CHAPTER 10: CONSTRUCTING THE REAL NUMBERS

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In this chapter we introduce the field of real numbers $\mathbb{R}$. There are several ways to introduce the real numbers. Three popular approaches are to introduce $\mathbb{R}$ with (i) new axioms, with (ii) Dedekind cuts of $\mathbb{Q}$, or with (iii) Cauchy sequences in $\mathbb{Q}$. We will use the third approach and construct real numbers as equivalence classes of Cauchy sequences of rational numbers. This approach is chosen since it avoids the need for additional axioms by building on the previously developed number systems, and it gives students practice with sequences in general and Cauchy sequences in particular.

The main theorem of this chapter is that $\mathbb{R}$, as constructed from Cauchy sequences, is a complete ordered field.

1. The real numbers

Our idea for constructing $\mathbb{R}$ is based on two intuitive principles: (1) every Cauchy sequence in $\mathbb{Q}$ should determine a real number, and (2) equivalent sequences should determine the same real number. The second principle can be reexpressed as the requirement that every Cauchy sequence in an equivalence class $\big[(a_i)\big]$ should determine the same real number. This idea leads us to the idea that real numbers correspond to equivalence classes $\big[(a_i)\big]$ of Cauchy sequences.

Finally we take one more conceptual step: real numbers don't merely *correspond* to equivalence classes of Cauchy sequence, but can be *defined* as equivalence classes of Cauchy sequences. In other words, if we wish to construct the real numbers then they have to be defined somehow, why not define them via this intuitive correspondence?[1]

Definition 1 (Real number)**.** If (a_i) is a Cauchy sequence in $\mathbb{Q}$, then let $\big[(a_i)\big]$ be the equivalence class containing (a_i) under the equivalence relation $\sim$ on the set of sequences in $\mathbb{Q}$. We call $\big[(a_i)\big]$ a *real number.*

Definition 2. The set of real numbers $\mathbb{R}$ is defined as follows:

$$\mathbb{R} \stackrel{\text{def}}{=} \left\{ \, \big[(a_i)\big] \; \big| \, (a_i) \textit{ is a Cauchy sequence in } \mathbb{Q} \, \right\}.$$

[1]As mentioned above, there is another intuitive correspondence with Dedekind cuts of rational numbers. So we could also define real numbers as Dedekind cuts. The Cauchy sequence approach and the Dedekind cut approach lead to isomorphic ordered fields, so from the mathematical point of view it does not matter which approach is followed.

In order to make $\mathbb{R}$ into a field we need to define an addition and multiplication operation on $\mathbb{R}$.

Definition 3 (Addition and multiplication)**.** Let $\big[(a_i)\big]$ and $\big[(b_i)\big]$ be real numbers. Then

$$\big[(a_i)_{i\geq n_0}\big] + \big[(b_i)_{i\geq m_0}\big] \stackrel{\text{def}}{=} \big[(a_i+b_i)_{i\geq l_0}\big]$$

and

$$\big[(a_i)_{i\geq n_0}\big] \cdot \big[(b_i)_{i\geq m_0}\big] \stackrel{\text{def}}{=} \big[(a_i b_i)_{i\geq l_0}\big].$$

Here l_0 is the maximum of n_0 and m_0. Our definitions give two binary operations $\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

In order to check that these definitions are well-defined we need to verify three facts that are not totally obvious: (i) (a_i+b_i) and (a_ib_i) are Cauchy sequences, (ii) if $(a'_i)\sim(a_i)$ then we can replace (a_i) with (a'_i) in the definition and the resulting sum and product will give the same real number, and (iii) if $(b'_i)\sim(b_i)$ then we can replace (b_i) with (b'_i) in the definition and the resulting sum and product will give the same real number.

The remainder of this section will be devoted to verifying the facts needed to confirm that the definition is well-defined.

Lemma 1. *Suppose that $(a_i)_{i\geq n_0}$ and $(b_i)_{i\geq m_0}$ are Cauchy sequences in an ordered field F. Then $(a_i+b_i)_{i\geq l_0}$ and $(a_ib_i)_{i\geq l_0}$ are also Cauchy. Here l_0 is the maximum of n_0 and m_0.*

Proof. First we prove the result for products, and leave the easier sum case to the reader.

Let $\varepsilon>0$. We must find a suitable N. Recall that Cauchy sequences are bounded (Chapter 9), so there is a bound A such that $|a_i|\leq A$ for all terms a_i of the first sequence. Likewise, there is a bound B such that $|b_i|\leq B$ for all terms b_i of the second sequence. Clearly we can assume that A and B are chosen to be positive.

Let $\varepsilon_1=\varepsilon/(2B)$. Since (a_i) is Cauchy, there is an integer N_1 such that $i,j\geq N_1$ implies $|a_i-a_j|<\varepsilon_1$. Similarly, if $\varepsilon_2=\varepsilon/(2A)$, there is an integer N_2 such that $i,j\geq N_2$ implies $|b_i-b_j|<\varepsilon_2$. Let N be the maximum of N_1 and N_2. If $i,j\geq N$, then

$$\begin{aligned} |a_ib_i-a_jb_j| &= |a_ib_i-a_ib_j+a_ib_j-a_jb_j| \\ &\leq |a_ib_i-a_ib_j|+|a_ib_j-a_jb_j|\,. \end{aligned}$$

Observe that

$$|a_ib_i-a_ib_j|=|a_i||b_i-b_j|\leq A|b_i-b_j|<A\varepsilon_2=\varepsilon/2.$$

Similarly,

$$|a_ib_j-a_jb_j|=|a_i-a_j||b_j|\leq|a_i-a_j|B<\varepsilon_1B=\varepsilon/2.$$

Thus $|a_ib_i-a_jb_j|<\varepsilon/2+\varepsilon/2=\varepsilon$ as desired. ◻

Exercise 1. Finish the above proof for the case of $(a_i + b_i)_{i \geq l_0}$. Hint: for ε' just use $\varepsilon/2$.

