convergent sequences are bounded. In other words, show that if $(a_i)_{i\ge n}$ is a convergent sequence in an ordered field, then there is a bound $M \in F$ such that $|a_i| \le M$ for all $i \ge n$. Show that if F is an Archimedean ordered field, then we can choose $M \in \mathbb{N}$.

Hint: Using the definition of convergence and a choice of ε (such as $\varepsilon = 1$) first find upper and lower bounds for (a_i) for $i \ge N$ for some N. These bounds work for most terms, but not necessarily when $i < N$. Then find upper and lower bounds for the case when $i < N$. There are only a finite number of terms in this case. Now take the largest of the absolute values of the upper and lower bounds you found above. Prove that this number works. Drawing a picture might help to visualize what is happening.

7. Equivalence relation for sequences

Definition 13 (Equivalent sequences)**.** Suppose that $(a_i)_{i\ge n_1}$ and $(b_i)_{i\ge n_2}$ are two sequences in an ordered field F. We write $(a_i) \sim (b_i)$ if the following occurs: for all positive $\varepsilon \in F$ there is a $N \in \mathbb{N}$ such that, for all $i \in \mathbb{N}$,

$$i \ge N \quad \Rightarrow \quad |a_i - b_i| < \varepsilon.$$

Remark 12. Informally the above definition says that the terms of the two sequences get and stay arbitrarily close to each other. Note the N in the above definition should be greater than or equal to both n_1 and n_2.

Lemma 38. *The relation $\sim$ is reflexive on the set of all sequences in F.*

Proof. We need to show $(a_i) \sim (a_i)$ for any given sequence $(a_i)_{i\ge n_0}$ in F. Let ε be an arbitrary positive element of F. We must find an $N \in \mathbb{N}$ such that

$$i \ge N \quad \Rightarrow \quad |a_i - a_i| < \varepsilon.$$

Let us propose $N = n_0$. If $i \ge N$ then $|a_i - a_i| < \varepsilon$ since $|a_i - a_i| = 0$ and ε is positive. So N has the desired property. □

Lemma 39. *The relation $\sim$ is transitive on the set of all sequences in F.*

Proof. Suppose $(a_i) \sim (b_i)$ and $(b_i) \sim (c_i)$. We need to show $(a_i) \sim (c_i)$. Let ε be an arbitrary positive element of F. We must find an $N \in \mathbb{N}$ such that

$$i \geq N \quad \Rightarrow \quad |a_i - c_i| < \varepsilon.$$

To find N we need to use the fact that $(a_i) \sim (b_i)$ and $(b_i) \sim (c_i)$. We work with $\varepsilon' = \varepsilon/2$. Since $(a_i) \sim (b_i)$ there is a $N_1 \in \mathbb{N}$ such that

$$i \geq N_1 \quad \Rightarrow \quad |a_i - b_i| < \varepsilon'.$$

Likewise, there is a $N_2 \in \mathbb{N}$ such that

$$i \geq N_2 \quad \Rightarrow \quad |b_i - c_i| < \varepsilon'.$$

Let us propose for N the larger of N_1 or N_2. Note that if $i \geq N$ then, since $N \geq N_1$, by transitivity we get $i \geq N_1$. As above we get $|a_i - b_i| < \varepsilon'$. Similarly if $i \geq N$ then $i \geq N_2$, so $|b_i - c_i| < \varepsilon'$. Thus if $i \geq N$ then

$$|a_i - c_i| = |(a_i - b_i) + (b_i - c_i)| \leq |a_i - b_i| + |b_i - c_i| < \varepsilon' + \varepsilon'.$$

Here we use the triangle inequality and the fact that $i \geq N_1$ and $i \geq N_2$. Since $\varepsilon = 2\varepsilon'$,

$$|a_i - c_i| < \varepsilon.$$

Thus N has the desired property. □

Remark 13. In the proof above we chose $N = \max\{N_1, N_2\}$. This is a common trick in proofs of this type. We are often looking for an N such that if $i \geq N$ then both Condition 1 and Condition 2 hold. If we know that Condition 1 holds when $i \geq N_1$ and Condition 2 holds when $i \geq N_2$, then we choose $N = \max\{N_1, N_2\}$. Note that the maximum of N_1 and N_2 is greater than or equal to both of them. Then when $i \geq N$, since $N \geq N_1$, by transitivity $i \geq N_1$ and so Condition 1 holds. Similarly when $i \geq N$ we also have $i \geq N_2$ and so Condition 2 holds as well.

