CHAPTER 11: EXPLORING $\mathbb{R}$

WAYNE AITKEN AND LINDA HOLT

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In this chapter we investigate some important properties of $\mathbb{R}$ that are a consequence of its completeness, and which fail for $\mathbb{Q}$. For example, every decimal expansion defines a real number, but not always a rational number. Also, for every positive integer n and every nonnegative real number x, there is a unique nonnegative nth root $x^{1/n}$. The existence of such roots often fails for rational numbers. We end the chapter by showing that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is uncountable.

1. Summary of properties of $\mathbb{R}$

We begin by collecting together for convenience some results about $\mathbb{R}$ that have already been proved.

Theorem 1. *The field $\mathbb{R}$ is a complete ordered field. In particular, every nonempty subset $S \subseteq \mathbb{R}$ which is bounded from above has a supremum (least upper bound). Likewise, every nonempty subset $S \subseteq \mathbb{R}$ which is bounded from below has a infimum (greatest lower bound).*

Proof. See Chapter 10 for the proof. The second statement is our definition of completeness from Chapter 9, and the third statement was proved to be a consequence of this definition. □

Theorem 2. *Every Cauchy sequence in $\mathbb{R}$ converges.*

Proof. See Chapter 10 for the proof (and Chapter 9 for the definition of Cauchy). □

Theorem 3. *The field $\mathbb{R}$ is an archimedian ordered field, and $\mathbb{Q}$ is a dense subfield of $\mathbb{R}$.*

Proof. In Chapter 9 we showed that complete ordered fields are archimedian. (We also proved this in Chapter 10 for $\mathbb{R}$ in particular). In Chapter 8 we proved that $\mathbb{Q}$ must be a dense subfield of any archimedian ordered field. □

Theorem 4 (Intermediate Value Theorem)**.** *Let $[a, b]$ be a closed interval in $\mathbb{R}$ where $a < b$ are elements of $\mathbb{R}$. Suppose $f\colon [a, b] \to \mathbb{R}$ is continuous. If $C \in \mathbb{R}$ is any value between $A = f(a)$ and $B = f(b)$ then there is an element $c \in [a, b]$ such that $f(c) = C$.*

Proof. We proved this in Chapter 9 for complete ordered fields containing $\mathbb{Q}$ as a subfield. □

Theorem 5. *Suppose $C \in \mathbb{R}$ and that $C \geq 0$. Then there is a $c \in \mathbb{R}$ such that $c^2 = C$.*

Proof. We proved this in Chapter 9 as a corollary to the intermediate value theorem. □

Theorem 6. *Let (a_i) be a bounded monotonic sequence in $\mathbb{R}$. Then (a_i) converges.*

Proof. We proved this in Chapter 9 for complete ordered fields. □

2. More results about sequences

In Chapter 8 we proved some limit laws. Here we add a few more.

Theorem 7. *Let $x \in \mathbb{R}$. If $x > 1$ then the sequence $(x^i)_{i \geq 1}$ of powers is an unbounded strictly increasing sequence of positive terms.*

Proof. By induction we can show $x^i > 1$ for all $i \geq 1$. This induction uses the following inequality:

$$x^{i+1} = x^i x > x^i 1 = x^i$$

This inequality also shows that (x_i) is a strictly increasing sequence.

Now we show that the sequence is unbounded. Suppose instead that (x^i) is bounded, so it is a monotonic bounded sequence. Since $\mathbb{R}$ is complete, this sequence would then converge (Theorem 6). All convergent sequences are Cauchy, so (x^i) would have to be Cauchy. Now observe that, for all $i \in \mathbb{N}$,

$$|x^{i+1} - x^i| = x^{i+1} - x^i = x^i(x-1) > 1(x-1) = x - 1.$$

This implies that (x^i) is not Cauchy (take $\varepsilon = x - 1$). This gives a contradiction. Thus the sequence (x^i) cannot be bounded. □

Exercise 1. Complete the above proof. (1) Show that if $i \geq 1$ then $x^i > 1$ by induction. (2) Explain why the sequence is not Cauchy.

Theorem 8. *If (x_i) is an unbounded increasing sequence of positive terms in $\mathbb{R}$ (or in any ordered field F), then the sequence $\left(x_i^{-1}\right)$ converges to 0.*

Proof. Let $\varepsilon > 0$ be given. Since (x_i) is unbounded, ε^{-1} cannot be an upper bound of (x_i). So there is a $k \in \mathbb{N}$ such that $x_k > \varepsilon^{-1}$. Hence $x_k^{-1} < \varepsilon$. If $i \geq k$ then $x_i \geq x_k$ since the sequence is increasing. So

$$\left|x_i^{-1} - 0\right| = x_i^{-1} \leq x_k^{-1} < \varepsilon.$$

We conclude that $\left(x_i^{-1}\right)$ converges to 0. □

Exercise 2. Combine the above theorems to show the following for $x \in \mathbb{R}$: (1) if $x > 1$ then the sequence (x^{-i}) converges to 0. (2) If $0 < x < 1$ then the sequence (x^i) converges to 0. Hint: the second follows from the first.

3. Decimal sequences

It is common to think of a real number as something that can be written as an infinite decimal, such as $3.14159\ldots$ or $1.41421\ldots$. Even rational numbers can be written in this way: $3/2 = 1.5000\ldots$ or $2/3 = 0.666666\ldots$. Our goal in this and the next few sections is to formally justify this view of real numbers. For convenience, we typically restrict our attention to nonnegative real numbers.

Definition 1 (Decimal sequence)**.** Suppose $n \in \mathbb{N}$ and $(d_i)_{i\geq 1}$ is a sequence where $d_i \in \{0, \ldots, 9\}$ for all $i \geq 1$. Then the sequence (s_i) whose ith term is

$$s_i = n + \sum_{j=1}^{i} \frac{d_j}{10^j}$$

is called a *decimal sequence*. It is a sequence of rational numbers.