Lemma 2. *Let $(a_i), (a'_i), (b_i), (b'_i)$ be Cauchy sequences with values in an ordered field F. If $(a_i) \sim (a'_i)$ then*

$$(a_i + b_i) \sim (a'_i + b_i) \qquad \textit{and} \qquad (a_i b_i) \sim (a'_i b_i).$$

If $(b_i) \sim (b'_i)$ then

$$(a_i + b_i) \sim (a_i + b'_i) \qquad \textit{and} \qquad (a_i b_i) \sim (a_i b'_i).$$

Proof. We leave the (easier) case of sums to the reader. We prove the first statement for products; the second statement is similar. So we assume that $(a_i) \sim (a'_j)$, and we aim to prove that $(a_i b_i) \sim (a'_i b_i)$. In other words, for each $\varepsilon > 0$ in F we aim to find a $N \in \mathbb{N}$ such that

$$i \geq N \;\Rightarrow\; |a_i b_i - a'_i b_i| < \varepsilon.$$

To find N, we use the fact that (b_i) is Cauchy, so is bounded (Chapter 9). So there is a $B \in F$ such that $|b_i| \leq B$ for all terms of the sequence. Clearly we can choose B to be positive. Let $\varepsilon' = \varepsilon/B$. Since $(a_i) \sim (a'_j)$, there is an $N \in \mathbb{N}$ such that

$$i \geq N \implies |a_i - a'_i| < \varepsilon'.$$

Observe that if $i \geq N$ then

$$|a_i b_i - a'_i b_i| = |a_i - a'_i| \cdot |b_i| \;\leq\; |a_i - a'_i| B \;<\; \varepsilon' B = \varepsilon.$$

So N is as desired. □

Exercise 2. Complete the proof by proving the case of $(a_i + b_i) \sim (a'_i + b_i)$. Hint: you do not need boundedness for Cauchy sequences in that case. In fact, it is possible to just use $\varepsilon' = \varepsilon$.

Remark 1. Because of the above lemmas, we now know that addition and multiplication are well-defined operations on $\mathbb{R}$.

2. The finite modification lemma

By definition, a real number $x \in \mathbb{R}$ can be designated by giving a Cauchy sequence (a_i) of rational numbers. More precisely, x is the equivalence class $[(a_i)]$, but informally it is good to think of x more as the real number determined by (a_i) "in the limit". We think of each a_i as a rational approximation which gets closer to x. All of this will be made precise later in the chapter, and we will see that in some sense x is really the limit of (a_i).

This idea leads to the following useful lemma, which was already discussed in Chapter 8.

Lemma 3 (Finite modification lemma)**.** *Suppose $(a_i)_{i \geq n_0}$ and $(b_i)_{i \geq m_0}$ differ in only in a finite number of terms. In other words, suppose that there is an integer k greater than or equal to both n_0 and m_0 such that $a_i = b_i$*

if $i \geq k$. Then $(a_i) \sim (b_i)$. In particular, (a_i) and (b_i) determine the same real number (in other words $[(a_i)]] = [(b_i)]$ in $\mathbb{R}$).

Proof. Let $\varepsilon > 0$ be given. If $i \geq k$ then

$$|a_i - b_i| = |a_i - a_i| = |0| = 0 < \varepsilon.$$

This shows $(a_i) \sim (b_i)$, and so $[(a_i)] = [(b_i)]$ in $\mathbb{R}$. □

Remark 2. Because of this remark we will rarely indicate the starting point of sequences. What is important is what happens long-term after any finite number of terms. With this convention, we can define addition and multiplication without reference to the starting point:

$$[(a_i)] + [(b_i)] = [(a_i + b_i)], \qquad \text{and} \qquad [(a_i)][(b_i)] = [(a_i b_i)].$$

In the next two sections we will use constant sequences (c) where $c \in \mathbb{Q}$. This is a sequence where $c_i = c$ for all indices i. Because of the above lemma, we will not usually need to state the starting index i (but a convenient default is to start with $i = 0$).

3. The real numbers $\mathbb{R}$ as a commutative ring

Our next step is to prove that $\mathbb{R}$ is a commutative ring. It is a bit harder to show it is a field, and so we will postpone that for a later section.

Theorem 4. *Addition and multiplication on $\mathbb{R}$ are commutative and associative.*

Exercise 3. Prove the above theorem.

Recall that in Chapter 8 we proved that the constant sequences (c) converge to c. Since constant sequences converge, such sequences are Cauchy. So if $c \in \mathbb{Q}$, the constant sequence gives a real number $[(c)] \in \mathbb{R}$. We are particularly interest in $[(0)]$ and $[(1)]$:

Theorem 5. *An additive identity for $\mathbb{R}$ exists and is $[(0)]$. A multiplicative identity for $\mathbb{R}$ exists and is $[(1)]$.*

Remark 3. Identities, if they exist, are unique.[2] Thus we can say "*the* additive identity" and "*the* multiplicative identity" of $\mathbb{R}$.