Lemma 40. *The relation $\sim$ is symmetric on the set of all sequences in F.*

Exercise 19. Prove the above.

From the above lemmas we get the following:

Theorem 41. *Let F be an ordered field. Then the relation $\sim$ is an equivalence relation on the set of sequences of (a_i) in F.*

Exercise 20. Show that if (a_i) and (b_i) have the same limit, then $(a_i) \sim (b_i)$. Hint: use $\varepsilon' = \varepsilon/2$ to find N_1 and N_2. Choose N to be the maximum of N_1 and N_2.

Theorem 42. *Let (a_i) and (b_i) be sequences in an ordered field F and suppose $(a_i) \sim (b_i)$. If (a_i) has a limit, then (b_i) converges and has the same limit as (a_i).*

Exercise 21. Prove the above theorem. Hint: Let a be the limit of (a_i). Given an $\varepsilon > 0$, you are looking for an $N \in \mathbb{N}$ such that $|b_i - a| < \varepsilon$ when $i \geq N$. What happens if you add and subtract a_i inside of the absolute value? How does that help you to use your two hypotheses?

Remark 14. Suppose (a_i) and (b_i) have the property that $a_i = b_i$ for sufficiently large i. In other words, suppose that there is a k such that $a_i = b_i$ for all $i \geq k$. That is, suppose the two sequences agree after some point. Then $(a_i) \sim (b_i)$. This is easily proved from the definition. So by Theorem 42, if one converges then both do with the same limit.

In particular, if we take a sequence (a_i) and change a finite number of terms, then the resulting sequence is equivalent to (a_i). Likewise, if we change the domain of $(a_i)_{i \geq n_0}$ by replacing n_0 with a larger integer, then the resulting sequence is equivalent.

Because of this, the limit of $(a_i)_{i \geq n_0}$ does not depend on the initial element n_0 of the domain of the sequence. So from now on we will ignore the starting point of sequences when considering limits: it does not matter where the sequence starts, or the behavior of any finite number of terms, but only what happens in the long term.

8. Limit laws

The concept of limits, and their basic laws, are extremely important in calculus. In this section we develop many of the basic limit laws.

Theorem 43. *Suppose $(a_i)_{i \geq n_0}$ is a constant sequence with value a. In other words, suppose $a_i = a$ for all $i \geq n_0$. Then (a_i) converges to a. In other words,*

$$\lim_{i \to \infty} a = a.$$

Exercise 22 (Easy)**.** Prove the above theorem. Hint, for all $\varepsilon > 0$, the choice $N = n_0$ will work.

Theorem 44. *Suppose (a_i) and (b_i) are two converging sequences in an ordered field F. Suppose there is a $N \in \mathbb{N}$ such that $a_i \leq b_i$ for all $i \geq N$. Then*

$$\lim_{i \to \infty} a_i \leq \lim_{i \to \infty} b_i$$

Proof. Let a be the limit of (a_i) and let b be the limit of (b_i). Suppose the conclusion fails, and that $a > b$. Let $\varepsilon = (a - b)/2$. Note: $\varepsilon > 0$. By the convergence hypothesis for (a_i), there is a N_1 such that $i \geq N_1$ implies $|a_i - a| < \varepsilon$. Likewise, there is a N_2 such that $i \geq N_2$ implies that $|b_i - b| < \varepsilon$. The existence of N_1 and N_2 follow from the definition of limit. Let N be as in the hypothesis of the theorem, and let i be the maximum of N, N_1, and N_2.

Since $i \geq N_1$ we have $|a_i - a| < \varepsilon$. Thus $-\varepsilon < a_i - a < \varepsilon$. Since $a_i - a > -\varepsilon$ we have $a_i > a - \varepsilon$. Using the rules of arithmetic for fractions developed in the previous chapter and the fact that $\varepsilon = (a-b)/2$, we get $a-\varepsilon = (a+b)/2$. So $a_i > (a+b)/2$.