Remark 1. The sequence (s_i) in the above definition is an example of a type of sequence called a *series*. Series are sequences defined in terms of summation. Each s_i is called a *partial sum* of the series, and the limit, if it exists, is called the *value* of the series.

Remark 2. The above definition describes the mathematical definition of a decimal sequence. Now we discuss notation used in practice. Let n, s_i, d_i be as in the above definition. Let N be the base 10 numeral representing n, and let D_i be the standard digit symbol representing d_i. Then the term s_1 is written $N.D_1$, the term s_2 is written $N.D_1D_2$, the term s_3 is written $N.D_1D_2D_3$ and so on. Notation such as

$$N.D_1D_2D_3D_4\ldots$$

is used to represent *the limit* of the above decimal sequence (s_i), which we will show always exists. It is the "$\ldots$" at the end indicates that we are referring to the limit (without the "$\ldots$", the express $N.D_1D_2D_3D_4$ would refer to s_4).

For example, $3.22222\ldots$ denotes the limit of (s_i) where

$$s_i = 3 + \sum_{j=1}^{i} \frac{2}{10^j}.$$

So $3.22222\ldots$ is the limit of the sequence with terms 3, 3.2, 3.22, 3.222, $\ldots$. Given facts about geometric series, one can show that $\sum_{j=1}^{i} \frac{2}{10^j}$ has limit equal $2/9$, so (s_i) has limit $3 + 2/9$. Thus $3.22222\ldots$ is $29/9$.

Obviously expressions such as $N.D_1D_2D_3D_4\ldots$ do not itself give full information about the decimal sequences or limits they represent. If there is an obvious pattern in the digits given, then the reader is expected to assume that the pattern continues. For example, the expression $3.1454545\ldots$ would suggest to the reader that $d_i = 4$ for even $i \geq 2$ and $d_i = 5$ for odd $i \geq 3$. Even if there is no such pattern, the the convention is that is that the number of digits expressed is enough to approximate the number at hand, and that further digits are not important (or not known) for the discussion.

If there is a $k \in \mathbb{N}$ such that $d_i = 0$ for all $i \geq k$ then s_i is constant for $i \geq k$ and we say that the decimal sequence "terminates". The limit is then equal to the kth term s_k, and we can write the number with a terminating decimal. For example, 7.3450 can be used to represent the limit when $n = 7, d_1 = 3, d_2 = 4, d_3 = 5, d_4 = 0$ and where we implicitly assume $d_k = 0$ for $k \geq 5$. Of course that same number could be written as 7.345 or 7.34500 or even 7.34500... . As we will see later, this number can also be written as 7.3449999....

Remark 3. There is nothing sacred about base 10. We can easily replace 10 with another integer $B > 1$ in Definition 1, and insist that

$$d_i \in \{0, \ldots, B-1\}.$$

This would give us base B expansions of real numbers.

Informal Exercise 3. What rational number is represented by 3.22000...? If we are using base $B = 4$ notation, which rational number does 3.22000... represent? (Write your answers as fractions in terms of two natural numbers given in base 10).

Informal Exercise 4. What rational number is represented by 2.2111... in base 10. If we are using base $B = 4$ notation, which rational number is expressed by 2.2111.... (Write your answers as fractions in terms of two natural numbers given in base 10).

Now we establish that any decimal sequence is bounded and monotonic.

Theorem 9. *Suppose $n \in \mathbb{N}$ and $(d_i)_{i \geq 1}$ is a sequence where $d_i \in \{0, \ldots, 9\}$ for all $i \geq 1$. Then the sequence (s_i) whose ith term is given by*

$$s_i = n + \sum_{j=1}^{i} \frac{d_j}{10^j}$$

is increasing with upper bound $n+1$ and lower bound n. In particular, it is is bounded and monotonic.

Proof. Since $d_i \leq 9$ we have

$$\sum_{j=1}^{i} \frac{d_j}{10^j} \quad \leq \quad \sum_{j=1}^{i} \frac{9}{10^j}.$$

(This can be rigorously shown using induction). So, by Lemma 10 (below),

$$\sum_{j=1}^{i} \frac{d_j}{10^j} \quad \leq \quad 1.$$

Adding n gives $s_i \leq n + 1$. So $n + 1$ is an upper bound for the sequence.

Since $d_{i+1} \geq 0$, we can show that n is a lower bound and that

$$s_{i+1} = s_i + \frac{d_{i+1}}{10^{i+1}} \geq s_i$$

for all i. Thus (s_i) is increasing. □

The following is a special case of the formula for geometric series.

Lemma 10. *For all i,*

$$\sum_{j=1}^{i} \frac{9}{10^j} = 1 - \frac{1}{10^i}.$$

Proof. Let $s = \sum_{j=1}^{i} \frac{9}{10^j}$. By properties of summations and powers,

$$\frac{s}{10} = \frac{1}{10}\sum_{j=1}^{i} \frac{9}{10^j} = \sum_{j=1}^{i} \frac{9}{10^{j+1}} = \sum_{j=2}^{i+1} \frac{9}{10^j} = \sum_{j=2}^{i} \frac{9}{10^j} + \frac{1}{10}\frac{9}{10^i}.$$

This is similar to the expression for s. In fact, we can write s as follows:

$$s \quad = \quad \frac{9}{10} + \sum_{j=2}^{i} \frac{9}{10^j}.$$

When we take the difference, the term $\sum_{j=2}^{i} \frac{9}{10^j}$ cancels giving us

$$s - \frac{s}{10} = \frac{9}{10} - \frac{1}{10}\frac{9}{10^i} = \frac{9}{10}\left(1 - \frac{1}{10^i}\right).$$

Multiply both sides by 10, then divide by 9. The result follows. □

Corollary 11. *Suppose $n \in \mathbb{N}$ and $(d_i)_{i \geq 1}$ is a sequence where $d_i \in \{0, \ldots, 9\}$ for all $i \geq 1$. Then the sequence (s_i) whose ith term is given by*

$$s_i = n + \sum_{j=1}^{i} \frac{d_j}{10^j}$$

converges to a real number $\mathbb{R}$. More specifically, it converges to a real number x with $n \leq x \leq n+1$.