Proof. Let $x = \big[(a_i)\big]$ be an arbitrary real number. By definition of $+$ in $\mathbb{R}$,

$$x + [(0)] = \big[(a_i)\big] + \big[(0)\big] = \big[(a_i + 0)\big] = \big[(a_i)\big] = x$$

where the next-to-last equality is due to the fact that 0 is the additive identity of $\mathbb{Q}$ (Chapter 7). By the commutative law (Theorem 4) we get

$$[(0)] + x = x + [(0)] = x.$$

Thus $[(0)]$ is the additive identity.

The proof that $[(1)]$ is the multiplicative identity is similar. □

[2]For any binary operation on a set S, one can show that if there is an identity, it must be unique. For example, if 0 and $0'$ are additive identities, $0 = 0 + 0' = 0'$.

We now consider inverses.

Lemma 6. *If (a_i) is a Cauchy sequence in an ordered field F, then $(-a_i)$ is also Cauchy.*

Proof. Suppose (a_i) is Cauchy. Let $\varepsilon > 0$ be given. In order to show that $(-a_i)$ is Cauchy, we must find an $N \in \mathbb{N}$ such that $\big|(-a_i) - (-a_j)\big| < \varepsilon$ for all $i, j \geq N$. Since (a_i) is Cauchy, there is a $N \in \mathbb{N}$ such that $|a_i - a_j| < \varepsilon$ for all $i, j \geq N$. If $i, j \geq N$ then

$$\big|(-a_i) - (-a_j)\big| = \big|(-1)(a_i - a_j)\big| = |-1||a_i - a_j| = |a_i - a_j|.$$

But $|a_i - a_j| < \varepsilon$, so $\big|(-a_i) - (-a_j)\big| < \varepsilon$ as desired. Thus $(-a_i)$ is Cauchy. □

Theorem 7. *Every element of $\mathbb{R}$ has an additive inverse. More specificially, if $x = \big[(a_i)\big]$, then $-x = \big[(-a_i)\big]$.*

Proof. Let $x \in \mathbb{R}$. Write x as $\big[(a_i)\big]$ where (a_i) is Cauchy in $\mathbb{Q}$. By Lemma 6 the sequence $(-a_i)$ is also Cauchy, so $y = \big[(-a_i)\big]$ is a real number. We leave it to the reader to show that y is the additive inverse of x. □

Exercise 4. Complete the proof of the above theorem.

As we will see, *multiplicative* inverses are trickier. Fortunately we do not need multiplicative inverses to conclude that $\mathbb{R}$ is a ring:

Theorem 8. *The real numbers $\mathbb{R}$ form a commutative ring.*

Exercise 5. Prove the above. Hint: some steps have been proved above. What laws have not been proved yet?

Now that we know that $\mathbb{R}$ is a commutative ring, we can use all the familiar algebraic manipulations and laws valid in rings.

4. The canonical embedding

Now that we have constructed $\mathbb{R}$ we wish to regard $\mathbb{Q}$ as a subset of $\mathbb{R}$. To do so we need to embed $\mathbb{Q}$ into $\mathbb{R}$. This will require an injective map $\mathbb{Q} \to \mathbb{R}$. What we will do is send any $r \in \mathbb{Q}$ to the constant sequence (r).

Theorem 9. *Let $b, c \in F$ where F is an ordered field. Suppose $b \neq c$. Then (b) and (c) are not equivalent sequences.*

Proof. Observe that both sequences converge. If they were equivalent then they would have to have the same limit (Chapter 8). However, the first sequence converges to b and the second to c. A contradiction. □

Corollary 10. *Let $b, c \in \mathbb{Q}$ be distinct. Then $[(b)] \neq [(c)]$ in $\mathbb{R}$.*

Proof. By the above theorem, $(b) \not\sim (c)$. So by general properties of equivalence classes $[(b)] \neq [(c)]$. □

Definition 4 (Canonical embedding)**.** The *canonical embedding* $\mathbb{Q} \to \mathbb{R}$ is the function defined by the rule $c \mapsto [(c)]$.

Theorem 11. *The canonical embedding $\mathbb{Q} \to \mathbb{R}$ is injective.*

Exercise 6. Prove the above theorem using Corollary 10.

Now that we have a canonical embedding $\mathbb{Q} \to \mathbb{R}$, and have shown that it is injective, we can use this to identify elements of $\mathbb{Q}$ with their images in $\mathbb{R}$. Thus we can think of $\mathbb{Q}$ as being a subset of $\mathbb{R}$.

We can go further. We can think of $\mathbb{Q}$ as a *subfield* of $\mathbb{R}$ (as defined in Chapter 8). To do so, we need to check that the addition and multiplication of $\mathbb{R}$ extends the addition and multiplication of $\mathbb{Q}$ defined in Chapter 7. In other words, when we are working with addition and multiplication on $\mathbb{Q}$ we want to be assured that we get the same result whether we use the addition of $\mathbb{Q}$ (Chapter 7) or the addition of $\mathbb{R}$ (this chapter). This is demonstrated in the following lemma.

Lemma 12. *The definitions of addition and multiplication on $\mathbb{R}$ extend the definitions of addition and multiplication on $\mathbb{Q}$. So $\mathbb{Q}$ is a subfield of $\mathbb{R}$.*

Proof. We give the proof for addition; the proof for multiplication is similar. Let $a, b \in \mathbb{Q}$ be given and let $+_{\mathbb{Q}}$ be the addition defined in Chapter 7. Let $+_{\mathbb{R}}$ be the addition defined in the current chapter. We must show that $a +_{\mathbb{Q}} b$ is identified with the same real number as $a +_{\mathbb{R}} b$ (via the canonical embedding).