Since $i \geq N_2$ we have $|b - b_i| < \varepsilon$. Thus $-\varepsilon < b_i - b < \varepsilon$. Since $b_i - b < \varepsilon$ we have $b_i < b + \varepsilon$. Using the rules of arithmetic for fractions developed in the previous chapter and the fact that $\varepsilon = (a-b)/2$, we get $b+\varepsilon = (a+b)/2$. So $b_i < (a+b)/2$.

Combining the above inequalities, we get

$$b_i < (a+b)/2 < a_i,$$

and so we have $a_i > b_i$. But since $i \geq N$ this contradicts the hypothesis of the theorem that $a_i \leq b_i$. □

Remark 15. This result cannot be generalized to $<$. Without more information, given $a_i < b_i$ for all i we cannot conclude strict inequality in the limit. Consider for instance $a_i = 1 - 1/i$ and $b_i = 1 + 1/i$.

Corollary 45. *Suppose (a_i) is a converging sequence in an ordered field F with limit a. Suppose that b is an upper bound of (a_i). Then $a \leq b$. In other words,*

$$\lim_{i\to\infty} a_i \leq b.$$

Suppose instead that b is a lower bound of (a_i). Then

$$\lim_{i\to\infty} a_i \geq b.$$

Proof. Apply Theorem 44 to (a_i) and the constant sequence with terms b. □

Remark 16. In the above, b does not have to be a bound for all a_i. It is enough that it is a valid bound for a_i with $i \geq N$ for some fixed N. (The proof above generalizes to this case).

Remark 17. If $(a_i)_{i\geq n}$ and $(b_i)_{j\geq m}$ are two sequences, then we can define $(a_i+b_i)_{i\geq p}$ to be the sequences whose value is the sum. Here the starting index p can be defined as the maximum of n and m. As in Remark 14 the starting index is irrelevant to our investigation of limits, so we will usually not indicate it. Similar comments apply to sequences of products $(a_i b_i)$.

For the sequence $(1/b_i)$ we have to choose a starting index p so that $b_i \neq 0$ for $i \geq p$. The problem is that there might not be such a p. If there is such a p, we choose the smallest such p as our starting index, and then $(1/b_i)$ is well-defined. Similar comments apply to sequences of quotients (a_i/b_i).

Theorem 46 (Sum law)**.** *Let (a_i) and (b_i) be sequences with values an ordered field F. If (a_i) and (b_i) converge then the sequence (a_i+b_i) converges and*

$$\lim_{i\to\infty}(a_i + b_i) = \lim_{i\to\infty} a_i + \lim_{i\to\infty} b_i.$$

Proof. Let a be the limit of (a_i) and let b be the limit of (b_i). Let $\varepsilon > 0$ be in F. We must show that there is a $N \in \mathbb{N}$ such that $|(a_i + b_i) - (a + b)| < \varepsilon$ for all $i \geq N$.

Let $\varepsilon' = \varepsilon/2$. Since (a_i) converges to a, there is a N_1 such that $|a_i - a| < \varepsilon'$ for all $i \geq N_1$. Likewise, there is a N_2 such that $|b_i - b| < \varepsilon'$ for all $i \geq N_2$. Let N be the maximum of N_1 and N_2. Suppose $i \geq N$. Then

$$\begin{aligned}
|(a_i + b_i) - (a + b)| &= |(a_i - a) + (b_i - b)| && (F \text{ is a field}) \\
&\leq |a_i - a| + |b_i - b| && (\text{triangle inequality}) \\
&< \varepsilon' + \varepsilon' && (i \geq N_1 \text{ and } i \geq N_2) \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (\text{choice of } \varepsilon') \\
&= \left(\frac{1}{2} + \frac{1}{2}\right)\varepsilon && (F \text{ is a field}) \\
&= \varepsilon && (1 + 1 \stackrel{\text{def}}{=} 2 \text{ in } F).
\end{aligned}$$

□

Theorem 47. *Let (a_i) be a sequence with values in an ordered field F, and let $c \in F$. If (a_i) converges then (ca_i) converges and*

$$\lim_{i\to\infty} c\, a_i = c \lim_{i\to\infty} a_i.$$

Corollary 48. *Let (a_i) be a sequence with values in an ordered field F. If (a_i) converges, then so does $(-a_i)$, and*

$$\lim_{i\to\infty} -a_i = -\lim_{i\to\infty} a_i$$

Exercise 23. Prove the above theorem and corollary. Hint: in the case that $c \neq 0$ choose $\varepsilon' = \varepsilon/|c|$. What if $c = 0$?