Proof. By Theorem 9, (s_i) is bounded and monotonic. Since $\mathbb{R}$ is a complete field, this implies that (s_i) has a limit x (Theorem 6). Since

$$n \leq s_i \leq n + 1$$

we have $n \leq x \leq n+1$ by basic limit laws (Chapter 8). □

Remark 4. This shows that every decimal sequence defines a real number. For example, 3.17117111711117... defines a real number between 3 and 4.

4. Decimal expansions

In this section we consider the converse to the problem in the previous section. We establish that every nonnegative real number is the limit of a decimal sequence. The decimal sequence giving x as its limit is called the *decimal expansion* of x.

Theorem 12. *Every nonnegative real number is the limit of a decimal sequence. In other words, every nonnegative real number has a decimal expansion.*

Proof. Let x be a nonnegative real number. We divide the proof into three steps. First we define a sequence (a_i) of rational numbers recursively in terms of the given x. Next we show that (a_i) is a decimal sequence. Finally we show that (a_i) converges to x.

We know $\mathbb{R}$ is archimedean, so by a result of Chapter 8 there is a unique integer n such that $n \leq x < n+1$. We define a_0 to be n.

Now suppose that $a_i \in \mathbb{Q}$ has been defined. We will now define a_{i+1} in terms of a_i. Consider

$$y = 10^{i+1}(x - a_i).$$

By the aforementioned result of Chapter 8 there is a unique integer d such that $d \leq y < d+1$. Now define a_{i+1}:

$$a_{i+1} \stackrel{\text{def}}{=} a_i + \frac{d}{10^{i+1}}.$$

Next multiply the terms occurring in the inequality $d \leq y < d+1$ by $1/10^{i+1}$ and simplify to observe that

$$a_{i+1} \leq x < a_{i+1} + \frac{1}{10^{i+1}}.$$

This process recursively defines a sequence $(a_i)_{i \geq 0}$ of rational numbers. We did so in such a way that $a_i \leq x < a_i + 1/10^i$ holds for all $i \geq 0$. In other words, we have

$$0 \leq x - a_i < \frac{1}{10^i}, \qquad \text{so} \quad 0 \leq 10^{i+1}(x - a_i) < 10.$$

In order to identify the digits, we select d_{i+1} to be the integer d that arises in the above definition. So for $i \geq 0$, let d_{i+1} be the unique integer such that

$$d_{i+1} \leq 10^{i+1}(x - a_i) < d_{i+1} + 1.$$

Equivalently, d_{i+1} is the largest integer less than or equal to $10^{i+1}(x - a_i)$, and, as established above, $0 \leq 10^{i+1}(x - a_i) < 10$. In other words, for each $i \geq 1$, we have $d_i \in \{0, \ldots, 9\}$. This completes the first part of the proof: we have defined (a_i) for $i \geq 0$ and the associated digits d_i for $i \geq 1$.

Our next step is to show that (a_i) is a decimal sequence. We do so by establishing that

$$a_i = a_0 + \sum_{j=1}^{i} \frac{d_j}{10^j}.$$

Observe that this can be shown by induction. With this established, we see that (a_i) is a decimal sequence. This completes the second part of the proof.

Finally, we show that (a_i) converges to x. By Corollary 11, (a_i) has a real limit, call it s. Above we established

$$a_i \le x < a_i + 1/10^i$$

for all $i \ge 1$. By previously established limit laws, we get $s \le x \le s + 0$. Thus $x = s$. So (a_i) has limit x as desired. □

Exercise 5. Which limit laws were used to show that $s \le x \le s + 0$ follows from $a_i \le x < a_i + 1/10^i$? (Hint: one law used was an exercise from this Chapter).

5. Uniqueness of decimal expansions

Some numbers have two distinct decimal expansions, but in other cases the expansions are unique. For example, 0.13999999... and 0.1400000... are two representations for the same number 7/50. However, 0.22222222... represents the unique expansion of 2/9.

In what sense is the decimal expansion of a nonnegative $x \in \mathbb{R}$ unique? In this section we will consider this question of uniqueness of decimal expansions.

Definition 2. Suppose $n \in \mathbb{N}$ and $(d_i)_{i \ge 1}$ is a sequence where $d_i \in \{0, \dots, 9\}$ for all $i \ge 1$, and that (s_i) is the decimal sequence whose ith term is given by

$$s_i = n + \sum_{j=1}^{i} \frac{d_j}{10^j}.$$

We call this a *nine-sequence* if there is a k such that $d_i = 9$ for all $i \ge k$. We call this a *zero-sequence* if there is a k such that $d_i = 0$ for all $i \ge k$.

Remark 5. The number 14.563599999... is the limit of a nine-sequence, but (by the uniqueness result below) the number 14.599999999111... is not, nor is 14.999999000....

The number 14.56360000... is the limit of a zero-sequence. As mentioned above we can write 14.5636 or 14.56360 for this number. For such a terminating epression, it is assumed that the digits beyond the termination point are 0, and that the associated decimal sequence is then a zero-sequence.

Theorem 13 (Non-uniqueness). *Suppose the sequence with ith term*

$$s_i = d_0 + \sum_{j=1}^{i} \frac{d_j}{10^j}$$

is a nine-sequence, and let k be the least integer such that $d_i = 9$ for all $i > k$. Then (s_i) converges to the same number as the zero-sequence (s'_i) where

$$s'_i = d'_0 + \sum_{j=1}^{i} \frac{d'_j}{10^j}$$

and where (d_i) is defined as follows: $d'_i = d_i$ if $i < k$, but $d'_k = 1 + d_k$, and $d'_i = 0$ if $i > k$.