This is actually pretty trivial once we figure out what is involved. The canonical embedding maps the rational number $a +_{\mathbb{Q}} b$ to the equivalence class of the constant sequence $(a +_{\mathbb{Q}} b)$. Since a is identified with the equivalence class of the constant sequence (a) and b is identified with the equivalence class of (b), the sum $a +_{\mathbb{R}} b$ is equal to the sum $[(a)] +_{\mathbb{R}} [(b)]$. By the definition of $+_{\mathbb{R}}$

$$[(a)] +_{\mathbb{R}} [(b)] = \big[(a +_{\mathbb{Q}} b)\big].$$

The result follows. □

Remark 4. Since 0 in $\mathbb{Q}$ is identified with the equivalence class $[(0)]$ of the constant sequence (0), and since $[(0)]$ is the additive identity of $\mathbb{R}$, we usually write 0 for the additive identity of $\mathbb{R}$. This is consistent with the practice of writing 0 for the additive identity of any ring.

Similarly, we write 1 for the multiplicative identity of $\mathbb{R}$.

Remark 5. In a similar manner, we see that additive inverse in $\mathbb{R}$ extends additive inverse in $\mathbb{Q}$. This follows from the identity $-[(r)] = [(-r)]$ proved above.

Subtraction in $\mathbb{R}$ extends the subtraction in $\mathbb{Q}$. This follows from the definition of $r - s$ (in any ring) as $r + (-s)$, and the fact that addition and additive inverse in $\mathbb{R}$ extend the operations in $\mathbb{Q}$.

5. The real numbers $\mathbb{R}$ as a field

Our next main step is to show that $\mathbb{R}$ is a field by showing that every $x \neq 0$ has a multiplicative inverse. Since nonzero real numbers are represented by

Cauchy sequences that are not equivalent to the zero sequence (0), we begin by considering such Cauchy sequences.

Lemma 13. *Suppose (a_i) is a Cauchy sequence in an ordered field F such that $(a_i) \not\sim (0)$. Then there is $k \in \mathbb{N}$ and a positive $d \in F$ such that $|a_i| \geq d$ for all $i \geq k$.*

Proof. When we negate the definition of equivalence in the case of $(a_i) \sim (0)$ we find that there exists a positive $\varepsilon \in F$ such that for all $N \in \mathbb{N}$ there is an integer $i \geq N$ with $|a_i - 0| \geq \varepsilon$. Fix such an $\varepsilon_0 > 0$ for what follows.

Let $\varepsilon' = \varepsilon_0/2$. Since (a_i) is Cauchy, there is a $N' \in \mathbb{N}$ such that

$$i, j \geq N' \implies |a_i - a_j| < \varepsilon'.$$

As above, there is an $i_0 \geq N'$ with $|a_{i_0}| \geq \varepsilon_0$.

If $i \geq N'$ then since both $i, i_0 \geq N'$ we have

$$\varepsilon_0 \leq |a_{i_0}| = |a_i + (a_{i_0} - a_i)| \leq |a_i| + |a_{i_0} - a_i| \leq |a_i| + \varepsilon'.$$

Thus

$$|a_i| \geq \varepsilon_0 - \varepsilon' = \varepsilon_0 - \frac{\varepsilon_0}{2} = \frac{\varepsilon_0}{2}.$$

In other words, if we set $d = \varepsilon_0/2$ and $k = N'$ then for all $i \geq k$ we have $|a_i| \geq d$. □

Lemma 14. *Suppose (a_i) is a Cauchy sequence in an ordered field F such that $(a_i) \not\sim (0)$. Then there is an integer k_0 such that $a_i \neq 0$ for all $i \geq k_0$ and such that $\left(a_i^{-1}\right)_{i \geq k_0}$ is a Cauchy sequence.*

Proof. By the previous lemma there is an integer k_0 and a positive $d \in F$ such that $|a_i| \geq d$ for all $i \geq k_0$. In particular, $a_i \neq 0$ if $i \geq k_0$. So a_i has a multiplicative inverse in F for all $i \geq k_0$ since F is a field. By properties of ordered fields and absolute values (Chapter 8)

$$0 < \left|a_i^{-1}\right| = |a_i|^{-1} \leq d^{-1}$$

for all $i \geq k_0$.

Our goal is to show $\left(a_i^{-1}\right)_{i \geq k_0}$ is Cauchy. So let $\varepsilon \in F$ be positive; we want an N such that if $i, j \geq N$ then $\left|a_i^{-1} - a_j^{-1}\right| < \varepsilon$.

Let $\varepsilon' = \varepsilon d^2$. Since d and ε are positive, so is ε'. Since (a_i) is Cauchy, there is a N' such that $|a_i - a_j| < \varepsilon'$ for all $i, j \geq N'$. Let N be the maximum of N' and k_0. If $i, j \geq N$ then

$$\begin{aligned}
|a_i^{-1} - a_j^{-1}| &= \left|(a_j - a_i) a_i^{-1} a_j^{-1}\right| \quad (F \text{ is a field}) \\
&= |a_j - a_i| \left|a_i^{-1}\right| \left|a_j^{-1}\right| \quad (F \text{ is an ordered field}) \\
&\leq |a_j - a_i| \left|a_i^{-1}\right| d^{-1} \quad (j \geq k_0, \text{ so } |a_j^{-1}| \leq d^{-1}) \\
&\leq |a_j - a_i| \, d^{-1} d^{-1} \quad (i \geq k_0, \text{ so } \left|a_i^{-1}\right| \leq d^{-1}) \\
&< \varepsilon' d^{-2} = \varepsilon. \qquad (i, j \geq N')
\end{aligned}$$