Theorem 49 (Product law)**.** *Let (a_i) and (b_i) be sequences with values in an ordered field F. If (a_i) and (b_i) converge, then $(a_i b_i)$ converges and*

$$\lim_{i\to\infty} a_i b_i = \lim_{i\to\infty} a_i \cdot \lim_{i\to\infty} b_i.$$

Proof. Let a be the limit of (a_i) and let b be the limit of (b_i). Let $\varepsilon > 0$ be in F. We must show that there is an $N \in \mathbb{N}$ such that $|a_i b_i - ab| < \varepsilon$ for all $i \geq N$.

By Exercise 18 there is a bound M_1 such that $|a_i| \leq M_1$ for all i in the domain of (a_i). Let M be the maximum of $M_1, |b|$, and 1. Thus $|b| \leq M$ and $0 < M$ and $|a_i| \leq M$ for all a_i. Let $\varepsilon' = \varepsilon/(2M)$. Observe that ε' is positive. Since (a_i) converges to a, there is a N_1 such that $|a_i - a| < \varepsilon'$ for all $i \geq N_1$. Likewise, there is a N_2 such that $|b_i - b| < \varepsilon'$ for all $i \geq N_2$.

Let N be the maximum of N_1 and N_2. Suppose $i \geq N$. Then

$$\begin{aligned}
|a_ib_i - ab| &= |a_ib_i - a_ib + a_ib - ab| && (F \text{ is a field}) \\
&= |a_i(b_i - b) + b(a_i - a)| && (F \text{ is a field}) \\
&\leq |a_i(b_i - b)| + |b(a_i - a)| && (\text{triangle inequality}) \\
&= |a_i||b_i - b| + |b||a_i - a| && (|xy| = |x||y|) \\
&\leq M|b_i - b| + M|a_i - a| && (\text{bound on } |a_i| \text{ and } |b|) \\
&< M\varepsilon' + M\varepsilon' && (i \geq N_1 \text{ and } i \geq N_2) \\
&= M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} && (\text{choice of } \varepsilon') \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} && (MM^{-1} = 1) \\
&= \left(\frac{1}{2} + \frac{1}{2}\right)\varepsilon && (F \text{ is a field}) \\
&= \varepsilon && (1 + 1 \stackrel{\text{def}}{=} 2 \text{ in } F).
\end{aligned}$$

□

Corollary 50 (Power law)**.** *Let (a_i) be a sequence in an ordered field F and let $n \in \mathbb{N}$. If (a_i) converges, then so does (a_i^n) and*

$$\lim_{i\to\infty} a_i^n = \left(\lim_{i\to\infty} a_i\right)^n$$

Exercise 24. Prove the above corollary by a simple induction argument.

Lemma 51. *Let (b_i) be a sequence in an ordered field F. Suppose that (b_i) converges to a non-zero value b. Then there is an $N \in \mathbb{N}$ such that*

$$|b_i| \geq |b|/2$$

for all $i \geq N$.

Remark 18. In particular, in the above situation $b_i \neq 0$ if $i \geq N$. So $(1/b_i)$ is well-defined, and (a_i/b_i) is well-defined where (a_i) is any sequence in F. See Remark 17.

Proof. Let $\varepsilon = |b|/2$. Observe that $\varepsilon > 0$. Since (b_i) has limit b, there is an N such that the following holds: if $i \geq N$ then $|b_i - b| < \varepsilon$. In this case $|b_i - b| < |b|/2$ and so, with the triangle inequality,

$$|b| = |b_i + (b - b_i)| \leq |b_i| + |b - b_i| < |b_i| + |b|/2.$$

Thus if $i \geq N$

$$|b_i| > |b| - |b|/2 = (2-1)|b|/2 = |b|/2$$

(note we defined $2 = 1 + 1$ in any ordered field, so $2 - 1 = 1$). In particular, if $i \geq N$ then $|b_i| \geq |b|/2$. □