Proof. Let x be the limit of (s_i), and let x' be the limit of (s'_i). Observe that for $i > k$,

$$\begin{aligned} s_i &= \sum_{j=0}^{k-1} \frac{d_j}{10^j} + \frac{d_k}{10^k} + \sum_{j=k+1}^{i} \frac{9}{10^j} \\ &= \sum_{j=0}^{k-1} \frac{d'_j}{10^j} + \frac{d'_k - 1}{10^k} + \sum_{j=1}^{i-k} \frac{9}{10^{j+k}} \\ &= \left(\sum_{j=0}^{k-1} \frac{d'_j}{10^j} + \frac{d'_k}{10^k} \right) - \frac{1}{10^k} + \frac{1}{10^k} \sum_{j=1}^{i-k} \frac{9}{10^j} \\ &= \left(\sum_{j=0}^{k} \frac{d'_j}{10^j} \right) - \frac{1}{10^k} + \frac{1}{10^k} \left(1 - \frac{1}{10^{i-k}} \right) \\ &= s'_k - \frac{1}{10^k} + \frac{1}{10^k} - \frac{1}{10^k} \frac{1}{10^{i-k}} \\ &= x' - \frac{1}{10^i}. \end{aligned}$$

The limit of (10^{-i}) is 0 by an earlier exercise. So $x = x'$. □

Remark 6. The above shows that a nine-sequence can be replaced by a zero-sequence representing the same real number. For example, $11.34999\ldots$ gives the same real number as $11.35000\ldots$ (where $k = 2$) and $9.99999\ldots$ gives the same result as $10.0000\ldots$ (where $k = 0$).

Theorem 14. *Consider two convergent sequences (s_i) and (t_i) defined in terms of sums:*

$$s_i = \sum_{j=l}^{i} a_j, \qquad t_i = \sum_{j=l}^{i} b_j.$$

Write A for the limit of (s_i) and B for the limit of (t_i). If $a_j \le b_j$ for all $j \ge l$, then $A \le B$. If in addition some $a_j < b_j$, then $A < B$.

Proof. Observe that $s_k \le t_k$ for all $k \ge l$ by induction. So in the limit $A \le B$ (See Chapter 8).

Now assume in addition that $a_j < b_j$ for a specific integer j. If $j > l$ then

$$s_j + (b_j - a_j) = (s_{j-1} + a_j) + (b_j - a_j) = s_{j-1} + b_j \le t_{j-1} + b_j = t_j.$$

If, on the other hand, $j = l$, then

$$s_j + (b_j - a_j) = a_j + (b_j - a_j) = b_j = t_j.$$

In any case, $s_j + (b_j - a_j) \le t_j$. By induction, this extends to $j \ge k$:

$$s_k + (b_j - a_j) \le t_k.$$

So in the limit. $A + (b_j - a_j) \le B$. Thus $A < B$ since $b_j - a_j > 0$. □

Theorem 15 (Comparison)**.** *Suppose we have two decimal sequences that differ starting in the kth digit. More precisely, suppose*

$$s_i = d_0 + \sum_{j=1}^{i} \frac{d_j}{10^j}, \qquad s'_i = d'_0 + \sum_{j=1}^{i} \frac{d'_j}{10^j}$$

such that there is a $k \in \mathbb{N}$ *where* $d_i = d'_i$ *if* $i < k$ *but* $d_k > d'_k$*. Let* S *and* S' *be the respective limits of* (s_i) *and* (s'_i)*. Then* $S \ge S'$*.*

Equality $S = S'$ *holds if and only if (i)* $d_k = d'_k + 1$*, (ii)* (s_i) *is a zero-sequence with* $d_i = 0$ *for all* $i > k$ *and (iii)* (s'_i) *is a nine-sequence with* $d'_i = 9$ *for all* $i > k$*.*

Proof. Define (t_i) by

$$t_i = e_0 + \sum_{j=1}^{i} \frac{e_j}{10^j}$$

where $e_i = d_i$ if $i \le k$, but where $e_i = 0$ if $i > k$. Let T be the limit of (t_i). Then $S \ge T$ by Theorem 14 where equality holds if and only if $(s_i) = (t_i)$ as sequences.

Define (t'_i) by

$$t'_i = e'_0 + \sum_{j=1}^{i} \frac{e'_j}{10^j}$$

where $e_i = d'_i$ if $i < k$, but where $e'_k = d_k - 1$ and $e'_i = 9$ if $i > k$. Let T' be the limit of (t'_i). Then $T' \ge S'$ by Theorem 14 where equality holds if and only if $(s'_i) = (t'_i)$ as sequences.

Finally, $T = T'$ by Theorem 13. Thus

$$S \ge T = T' \ge S'$$

with equality $S = T = T' = S'$ if and only if $(s_i) = (t_i)$ and $(s'_i) = (t'_i)$. The result follows. □

This leads to the main uniqueness results described in the following three corollaries.

Corollary 16. *Every nonnegative real number has a unique non nine-sequence decimal expansion.*

Corollary 17. *Every nonnegative real number has a unique non zero-sequence decimal expansion.*

Corollary 18. *Suppose x is a nonnegative real number with a decimal expansion that is neither a nine-sequence nor a zero-sequence. Then the decimal expansion of x is unique.*

Informal Exercise 6. Write $3/2$ using two different decimal expansions. What is the unique non nine-sequence representing $3/2$?

Informal Exercise 7. Write $14.999999000\ldots$ in terms of a nine-sequence.

6. Basic inequalities for kth powers

We now establish a tool-kit of useful results used later in the chapter.

Lemma 19. *Let x, y be positive elements of an ordered field F. Then x^n and y^n are also positive for all $n \in \mathbb{N}$. Furthermore, if $n \geq 1$ then*

$$x^n \leq y^n \iff x \leq y$$

and

$$x^n < y^n \iff x < y.$$

Proof. If $x \in F$ is positive, then observe that x^n is positive for $n \geq 0$ (by induction using closure of the positive subset $P \subseteq F$, and using the fact that 1 is positive in the base case).