So $|a_i^{-1} - a_j^{-1}| < \varepsilon$ as desired. We conclude that $\left(a_i^{-1}\right)_{i \geq k_0}$ is Cauchy. □

Theorem 15. *Let $x \in \mathbb{R}$. If $x \neq 0$ then x has a multiplicative inverse.*

Proof. Write $x = [(a_i)]$ where (a_i) is a Cauchy sequence of rational numbers. By the previous lemma, there is a k_0 such that $\left(a_i^{-1}\right)_{i \geq k_0}$ is Cauchy. Thus

$$y = \left[(a_i^{-1})\right]$$

is a real number. By definition of multiplication in $\mathbb{R}$,

$$xy = [(a_i)] \left[(a_i^{-1})\right] = \left[(a_i a_i^{-1})\right] = [(1)].$$

Thus $xy = 1$. By the commutative law for multiplication, $yx = xy = 1$. We conclude that y is the multiplicative inverse of x. □

We now come to the next main theorems of this chapter.

Theorem 16. *The set of real numbers $\mathbb{R}$ is a field.*

Proof. We know that $\mathbb{R}$ is a commutative ring by Theorem 8. We know that $0 \neq 1$ by Corollary 10. Multiplicative inverses exist by Theorem 15. We conclude that $\mathbb{R}$ is a field. □

6. The real numbers $\mathbb{R}$ as an ordered field

In order to show that $\mathbb{R}$ is an ordered field, we need to define the set of positive real numbers P, and to show that this set has the required properties: closure and trichotomy.

Since each real number can be thought of as $[(a_i)]$ where (a_i) is Cauchy, we might be tempted to say that $x = [(a_i)]$ is positive if each a_i is positive. *This idea does not work.* For example, the sequence $(1/i^2)$ converges to 0, and so is equivalent to the constant sequence (0). Thus $[(1/i^2)]$ is zero even though all its terms are strictly positive.

Furthermore, if a sequence has a finite number of zero or negative terms, and the rest are positive, then the sequence could represent a positive number. Thus there are two ways in which the naive definition of positive is defective. The following definition corrects both deficiencies.

Definition 5 (Positive)**.** A *positive-type Cauchy sequence* in an ordered field F is a Cauchy sequence (a_i) with the following property: there is a positive $d \in F$ and an $N \in \mathbb{N}$ such that $a_i \geq d$ for all $i \geq N$.

A *positive* real number is a real number of the form $[(a_i)]$ where (a_i) is a positive-type Cauchy sequence with terms in $\mathbb{Q}$.

Exercise 7. Suppose (a_i) and (b_i) are Cauchy sequences where $(a_i) \sim (b_i)$. Show that (a_i) is positive-type Cauchy if and only if (b_i) is.

Hint: We will be assuming that (a_i) is positive-type and proving that i(b_i) is. Suppose there is a $d_0 > 0$ and a $N_0 \in \mathbb{N}$ such that $a_i \geq d_0$ for all $i \geq N_0$. We must find d and N that work for (b_i). Let $\varepsilon = d_0/2$ and choose a N_1 so that $|a_i - b_i| < \varepsilon$ for all $i \geq N_1$. Why does such a N_1 exist? Choose N as the maximum of N_0 and N_1. What do you think d should be? Prove that your choice of N and d work for (b_i).

Remark 6. The above exercise tells us that if we wish to decide if a real number x is positive, we can take *any* Cauchy sequence from the equivalence class defining x, and check the definition for that particular sequence.

For example, if $r \in \mathbb{Q}$ is thought of as a real number via the canonical embedding, then we can decide if r is positive in $\mathbb{R}$ just by looking at the constant sequence (r). From this we conclude that a rational number r is positive in $\mathbb{R}$ if and only if it is positive in $\mathbb{Q}$. Thus the present definition of positive for $\mathbb{R}$ is compatible with the definition of Chapter 7 for elements that happen to be in $\mathbb{Q}$.

Theorem 17 (Closure). *If $x, y \in \mathbb{R}$ are positive then so is $x + y$ and xy.*

Proof. Let $x = [(a_i)]$ and $y = [(b_i)]$ where (a_i) and (b_i) are positive-type Cauchy sequences of rational numbers. By definition there is a positive number $d_1 \in \mathbb{Q}$ and an integer $N_1 \in \mathbb{N}$ such that $a_i \geq d_1$ for all $i \geq N_1$. Likewise, there is a positive $d_2 \in \mathbb{Q}$ and a $N_2 \in \mathbb{N}$ such that $b_i \geq d_2$ for all $i \geq N_2$. Let $d = d_1 + d_2$. We know that $d > 0$ since it is the sum of positive elements. Let N be the maximum of N_1 and N_2. If $i \geq N$, then

$$a_i + b_i \geq d_1 + b_i \geq d_1 + d_2 = d.$$

Thus $x + y = [(a_i + b_i)]$ is positive.

The proof for xy is similar. □

Exercise 8. Prove the above for the case of multiplication.

We now want to prove a trichotomy law: for all $x \in \mathbb{R}$ exactly one of the following occurs (i) x is positive, (ii) $x = 0$, or (iii) $-x$ is positive. In the third case we also say that x is *negative*.

We divide the proof of this law into lemmas:

Lemma 18. *The real number* 0 *is neither positive nor negative.*

Proof. The real number 0 is defined by the constant sequence $(a_i) = (0)$. Since $a_i = 0$, there can be no $d > 0$ and N such that $a_i \geq d$ for all $i \geq N$. (Since if $a_i \geq d$ and $d > 0$ then $a_i > 0$, a contradiction). So 0 cannot be positive.