Theorem 52 (Inverse law)**.** *Let (b_i) be a sequence with values in an ordered field F. Suppose (b_i) converges and*

$$\lim_{i\to\infty} b_i \neq 0.$$

Then (b_i^{-1}) is a well-defined convergent sequence and

$$\lim_{i\to\infty} b_i^{-1} = \left(\lim_{i\to\infty} b_i\right)^{-1}.$$

Proof. The previous lemma shows (b_i^{-1}) is well-defined. Let $b \neq 0$ be the limit of (b_i). Let $\varepsilon > 0$ be in F. We must show that there is an $N \in \mathbb{N}$ such that $|1/b_i - 1/b| < \varepsilon$ for all $i \geq N$.

By the previous lemma, there exists an $N_1 \in \mathbb{N}$ such that if $i \geq N_1$ then $|b_i| \geq |b|/2$. In particular, if $i \geq N_1$ then

$$\frac{1}{|b_i|} \leq \frac{2}{|b|}.$$

Let $\varepsilon' = \varepsilon|b|^2/2$. Observe that $\varepsilon' > 0$. Since (b_i) converges to b, there is an $N_2 \in \mathbb{N}$ such that if $i \geq N_2$ then $|b_i - b| < \varepsilon'$.

Let N be the maximum of N_1 and N_2. Suppose $i \geq N$. Then

$$\begin{aligned}
\left|\frac{1}{b_i} - \frac{1}{b}\right| &= \left|\frac{b - b_i}{b_i b}\right| && (F \text{ is a field})\\
&= \frac{|b - b_i|}{|b_i||b|} && (|xy^{-1}| = |x||y|^{-1} \text{ and } |xy| = |x||y|)\\
&< \frac{\varepsilon'}{|b_i||b|} && (i \geq N_2)\\
&= \varepsilon' \frac{1}{|b|}\frac{1}{|b_i|} && (F \text{ is a field})\\
&\leq \varepsilon' \frac{1}{|b|}\frac{2}{|b|} && (i \geq N_1)\\
&= \frac{\varepsilon|b|^2}{2}\frac{1}{|b|}\frac{2}{|b|} && (\text{choice of } \varepsilon')\\
&= \varepsilon && (F \text{ is a field})
\end{aligned}$$

□

Exercise 25. Prove the following two corollaries.

Corollary 53 (Quotient law)**.** *Let (a_i) and (b_i) be sequences with values in an ordered field F. Suppose (a_i) and (b_i) converge. Suppose also that the limit of (b_i) is not zero. Then (a_i/b_i) converges and*

$$\lim_{i\to\infty} \frac{a_i}{b_i} = \frac{\lim_{i\to\infty} a_i}{\lim_{i\to\infty} b_i}.$$

Corollary 54 (Power law 2)**.** *Let (a_i) be a sequence in an ordered field F and let $n \in \mathbb{Z}$. If (a_i) converges to a nonzero limit, then so does (a_i^n) and*

$$\lim_{i\to\infty} a_i^n = \left(\lim_{i\to\infty} a_i\right)^n.$$

9. Suprema and infima

Recall that if a nonempty subset S of $\mathbb{Z}$ has an upper bound, then S actually has a maximum. Likewise, if such S has a lower bound then it has a minimum. This is *not* true for ordered fields such as $\mathbb{R}$ and $\mathbb{Q}$. Consider the interval $(0,1)$ in $\mathbb{Q}$ (or in $\mathbb{R}$). It is bounded above and below, but has no maximum or minimum. However, the interval $(0,1)$ has a least upper bound 1 and a greatest lower bound 0. Least upper bounds (suprema) and greatest lower bounds (infima) are the "next best thing" to maxima and minima. They differ from maxima and minima in the sense that they are sometimes not in the set S itself, but are always "arbitrarily close" to S.

When we construct the real numbers we will see that *any* nonempty subset with an upper bound has a least upper bound, and any nonempty subset with a lower bound has a greatest lower bound. According to the following example, $\mathbb{Q}$ lacks this property: it has bounded sets without least upper bounds.