Observe next that if $x \leq y$, then $x^n \leq y^n$ for $n \geq 0$ (also by induction). Similarly, if $x < y$ then $x^n < y^n$ for $n \geq 1$ (by induction starting at $n = 1$).

Suppose $x^n \leq y^n$ and $n \geq 1$. If $y < x$ then $y^n < x^n$ by the above. This is a contradiction to trichotomy. Thus $x \leq y$.

Suppose $x^n < y^n$ and $n \geq 1$. If $y \leq x$ then $y^n \leq x^n$ by the above. This is a contradiction to trichotomy. Thus $x < y$. □

Lemma 20. *Let x, y be nonnegative elements of an ordered field F. Let $n \in \mathbb{N}$. Then x^n and y^n are also nonnegative. Furthermore, if $n \geq 1$, then*

$$x^n \leq y^n \iff x \leq y.$$

Proof. The case where x and y are both positive is covered by Lemma 19. Observe that if one or both of x, y is zero then previously established facts about 0 give the conclusion. □

Exercise 8. Give details in the above proof where (i) $x = 0$, and (ii) $y = 0$.

Lemma 21. *Let x, y be nonnegative elements of an ordered field F. If n is a positive integer then*

$$x^n = y^n \iff x = y.$$

Proof. The direction $\Leftarrow$ follows from properties of equality: if $x = y$ then we have $x^n = y^n$ by substitution.

So suppose $x^n = y^n$. Then $x^n \leq y^n$, hence $x \leq y$ by the previous lemma. Likewise, $y \leq x$. So $x = y$. □

7. Existence of nth roots

In this section, we will establish that every nonnegative $x \in \mathbb{R}$ has a unique nonnegative nth root in $\mathbb{R}$.

Definition 3 (nth root)**.** Let F be a field, and let n be a positive integer. If $x^n = X$ where $x, X \in F$, then we say that x is an *nth root* of X.

In the special case where $n = 2$, then x is called a *square root* of X. In the special case where $n = 3$, then x is called a *cube root* of X.

First we show existence of an nth root for $C \geq 1$.

Lemma 22. *Let n be a positive integer. If $C \geq 1$ is a real number, then there is a positive real number c such that $c^n = C$.*

Proof. Let $b = C$. Observe, using induction for $n \geq 1$, that $f(x) = x^n$ defines a continuous function $[0, b] \to \mathbb{R}$ and that since $C \geq 1$,

$$C \leq C^n.$$

The above is for all $n \geq 1$. Now fix n, and let $f(x) = x^n$. Let $A = f(0)$ and let $B = f(b)$. Observe that $A = f(0) = 0$, so $A \leq C$. Since $b = C$, we have $B = f(b) = C^n$. So $C \leq B$, since $C \leq C^n$. Since $A \leq C \leq B$, there is a real $c \in [0, b]$ such that $f(c) = C$ by the Intermediate Value Theorem (Chapter 9). Since $f(c) = c^n$, the number c has the desired property. □

Exercise 9. Give details for two steps in the first paragraph of the above proof: (1) Show that $f(x) = x^n$ defines a continuous function (hint: consider $g(x) = x$ the identity function. What is g^n *in the ring of continuous functions*?) (2) Show that if $C \geq 1$ then $C^n \geq C$ for all $n \geq 1$.

We need another lemma to handle roots of real numbers less than one.

Lemma 23. *Let n be a positive integer. If $0 < X < 1$ is a real number, then there is a positive real number x such that $x^n = X$.*

Proof. Let $C = 1/X$. Observe that $C > 1$. So there is a positive real number c with $c^n = C$ (by the previous lemma). Let $x = 1/c$. Observe that $x^n = 1^n/c^n = 1/C = X$ as desired. □

Theorem 24 (nth roots)**.** *Let $X \in \mathbb{R}$ be nonnegative and let $n \in \mathbb{N}$ be postive. Then X has a unique nonnegative nth root x. If X is positive then so is x.*

Proof. If $X > 0$ then the existence of a positive nth root follows from the previous two lemmas. If $X = 0$ then $x = 0$ is an nth root.

To show uniqueness, suppose x_1 and x_2 are two nth roots. Then

$$x_1^n = X = x_2^n.$$

By Lemma 21, $x_1 = x_2$ as desired. So uniqueness holds. □

Definition 4. If x is a nonnegative real number and if $n \in \mathbb{N}$ is a positive integer, then $x^{1/n}$ is defined to be the unique nonnegative nth root of x. We sometimes write $x^{1/n}$ as $\sqrt[n]{x}$.

If x is a nonnegative real number, then $\sqrt{x}$ is defined to be the unique nonnegative square root of x. In other words, $\sqrt{x}$ is $x^{1/2}$.

Definition 5. An *irrational* real number is an element of $\mathbb{R}$ that is not in $\mathbb{Q}$.

Remark 7. Suppose that b is a natural number that is not of the form a^n for some $a \in \mathbb{N}$. Then one can show that $b^{1/n}$ is irrational. We will not prove this here, but it can be easily proved using basic number theory.

For example 6 is not of the form a^3 with $a \in \mathbb{N}$, in other words 6 is not a cube. So $6^{1/3}$ can be shown to be irrational.

8. Fractional powers

Here we give a few properties of fractional powers.

Theorem 25. *Let $x, y \in \mathbb{R}$ be nonnegative, and let $n \in \mathbb{N}$ positive. Then*

$$(xy)^{1/n} = x^{1/n}y^{1/n}.$$

Proof. Let $v = x^{1/n}$ and $w = y^{1/n}$. By Definition 4, $v^n = x$ and $w^n = y$, and v and w are nonnegative. By closure properties, vw is nonnegative, and by properties of commutative rings,

$$(vw)^n = v^n w^n = xy.$$

Thus vw is the nonnegative nth root of xy. So $(xy)^{1/n} = vw = x^{1/n}y^{1/n}$. □

Exercise 10. Prove the following.