We now show that 0 cannot be negative. Suppose 0 is negative. Then -0 is positive (by definition of negative). But $-0 = 0$, and we have already shown that 0 is not positive. □

Lemma 19. *Let $x \in \mathbb{R}$. It is not possible for both x and $-x$ to be positive.*

Exercise 9. Prove the above. Hint: suppose not. Write $x = [(a_i)]$. Observe that $-x = [(-a_i)]$ by Theorem 7. Define a d_1 and N_1 for (a_i) and d_2 and N_2 for $(-a_i)$. Let i be the maximum of N_1 and N_2, and show that a_i is both positive and negative in $\mathbb{Q}$.

Remark 7. Notice how we use a trichotomy law for $\mathbb{Q}$ from an earlier chapter to help prove a trichotomy law for $\mathbb{R}$.

The above two lemmas show part of the trichotomy law: together they show that at most one of the trichotomy conditions hold. We still need to show that at least one condition holds. This follows from the next lemma.

Lemma 20. *If $x \neq 0$ is a real number, then either x or $-x$ is positive.*

Proof. Write $x = \big[(a_i)\big]$ where (a_i) is a Cauchy sequence in $\mathbb{Q}$. By Theorem 7, $-x = \big[(-a_i)\big]$.

By Lemma 13, there is a $k_1 \in \mathbb{N}$ and a positive rational number $d \in \mathbb{Q}$ such that $|a_i| \geq d$ for all $i \geq k$. Since (a_i) is Cauchy, there is a $k_2 \in \mathbb{N}$ such that $|a_i - a_j| \leq d/2$ for all $i, j \geq k_2$. Let N be the maximum of k_1 and k_2.

In particular, we have $|a_N| \geq d > 0$, so either $a_N \geq d$ or $a_N \leq -d$. This gives us two cases.

We begin with the case $a_N \geq d$. For all $i \geq N$ we have $|a_i - a_N| \leq d/2$ since $i, N \geq k_2$. This means

$$-\frac{d}{2} \leq a_i - a_N \leq \frac{d}{2}.$$

So

$$a_i \geq a_N - \frac{d}{2} \geq d - \frac{d}{2} = \frac{d}{2}.$$

This shows the for all $i \geq N$ we have $a_i \geq d/2$. Thus $x = \big[(a_i)\big]$ is positive by Definition 5.

Finally consider the case $a_N \leq -d$. For all $i \geq N$ we have $|a_i - a_N| \leq d/2$ since $i, N \geq k_2$. This means

$$-\frac{d}{2} \leq a_N - a_i \leq \frac{d}{2}.$$

So

$$-a_i \geq -a_N - \frac{d}{2} \geq d - \frac{d}{2} = \frac{d}{2}.$$

This shows the for all $i \geq N$ we have $-a_i \geq d/2$. Note that since $d > 0$, it follows that $d/2 > 0$. Thus $x = \big[(-a_i)\big]$ is positive by Definition 5. □

Putting these lemmas together, we conclude the following:

Theorem 21. *For every $x \in \mathbb{R}$ exactly one of the following occurs: (i) x is positive, (ii) $x = 0$, or (iii) $-x$ is positive.*

We now come to the next main theorem of this chapter.

Theorem 22. *The set of real numbers $\mathbb{R}$ is an ordered field.*

Proof. We know that $\mathbb{R}$ is a field by Theorem 16. To show that $\mathbb{R}$ is an ordered field we need to check that (i) the positive elements are closed under addition and multiplication, and (ii) the positive elements satisfy the trichotomy law. Both these were done in Theorems 17 and 21 respectively. □

Remark 8. As mentioned above, positivity defined for $\mathbb{R}$ is compatible with the earlier concept of positivity defined for $\mathbb{Q}$. Since $x < y$ means $y - x$ is positive, if follows that inequality in $\mathbb{R}$ is compatible with inequality in $\mathbb{Q}$. In other words, we can show that if $x, y \in \mathbb{Q}$ then $x < y$ holds for $\mathbb{Q}$ if and only if it holds for $\mathbb{R}$.

We end this section with a few lemmas concerning the order relation of $\mathbb{R}$.

Lemma 23. *Suppose (a_i) is a Cauchy sequence of rational numbers. Suppose there is a $k \in \mathbb{N}$ such that $a_i \geq 0$ for all $i \geq k$. Then $x \geq 0$ where $x = [(a_i)]$ is the corresponding real number.*

Proof. The only way for $x \geq 0$ to fail is for $-x > 0$. Suppose this happens. Since $-x = [(-a_i)]$ the sequence $(-a_i)$ is of positive type. So there is a $N \in \mathbb{N}$ and a positive $d \in \mathbb{Q}$ such that $-a_i \geq d$ for all $i \geq N$. Let i be the maximum of N and k. Then

$$d \leq -a_i \leq 0,$$

which is a contradiction since d is positive. □

Lemma 24. *Suppose (a_i) and (b_i) are two Cauchy sequences of rational numbers, and let $x = [(a_i)]$ and $y = [(b_i)]$ be the corresponding real numbers. If there is a $k \in \mathbb{N}$ such that $a_i \leq b_i$ for all $i \geq k$, then $x \leq y$.*

Proof. Observe that $y - x = [(b_i - a_i)]$. For all $i \geq k$, we have $b_i - a_i \geq 0$. By the previous lemma, $y - x \geq 0$. The result follows. □

Remark 9. If we have $a_i < b_i$ instead, we cannot necessarily conclude that $x < y$. Without extra information, we can only conclude that $x \leq y$.