Example (Informal). Consider the set $S = \{x \in \mathbb{Q} \mid x^2 \le 2\}$. This set has many upper bounds, such as 3 or even $3/2$, but it turns out that no $b \in \mathbb{Q}$ can be a *least* upper bound. In fact, a positive rational number r is an upper bound if and only if $r^2 \ge 2$. But if $r^2 \ge 2$ then, since $\mathbb{Q}$ has no square root of 2, we know $r^2 > 2$. It turns out that given such an r one can show that there is smaller positive rational number r' such that $(r')^2 > 2$. So no given r is a least upper bound.

On the other hand, $S = \{x \in \mathbb{R} \mid x^2 \le 2\}$ has a least upper bound in $\mathbb{R}$. It turns out to be $b = \sqrt{2}$.

Definition 14 (Supremum)**.** Let S be a nonempty subset of an ordered field F. A *supremum* M of S is a least upper bound in the following sense.

(1) $M \ge x$ for all $x \in S$.
(2) If $B \in F$ is such that $B \ge x$ for all $x \in S$ then $M \le B$.

Definition 15 (Infimum)**.** Let S be a nonempty subset of an ordered field F. An *infimum* m of S is a greatest lower bound in the following sense.

(1) $m \le x$ for all $x \in S$.
(2) If $b \in F$ is such that $b \le x$ for all $x \in S$ then $b \le m$.

Remark 19. The plural of "supremum" is "suprema" and the plural of "infimum" is "infima". This reflects the Latin origin of these words.

Suprema are similar to maxima, but they do not have to be in the set S. Likewise infima are like minima. For example, if they exist then they are unique.

Exercise 26 (Easy)**.** Show that if S is a nonempty unbounded subset of an ordered field in the sense that S has no upper bounds, then S has no supremum. Note: sometimes in this case the supremum is defined to be ∞. Likewise a nonempty set with no lower bound is sometimes said to have infimum $-\infty$. *In this course we will say that the supremum or infimum do not exist in these cases.*

Exercise 27. Prove the following two theorems.

Theorem 55. *Let S be a nonempty subset of an ordered field F. If S has a supremum then it has a unique supremum. Similarly, if S has an infimum then it has a unique infimum.*

Theorem 56. *Let S be a nonempty subset of an ordered field F. If S has a maximum, then the supremum of S is the maximum. If S has a minimum, then the infimum of S is the minimum.*

Exercise 28. Prove the following corollary.

Corollary 57. *Let S be a nonempty subset of an ordered field F. If a bound is in the set S itself, then it must be an supremum or infimum. More precisely, if $M \in S$ is an upper bound, it must be the supremum and the maximum of S. If $m \in S$ is a lower bound, it must be the infimum and the minimum of S.*

Suprema and infima do not have to be in the set they bound, but they are "arbitrarily close" to the set. This is made precise in the following theorem:

Theorem 58. *Let S be a nonempty subset of an ordered field F. Then an upper bound M is the supremum of S if and only if the following is true: for all $\varepsilon > 0$ there is an element of S in the interval $(M - \varepsilon, M]$. Similarly a lower bound m is the infimum of S if and only if the following is true: for all $\varepsilon > 0$ there is an element of S in the interval $[m, m + \varepsilon)$.*

Proof. We prove the claim for upper bounds. The claim for lower bounds is similar. So let M be an upper bound. First we assume that for all $\varepsilon > 0$ there is an element of S in the interval $(M - \varepsilon, M]$. We will show that M is a supremum using the definition. This requires us to show that if B is an upper bound then $M \leq B$. Suppose otherwise that such an upper bound of S has the property that $B < M$. Since B is an upper bound, there are no elements of S in $(B, M]$. Let $\varepsilon = M - B$. So the interval $(B, M]$ is just the interval $(M - \varepsilon, M]$. But this interval has no elements of S, a contradiction.

Now suppose M is a supremum. We will show that if $\varepsilon > 0$ then the interval $(M - \varepsilon, M]$ intersects S. By the definition of supremum, $M - \varepsilon$ cannot be an upper bound since $M - \varepsilon < M$. Since $M - \varepsilon$ is not an upper bound, there is an element $x \in S$ such that $x > M - \varepsilon$ for some $x \in S$. Since M is an upper bound, $x \leq M$. Combining the inequalities gives us $x \in (M - \varepsilon, M]$ as desired. □

The above shows that suprema and infima are on the "boundary" of S in some sense. The following illustrates this as well with sequences. We state and prove the version for suprema, but of course there is a version for infima as well. We will need this theorem in the next chapter when we prove the intermediate value theorem.