Theorem 26. *Let $x, y \in \mathbb{R}$ be nonnegative, and let $n \in \mathbb{N}$ positive. Then*

$$\left(x^{1/n}\right)^n = x, \qquad \text{and} \qquad (x^n)^{1/n} = x.$$

Definition 6 (Fractional powers)**.** Suppose x is a nonnegative real number, and p/q is a positive rational number with p, q positive integers. Then

$$x^{p/q} \stackrel{\text{def}}{=} (x^p)^{1/q}.$$

Lemma 27. *The above definition is well-defined: it does not depend on the choice of numerator and denominator used to represent the given rational number.*

Proof. Suppose that $p/q = r/s$ where p, q, r, s are positive integers. We must show that

$$(x^p)^{1/q} = (x^r)^{1/s}.$$

Let $v = (x^p)^{1/q}$ and $w = (x^r)^{1/s}$. Observe that

$$v^{qs} = x^{ps} \qquad \text{and} \qquad w^{qs} = x^{rq}.$$

Since $p/q = r/s$, we have $ps = qr$. Thus $v^{qs} = w^{qs}$. By Lemma 21, $v = w$ as desired. □

Theorem 28. *Suppose x is a nonnegative real number, and p/q is a positive rational number (with p, q positive integers). Then*

$$x^{p/q} = (x^p)^{1/q} = \left(x^{1/q}\right)^p$$

Proof. The first equality is true by definition. To establish $(x^p)^{1/q} = \left(x^{1/q}\right)^p$, raise both sides to the same power q. Both sides simplify to give the same answer, namely x^p. Now use Lemma 21. □

Remark 8. For example, you can compute $8^{2/3}$ in two ways. The first method starts with $8^2 = 64$. Then you take the cube root, which is 4. In the second method you take the cube root of 8. This is 2. Next square it. This gives 4. Of course, both methods give the same answer.

9. Roots in $\mathbb{R}$

Above we considered only nonnegative nth roots of nonnegative real numbers. In this case we have existence and uniqueness. When we look at *all* real numbers we experience problems with existence and uniqueness when n is even. The case when n is odd works out better.

Lemma 29. *Suppose n is a positive integer. Then 0 has exactly one nth root. That root is 0.*

Exercise 11. Use the fact that $\mathbb{R}$ is a field to show that if $x \neq 0$ then $x^n \neq 0$ for all $n \geq 1$. Now prove the above lemma.

Theorem 30 (Roots for even exponents)**.** *Suppose n is a positive even integer, and that $x \in \mathbb{R}$.*

If $x > 0$, then x has exactly two nth roots: $x^{1/n}$ and $-x^{1/n}$.
If $x = 0$ then x has exactly one nth root. That root is 0.
If $x < 0$ then x has no nth roots.

Proof. Write n as $2m$.

First suppose that x is positive. Then $x^{1/n}$ is a positive nth root. Consider the negative real number $-x^{1/n}$. Then

$$\left(-x^{1/n}\right)^n = (-1)^n \left(x^{1/n}\right)^n = \left((-1)^2\right)^m x = 1^m \cdot x = x.$$

So $-x^{1/n}$ is a second nth root. Suppose y is a third nth root. Observe that y cannot be positive by the uniqueness claim of Theorem 24. $y \neq 0$ since $x \neq 0$. So y is negative. This implies that $-y$ is positive. Observe that

$$(-y)^n = y^n = x.$$

Thus $-y$ is the positive nth root $x^{1/n}$. This implies that $y = -x^{1/n}$. So there is no distinct third nth root.

The case of $x = 0$ is covered by the previous lemma.

Finally, suppose $x < 0$. If $y \in \mathbb{R}$, then y^2 is nonnegative. So $y^n = \left(y^2\right)^m$ is nonnegative. So $y^n \neq x$. Thus x has no nth roots. □

Lemma 31. *Suppose n is a positive odd integer. If $x < 0$ then $x^n < 0$.*

Exercise 12. Prove the above. Hint: write $n = 2m + 1$. (There is no need to use induction).

Theorem 32 (Roots for odd exponents)**.** *Suppose n is an odd integer, and $x \in \mathbb{R}$. Then x has a unique nth root.*

If $x < 0$ then the unique nth root of x is $-|x|^{1/n}$.

Proof. As in the above lemma, if $y < 0$ then y^n is negative. This shows that if $x \geq 0$, then x cannot have any negative nth roots. But we know x has a unique nonnegative square root. Thus the theorem holds for $x \geq 0$.

If $x < 0$, then the nth power of $-|x|^{1/n}$ is equal to $-|x|$. But $-|x| = x$ in this case. So $-|x|^{1/n}$ is an nth root. Suppose y is another nth root. Then $(-y)^n = -x = |x|$. By the uniqueness claim for nonnegative reals demonstrated above, $-y = |x|^{1/n}$. Thus $y = -|x|^{1/n}$. So the nth root is unique. □

10. Countability and uncountability

We have encountered several differences between $\mathbb{Q}$ and $\mathbb{R}$. We consider one more very important difference: $\mathbb{Q}$ is countable, but $\mathbb{R}$ is not.

Definition 7. A nonempty set S is *countable* if there is a surjection $\mathbb{N} \to S$. The empty set is also considered to be countable. If S is nonempty and no such surjection exists, then S is said to be *uncountable*.

Remark 9. The surjective function $\mathbb{N} \to S$ described above is called a *counting function*. We do not require it to be a bijective since we want to consider finite S where a function $\mathbb{N} \to S$ cannot be a bijection.

However, it turns out that if S is infinite, one can show that the existence of a surjective $f\colon \mathbb{N} \to S$ implies the existence of a bijective $f'\colon \mathbb{N} \to S$. We will not need this result here, though.

Exercise 13. Show that every finite set is countable. Show that $\mathbb{N}$ and $\mathbb{Z}$ are countable. Hint: for $\mathbb{Z}$, define a function $\mathbb{N} \to \mathbb{Z}$ that sends each even $2k \in \mathbb{N}$ to k and sends each odd $2k - 1 \in \mathbb{N}$ to $-k$.