7. Relationship between $\mathbb{R}$ and $\mathbb{Q}$

In this section we will consider a few useful results relating $\mathbb{R}$ and $\mathbb{Q}$. For example, we will see that Cauchy sequences of rational numbers always converge to real numbers, and that all real numbers are limits of rational sequences. We will also see that $\mathbb{R}$ is an Archimedean ordered field. Recall from Chapter 8 that this implies that $\mathbb{Q}$ is dense in $\mathbb{R}$; in other words, between any two distinct real numbers we can always find a rational number.

We begin with a lemma that can be used to compare rational numbers to real numbers.

Lemma 25. *Suppose $x \in \mathbb{R}$ is given by $x = [(a_i)]$. Suppose that b is a rational number. If there is a $k \in \mathbb{N}$ such that $a_i \leq b$ for all $i \geq k$, then $x \leq b$ (where here we are thinking of b as a real number). If, instead, there is a $k \in \mathbb{N}$ such that $a_i \geq b$ for all $i \geq k$, then $x \geq b$.*

Proof. For the first statement, apply Lemma 24 to the sequences (a_i) and (b). For the second statement, switch the order and apply Lemma 24 again. □

Theorem 26. *Suppose $y > 0$ is a real number. Then there is a positive integer n such that $1/n \leq y$.*

Proof. Write y as $[(a_i)]$ where (a_i) is a positive-type Cauchy sequence of rational numbers. By Definition 5, there is a $N \in \mathbb{N}$ and a positive $d \in \mathbb{Q}$ such that $a_i \geq d$ for all $i \geq N$. Write $d = m/n$ where $m, n \in \mathbb{N}$ are positive. Thus $a_i \geq d \geq 1/n$ for all $i \geq N$.

Since $1/n \leq a_i$ for all $i \geq N$, we get $1/n \leq y$ by the above Lemma. □

Theorem 27. *The real numbers $\mathbb{R}$ form an Archimedean ordered field.*

Proof. This follows from the previous theorem (by a result in Chapter 8). □

Corollary 28. *The field $\mathbb{Q}$ is dense in $\mathbb{R}$. In other words, if $x, y \in \mathbb{R}$ are such that $x < y$, there is a $r \in \mathbb{Q}$ with $x < r < y$.*

Exercise 10. Which theorem in Chapter 8 yields the above corollary?

The following theorem says that if a Cauchy sequence of rational numbers represents a certain real number, then the Cauchy sequence (as a sequence in $\mathbb{R}$) converges to the real number.

Theorem 29. *Let (a_i) be a Cauchy sequence of rational numbers. Then (a_i) considered as a sequence of real numbers converges to the real number x where $x = [(a_i)]$.*

Proof. Let ε be an arbitrary positive real number. We must find a $N \in \mathbb{N}$ such that $|a_i - x| < \varepsilon$ for all $i \geq N$. It seems like we should be able to use the definition of Cauchy sequence to find such a N. There is a slight problem: (a_i) is a Cauchy sequence in $\mathbb{Q}$, but $\varepsilon > 0$ is in $\mathbb{R}$.

We solve the problem by choosing a positive integer n such that $1/n \leq \varepsilon$ (Theorem 26). Let $\varepsilon' = 1/n$, and note that ε' is a positive element of $\mathbb{Q}$ such that $\varepsilon' < \varepsilon$. By definition of Cauchy sequence in $\mathbb{Q}$, we have an $N \in \mathbb{N}$ such that $|a_i - a_j| < \varepsilon'$ for all $i, j \geq N$. We will show that this N has the desired property for convergence: that $|a_i - x| < \varepsilon$ for all $i \geq N$. (We will actually show $|a_i - x| \leq \varepsilon'$, which is even stronger.)

So fix $i \geq N$. Recall that a_i is thought of as both a rational number and a real number via the canonical embedding $\mathbb{Q} \to \mathbb{R}$. More precisely, a_i as a real number is defined by $[(c_j)]$ where (c_j) is the constant sequence whose terms are all equal to the rational number a_i. Recall also that $x = [(a_j)]$.

So let (c_j) be the constant sequence whose terms are equal to a_i, and assume $j \geq N$. By our choice of N we have $|c_j - a_j| < \varepsilon'$ since $c_j = a_i$. By properties of absolute values (in $F = \mathbb{Q}$),

$$-\varepsilon' < c_j - a_j < \varepsilon'.$$

The above holds for all $j \geq N$, so we can use Lemma 25 to conclude that

$$-\varepsilon' \leq [(c_j - a_j)] \leq \varepsilon'.$$

In other words (using properties of absolute values in $F = \mathbb{R}$),

$$\big|[(c_j - a_j)]\big| \leq \varepsilon'.$$

This implies that

$$|a_i - x| = \big|[(c_j)] - [(a_j)]\big| = \big|[(c - a_j)]\big| \leq \varepsilon' < \varepsilon.$$

This completes the proof that (a_i) converges to x. □

Corollary 30. *Every Cauchy sequence of rational numbers converges to some real number.*

Proof. Let (a_i) be a Cauchy sequence of rational numbers. Let $x = [(a_i)]$. By Theorem 29, (a_i) has limit x. □

Note. Our goal is to show all Cauchy sequences in $\mathbb{R}$ converge. The above is a nice step in this direction, but we still more to show.

Remark 10. Let (a_i) be a sequence of rational numbers. There is some ambiguity of what *Cauchy* means for (a_i) when we embed $\mathbb{Q}$ into $\mathbb{R}$. We can mean the Cauchy condition holds for all positive $\varepsilon \in \mathbb{Q}$. Call this $\mathbb{Q}$-Cauchy. Or we can mean that the Cauchy condition holds for all positive $\varepsilon \in \mathbb{R}$. Call this $\mathbb{R}$-Cauchy.