Theorem 59. *Let S be a nonempty subset of an Archimedean ordered field F. If M is the supremum of S then M is the limit of a sequence (a_i) of elements $a_i \in S$ and is the limit of a sequence (b_i) of elements $b_i \notin S$. If $a < M$ is given we can assume that each a_i is in the interval $[a, M]$. Similarly, if $b > M$ is given we can assume that each b_i is in the interval $[M, b]$.*

Exercise 29. Prove the above theorem. Hint: to define a_i use the previous theorem with $\varepsilon = 1/i$. (For the refinement, choose $a_i = a$ if $M - 1/i < a$). Note: this a long exercise since it involves (i) defining a sequence (a_i), (ii) proving that (a_i) in in the correct interval and converges to M, (iii) showing that in step (i) we can assume $a \le a_i$, and (iv)-(vi) proving something similar for a sequence (b_i).

Suprema and infima do not always exist for bounded sets in $\mathbb{Q}$. So we cannot prove they exist in general (at least not without the extra completeness property, see Chapter 9). However, we can prove a partial result that will be useful when we study Cauchy sequences in Chapter 9. We prove the version for suprema only, but obviously an analogous result holds for infima as well. First a definition.

Definition 16. Let S be a nonempty subset of an ordered field F, and let $\varepsilon > 0$ be in F. An *ε-almost-supremum* A of S is an upper bound of S such that there is an $x \in S$ in the interval $(A - \varepsilon,\, A]$.

Theorem 60. *Let S be a nonempty subset of an Archimedean ordered field F, and let $\varepsilon > 0$ be in F. If S is bounded from above, then S has an ε-almost-supremum.*

Proof. Let $x_0 \in S$. Such a element exists since S was assumed to be nonempty. Let B be an upper bound of S. Since F is an Archimedean ordered field, there is an integer $n \in \mathbb{N}$ such that $n\,\varepsilon \ge (B - x_0)$ (Definition 10). This implies that $x_0 + \varepsilon n \ge B$, so $x_0 + \varepsilon n$ is an upper bound of S. By the well-ordering principle, there is a smallest $n_0 \in \mathbb{N}$ such that $x_0 + \varepsilon n_0$ is an upper bound of S. Let $A = x_0 + \varepsilon n_0$ for such an n_0. So A is an upper bound.

Observe that $A - \varepsilon = x_0 + \varepsilon(n_0 - 1)$ is not an upper bound of S. (There are two cases. If $n_0 = 0$ this is true since $x_0 > A - \varepsilon$. If $n_0 > 0$, this is true based on the choice of n_0). Thus there is an element x with $A - \varepsilon < x$. Since A is an upper bound, $x \le A$. So $x \in (A - \varepsilon, A]$. This means that A is an ε-almost-supremum. □

10. Embedding of $\mathbb{Q}$ (optional)

Let F be an ordered field. We can embed $\mathbb{N}$ into F using the idea mentioned in Remark 5. Informally, $n \in \mathbb{N}$ is mapped to $1 + 1 + \ldots + 1$ where the sum has n terms, and where $1 \in F$ is the multiplicative identity. Since F has additive inverses, we can extend this embedding to give an embedding of $\mathbb{Z}$ into F. Finally since F has multiplicative inverses for nonzero elements, we can extend this to an embedding of $\mathbb{Q} \to F$. These embeddings can all be shown to be injective, and so we can think of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ as subsets of F. We can show that $\mathbb{Q}$ is actually an ordered subfield of F. This gives a sketch of the argument that every ordered field contains $\mathbb{Q}$ as an ordered subfield.

Instead of making this a formal theorem, we just make the assumption that F containes $\mathbb{Q}$ whenever it is convenient. We can get away with this short-cut since the only ordered fields we will see, $\mathbb{Q}$ and $\mathbb{R}$, obviously contain $\mathbb{Q}$. However, it is nice to know that it is automatically true for ordered fields in general.

Once we know that every ordered field contains $\mathbb{Q}$ as an ordered subfield, we can simplify the definition of *Archimedean* by replacing the definition with the second requirement.