Theorem 33. *The set or real numbers $\mathbb{R}$ is uncountable.*

Proof. We show that no function $f\colon \mathbb{N} \to \mathbb{R}$ can be surjective. We do so by taking any given $f\colon \mathbb{N} \to \mathbb{R}$, and finding a real number that is not in the image. So let $f : \mathbb{N} \to \mathbb{R}$ be given. We can think of f as providing a list or sequence of real numbers: $f(0), f(1), f(2), \ldots$. The goal is to construct a decimal sequence giving a new real number not on the list.

Let n be a positive integer greater than $f(0)$. For each $i \geq 1$ let d_i be the ith digit in the decimal expansion of $|f(i)|$. In other words d_i serves as the digit for the 10^{-i} place of the expansion. (For definiteness, we choose the decimal expansion to be the unique non nine-sequence).

Now define $d'_i = 5$ if $d_i \neq 5$, but choose $d'_i = 7$ if $d_i = 5$. Consider the decimal sequence defined by

$$s_i = n + \sum_{j=1}^{i} \frac{d'_j}{10^j}.$$

As we proved above, this defines a real number x between n and $n+1$. Note that x is positive since $x > n > 0$.

Observe that $x \neq f(0)$ since $x \geq n > f(0)$. If $i \geq 1$ and $f(i) \geq 0$ then the decimal expansion of x and $f(i)$ differ in the 10^{-i} position: the first has coefficient d'_i and the second d_i. By uniqueness of decimal expansions for non nine-sequences, $x \neq f(i)$. If $f(i)$ is negative then $x \neq f(i)$ since x is positive.

In any case, $x \neq f(i)$ for each $i \geq 0$. Thus x is not in the image of f. Thus f cannot be surjective. □

In contrast $\mathbb{Q}$ is countable. This is surprising at first since $\mathbb{Q}$ is dense in $\mathbb{R}$. The first step is to recall that each $\mathbb{Q}$ can be written as a fraction a/b where $b \geq 0$ and where $a, b \in \mathbb{Z}$. It is somewhat surprising at first that $\mathbb{Z} \times \mathbb{Z}$ is countable. For our needs, we can focus on the subset Q, and show it is countable:

$$Q \stackrel{\text{def}}{=} \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \geq 1\}.$$

Lemma 34. *There is a bijection $\mathbb{N} \to Q$. In particular, the set Q is countable.*

Proof. (Informal) We show this by constructing a sequence of points in Q in such a way that every element of Q occurs eventually in the sequence. There are several reasonable ways of doing this. One way is to proceed as follows: start with

$$(-1, 1), (0, 1), (1, 1)$$

then continue with

$$(-2, 1), (-2, 2), (-1, 2), (0, 2), (1, 2), (2, 2), (2, 1)$$

then continue with

$$(-3, 1), (-3, 2), (-3, 3), (-2, 3), \ldots, (2, 3), (3, 3), (3, 2), (3, 1)$$

and so on. In the nth group we consider the subset of pairs (a, b) where the max of $|a|$ and b is equal to n. One can show that there are $4n - 1$ terms in each group, but what is important in this proof is that each has a finite number of terms. Combine these finite sequences into one infinite sequence:

$$(-1, 1), (0, 1), (1, 1), (-2, 1), (-2, 2), (-1, 2), (0, 2), (1, 2), (2, 2), \ldots,$$

If we define $f \colon \mathbb{N} \to \mathbb{Q}$ by sending $f(n)$ to the nth term of the sequence (where the 0th term is the start of the sequence), then this gives a bijection $\mathbb{N} \to \mathbb{Q}$. □

Theorem 35. *The set $\mathbb{Q}$ is countable.*

Proof. Start with any surjection $f\colon \mathbb{N} \to Q$, for example the map from the previous lemma. Define a function $g\colon \mathbb{N} \to \mathbb{Q}$ by the rule that $g(n) = a/b$ where $(a, b) = f(n)$.

We show that every $r \in \mathbb{Q}$ is in the image of g, so g is surjective. Since r is rational, it can be written as a/b for some $(a, b) \in Q$. Since f is surjective, there is an $n \in \mathbb{N}$ such that $f(n) = (a, b)$. Thus

$$g(n) = a/b = r.$$

In other words, r is in the image of g. □

Appendix: a rigorous proof of Lemma 34

In this appendix we outline a rigorous proof of the existence of a bijection $\mathbb{N} \to Q$ where

$$Q \stackrel{\text{def}}{=} \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \geq 1\}.$$

First we define an order on Q. It will be a bit different than the order described informally above, but it is easier to describe and work with. Given pairs $(a_1, b_1), (a_2, b_2) \in Q$ we define $(a_1, b_1) < (a_2, b_2)$ to hold if and only if one of the following occurs

(i) $\max(|a_1|, b_1) < \max(|a_2|, b_2)$.

(ii) $\max(|a_1|, b_1) = \max(|a_2|, b_2)$ and $a_1 < a_2$.

(iii) $\max(|a_1|, b_1) = \max(|a_2|, b_2)$ and $a_1 = a_2$ and $b_1 < b_2$.

Observe that $(-1, 1)$ is the minimum element of Q, but that Q has no maximum with this order. We show that $<$ is indeed an order relation:

Lemma 36. *The relation $<$ is a strict linear order on Q.*

Proof. For transitivity, assume $(a_1, b_1) < (a_2, b_2)$ and $(a_2, b_2) < (a_3, b_3)$. The condition $(a_1, b_1) < (a_2, b_2)$ divides into three cases, and the condition $(a_2, b_2) < (a_3, b_3)$ divides into three cases. For each of the nine combined possibilities, it is immediate that $(a_1, b_1) < (a_3, b_3)$.