In the above theorem and corollary we are thinking of $\mathbb{Q}$-Cauchy. We proved that any $\mathbb{Q}$-Cauchy sequence gives a convergent sequence in $\mathbb{R}$. But convergent sequences are automatically Cauchy (Chapter 8). Thus any $\mathbb{Q}$-Cauchy sequence is automatically a $\mathbb{R}$-Cauchy sequence.

Conversely, any $\mathbb{R}$-Cauchy sequence whose terms are in $\mathbb{Q}$ is a $\mathbb{Q}$-Cauchy sequence. (If a condition holds for all $\varepsilon > 0$ in $\mathbb{R}$ then it will certainly hold for all $\varepsilon > 0$ in $\mathbb{Q}$ since $\mathbb{Q} \subseteq \mathbb{R}$). We conclude that if (a_i) is a sequence of rational numbers, there is no difference between being $\mathbb{Q}$-Cauchy or $\mathbb{R}$-Cauchy.

Corollary 31. *Every real number is the limit of a sequence of rational numbers*

Proof. Let $x = [(a_i)]$ be a real number. By Theorem 29, (a_i) has limit x. □

Corollary 32. *If $x \in \mathbb{R}$ and if $\varepsilon \in \mathbb{R}$ is positive, then there is a rational number $r \in \mathbb{Q}$ with $|x - r| < \varepsilon$.*

Proof. Since x is the limit of a sequence (a_i) of rational numbers, there is a $N \in \mathbb{N}$ such that $|a_i - x| < \varepsilon$ for all $i \geq N$. Let $r = a_N$. □

Remark 11. The proceeding two corollaries can also be proved as consequences of the Archimedean property of $\mathbb{R}$: they hold for all Archimedean ordered fields.

8. $\mathbb{R}$ is complete

As we have discussed before, $\mathbb{Q}$ has "holes". For example, $\mathbb{Q}$ is missing a square root for 2. Because of this, $\mathbb{Q}$ has Cauchy sequences that do not converge. We will now show that the real numbers $\mathbb{R}$ do not have Cauchy sequences that fail to converge. So $\mathbb{R}$ is complete, and does not have "holes".

We begin by showing that every Cauchy sequence of real numbers converges. We already know, from the previous section, that every Cauchy sequence of rational numbers converges in $\mathbb{R}$. But this is not enough for our current needs. We need to extend the result to Cauchy sequences with terms in $\mathbb{R}$. We begin with a lemma.

Lemma 33. *If (a_i) is a sequence of real numbers, then there is a sequence (b_i) of rational numbers such that $(a_i) \sim (b_i)$. (Equivalence is taken with $F = \mathbb{R}$.)*

Proof. For each a_i, we know by Corollary 32 that there is a rational number b_i such that $|a_i - b_i| < 1/i$. Consider the sequence (b_i) formed from such rational numbers.[3] We must show that $(a_i) \sim (b_i)$.

Let $\varepsilon \in \mathbb{R}$ be an arbitrary positive real number. We must find a $N \in \mathbb{N}$ such that $|a_i - b_i| < \varepsilon$ for all $i \geq N$. By Theorem 26 we can find a positive n such that $1/n < \varepsilon$. Let $N = n$. If $i \geq N$ then

$$\begin{aligned} |a_i - b_i| &< 1/i && \text{(choice of } b_i\text{)} \\ &\leq 1/n && (i \geq N \text{ and } N = n) \\ &< \varepsilon && \text{(choice of } n\text{)} \end{aligned}$$

Thus $(a_i) \sim (b_i)$ as desired. □

Theorem 34. *Every Cauchy sequence in $\mathbb{R}$ converges.*

Proof. Let (a_i) be a Cauchy sequence of real numbers. By Lemma 33 there is a sequence (b_i) of rational numbers such that $(b_i) \sim (a_i)$.

In Chapter 9 we proved that if two sequences are equivalent, and if one is Cauchy, then the other is. Since (a_i) is Cauchy, we conclude that (b_i) is also a Cauchy sequence. By Corollary 30 we conclude that (b_i) has a limit.

In Chapter 8 we proved that if two sequences are equivalent, and if one has a limit, then the other does as well. Since (b_i) has a limit, we conclude that (a_i) must have a limit. □

Now for the main theorem.

Theorem 35 (Main theorem)**.** *The field $\mathbb{R}$ is a complete ordered field.*

Proof. In Chapter 9 we proved that if an ordered field is Archimedean and if every Cauchy sequences converges in that field, then that field must be complete. So, since $\mathbb{R}$ is Archimedean, and since every Cauchy sequence in $\mathbb{R}$ converges, $\mathbb{R}$ is a complete ordered field. □

Dedekind cuts (draft)

Another approach to constructing the real numbers is to use Dedekind cuts.[4]

Nested intervals (draft)

Another approach to constructing the real numbers is to use certain nested sequences of intervals of Rational numbers.[5]

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[3] In order to avoid using the axiom of choice, we can select b_i to have the smallest possible denominator, and among fractions with the smallest possible denominator we choose the smallest possible numerator.

[4] Definitions and discussion will be provided in a future draft.

[5] Definitions and discussion will be provided in a future draft.

Uniqueness (draft)

Any two complete ordered fields are isomorphic as ordered fields.[6] One can define $\mathbb{R}$ to be *any* complete ordered field, and use the fact that any two constructions are isomorphic to conclude that this definition is in some sense well-defined.

[6] Definition of isomorphism, and discussion will be provided in a future draft.