In order to prove the trichotomy property for $(a_1, b_1), (a_2, b_2) \in Q$ divide into cases: $\max(|a_1|, b_1) \neq \max(|a_2|, b_2)$ or $\max(|a_1|, b_1) = \max(|a_2|, b_2)$. In the later case divide further into subcases: $a_1 \neq a_2$ or $a_1 = a_2$. □

By the above lemma, Q is an ordered set with $<$.

Lemma 37. *With the order $<$ defined above, Q is well-ordered.*

Proof. Let S be a nonempty subset of Q. We will show that S has a minimum. First consider the set T_1 of natural numbers that can be written as $\max(|a|, b)$ for some $(a, b) \in S$. Note that T_1 has a minimum t since $\mathbb{N}$ is well-ordered (Chapter 2). Let S_1 be the subset of S consisting of pairs $(a, b) \in S$ such that $t = \max(|a|, b)$. Observe that if $(a, b) \in S_1$ then $(a, b) < (a', b')$ for each $(a', b') \in S - S_1$ since $\max(|a'|, b')$ must be strictly greater than t.

Now let A be the following set:

$$A \stackrel{\text{def}}{=} \{a \in \mathbb{Z} \mid \text{there is a } b \text{ with } (a,b) \in S_1\}.$$

Observe that A is a nonempty subset of $\mathbb{Z}$ and has lower bound $-t$. Thus it must have a minimum a_0 (by a theorem of Chapter 4). Let S_2 be the set of pairs $(a_0, b) \in S_1$. Observe that if $(a_0, b) \in S_2$ then $(a_0, b) < (a', b')$ for each $(a', b') \in S_1 - S_2$ since $a_0 < a$. Combining with a previous result we get, in fact, that $(a_0, b) < (a', b')$ for each $(a_0, b) \in S_2$ and $(a', b') \in S - S_2$.

Observe also that S_2 is nonempty, so the set

$$B \stackrel{\text{def}}{=} \{b \in \mathbb{N} \mid (a_0, b) \in S_2\}$$

is nonempty. Let b_0 be its minimum. Observe that if (a_0, b_0) is the minimum of S_2. Conclude that (a_0, b_0) is actually the minimum of all of S. □

Now we are ready to define a bijection $f\colon \mathbb{N} \to Q$. We define this recursively by the conditions (i) $f(0) = (-1, 1)$ and (ii) $f(n+1)$ is the smallest element of Q strictly greater than $f(n)$. Condition (ii) is well-defined since Q is well-ordered and has no maximum. So by the principle of recursive definition, f is defined.

We still need to show that f is bijective. By induction we can show, for any fixed $n \in \mathbb{N}$, that $f(n+k) > f(n)$ for all $k \geq 1$. A corollary of this is that f is injective.

Suppose that $(a, b) \in Q$ is not in the image of f. By induction we can show that $f(n) < (a, b)$ for all $n \in \mathbb{N}$. Since f is injective, and since every element in the image is less than (a, b), this shows (as seen in Chapter 3) that the set of elements S less than (a, b) is infinite. However, if $t = \max(|a|, b)$

$$S \subseteq \{-t, \ldots, t\} \times \{1, \ldots, t\}$$

which is finite. A contradiction. We conclude that every element of Q is in the image of f.

Thus $f\colon \mathbb{N} \to Q$ is a bijection as desired.

Appendix: more on countability

We will give yet another argument that

$$Q \stackrel{\text{def}}{=} \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \geq 1\}.$$

is countable. While doing so we will establish some additional important properties of countable sets.

Theorem 38. *Every subset of a countable set is countable.*

Proof. Suppose $S \subseteq T$ and T is countable. Our goal is to show S is countable, and if S is empty, it is countable by definition. So we will assume S is nonempty, and fix an element $s_0 \in S$. In this case T is nonempty as well.

Since, in this case, T is countable and nonempty, there is a surjective function $f: \mathbb{N} \to T$. Define a function $g: \mathbb{N} \to S$ as follows:

$$g(n) = \begin{cases} f(n), & \text{if } f(n) \in S \\ s_0 & \text{otherwise} \end{cases}$$

Given $s \in S$ we have $s \in T$, so there is an $n \in \mathbb{N}$ so that $f(n) = s$ since f is surjective. Observe that $g(n) = f(n) = s$. We have established that g is surjective, and so S is countable. □

Theorem 39. *Let $f: S \to T$ be a function. If f is surjective and S is countable then T is countable. If f is injective and T is countable then S is countable. Hence, if f is bijective then one of S and T is countable if and only if both are countable.*

Proof. First assume that f is surjective. If S is empty, T is empty so countable. Otherwise there is a surjection $\mathbb{N} \to S$ by definition of countable. Thus there is a surjection $\mathbb{N} \to T$ by composition.

Now assume that $f: S \to T$ is injective, and let T_0 be the image of f in T. From f we get a bijection $S \to T_0$. Let $g: T_0 \to S$ be the inverse. By the previous theorem, T_0 is countable, and note that $g: T_0 \to S$ is surjective. Thus S must also be countable by the first part of the current theorem. □

Now we give a third proof for the countability of Q:

Theorem 40. *The set $Q \overset{\text{def}}{=} \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid b \geq 1\}$ is countable.*

Proof. Let $f: Q \to \mathbb{Z}$ be defined by the rule

$$f(a, b) = 2^b(2a + 1).$$

We begin by showing that f is injective. Assume

$$2^{b_1}(2a_1 + 1) = 2^{b_2}(2a_2 + 1)$$

with $b_2 \geq b_1$. Then

$$(2a_1 + 1) = 2^{b_2 - b_1}(2a_2 + 1).$$

The left side is odd, so the right must be as well. Thus $b_2 = b_1$ and so we also have $(2a_1 + 1) = (2a_2 + 1)$. From the later equation we get $2a_1 = 2a_2$, and by cancelling we get $a_1 = a_2$. Thus $(a_1, b_1) = (a_2, b_2)$.

We have established the injectivity of the function $f: Q \to \mathbb{Z}$. Since $\mathbb{Z}$ is countable, we use the previous theorem to conclude Q is as well. □