APPENDIX B: EXPLORING $\mathbb{C}$

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In this appendix we consider further properties of $\mathbb{C}$ which every mathematician should know. The reason why this material is in an appendix, and not the main text, is that it relies on basic trigonometry as well as the (real) exponential function. This is a departure from the main text which is a self-contained axiomatic development of the number systems.

Although we certainly could define and develop these important functions using our usual rigorous methodology, this would take us too far afield. So we take a few facts from precalculus as given. Consequently, for this appendix all the concepts and results can be thought of as "informal" since we draw on basic mathematical knowledge developed outside this course.

In this appendix we will consider the polar form of complex numbers and De Moivre's law. We will also consider the complex exponential function. We will establish that every nonzero element $z \in \mathbb{C}$ has n distinct nth roots for every positive integer n. In other words, if $z \neq 0$ then the polynomial $X^n - z$ has n roots in $\mathbb{C}$. In the next appendix, we will consider the important *fundamental theorem of algebra* concerning complex roots of more general polynomials.

1. Trigonometric and the real exponential functions

In order to study the complex numbers in more depth, we need some basic trigonometry and some basic facts about the real exponential function e^x. Actually the only trig functions we will need are the sine and cosine functions, $\sin x$ and $\cos x$. We will use radians, so will need to assume we have the positive real number π.

We will list here a few basic facts that will be used in this appendix. We take the following propositions as given, and assume the reader is familiar with them (but not necessarily their rigorous proofs).

Proposition 1 (Sine and cosine)**.** *The sine and cosine functions are both continuous functions $\mathbb{R} \to [-1, 1]$. The number π is a positive real number such that the sine and cosine functions are periodic with period 2π in the sense that*

$$\sin(x + 2\pi k) = \sin x, \qquad \cos(x + 2\pi k) = \cos x$$

for all $x \in \mathbb{R}$ *and* $k \in \mathbb{Z}$. *Some special values of the sine function are as follows:* $\sin(0) = 0, \sin(\pi/2) = 1$, *and* $\sin \pi = \sin 2\pi = 0$. *Some special values of the cosine function are as follows:* $\cos(0) = 1, \cos(\pi/2) = 0$, *and* $\cos \pi = -1$. *In fact,* $\cos x = 1$ *if and only if* $x = 2k\pi$ *for some* $k \in \mathbb{Z}$.

The sine function is odd and the cosine function is even in the sense that

$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x$$

for all $x \in \mathbb{R}$.

Remark 1. Sometimes we will use Greek letters such as $\alpha, \beta, \theta \in \mathbb{R}$ for our inputs, where we think the inputs as angles in radians. So we will sometimes write expressions such as $\sin \theta$ or $\cos \alpha$. We adopt the usual convention that $\sin^2 \theta$ is $(\sin \theta)^2$ and $\cos^2 \theta$ is $(\cos \theta)^2$. So here these expressions do *not* refer to iteration.

Proposition 2 (Trig identities)**.** *For all* $\theta, \alpha, \beta \in \mathbb{R}$, *we have*

$$\sin^2 \theta + \cos^2 \theta = 1,$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta,$$

$$\sin(\alpha + \beta) = \cos \alpha \sin \beta + \cos \beta \sin \alpha,$$

Proposition 3 (The real exponential function)**.** *The real exponential function* $\exp x$, *also written* e^x, *is a continuous function* $\mathbb{R} \to (0, \infty)$. *The exponential function has special value* $\exp 0 = 1$. *Also*

$$e^{x+y} = e^x e^y$$

for all $x, y \in \mathbb{R}$.

Proposition 4 (Polar coordinates)**.** *Let* $(x, y) \in \mathbb{R}^2$ *be a point on the real plane. Then there are real numbers* r *and* θ *such that*

$$(x, y) = (r \cos \theta, r \sin \theta).$$

Furthermore, if we require that r *be nonnegative, then* r *is unique. If we restrict* θ *in the range* $0 \leq \theta < 2\pi$, *or some other half-open interval of length* 2π, *and if* $(x, y) \neq (0, 0)$, *then* θ *is also unique.*

Remark 2. We call the r in the above proposition the *radius of* (x, y). When it is unique, we call the θ in the above proposition the *angle of* (x, y). We call (r, θ) *polar coordinates* of the point (x, y).

2. Polar form of complex numbers

From Proposition 4 it easily follows that every complex number z can be expressed as $r \cos \theta + r \sin \theta \cdot i$ where r and θ are real numbers, and where r is nonnegative. We call this a *polar form* of z.

Theorem 5 (Polar form of a complex number)**.** *Let* $z \in \mathbb{C}$. *Then there are numbers* $r, \theta \in \mathbb{R}$ *with* $r \geq 0$ *such that*

$$z = r \cos \theta + r \sin \theta \cdot i.$$

Proof. Write z as $x + yi$ where $x, y \in \mathbb{R}$. By Proposition 4, there is a $\theta \in \mathbb{R}$ and a nonnegative $r \in \mathbb{R}$ such that $x = r\cos\theta$ and $y = r\sin\theta$. So

$$z = x + yi = r\cos\theta + r\sin\theta \cdot i.$$

□

Remark 3. To help with readability, the polar form is sometimes written with the i before $r\sin\theta$:

$$z = x + yi = r\cos\theta + i\,r\sin\theta.$$

The radius of a complex number is just the absolute value:

Theorem 6 (Radius formula). *If $z = r\cos\theta + ir\sin\theta$ then $r = |z|$.*

Proof. Observe

$$|z|^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2.$$

The first equality uses the formula for absolute value of complex numbers from Chapter 12. The last equality uses Proposition 2. □

Informal Exercise 1. Let $z = r\cos\theta + ir\sin\theta$ be a complex number written in polar form. Show that $\overline{z} = r\cos(-\theta) + ir\sin(-\theta)$.

The above gives a nice description of complex conjugation in polar form. There is also a nice description of multiplication for complex numbers in polar form:

Theorem 7 (Product formula). *Let $z, w \in \mathbb{C}$. If z, w are written in polar coordinates as*

$$z = r_1\cos\theta_1 + r_1\sin\theta_1 \cdot i \qquad w = r_2\cos\theta_2 + r_2\sin\theta_2 \cdot i$$

then the product can be written as

$$zw = r\cos\theta + r\sin\theta \cdot i$$

where $r = r_1 r_2$ and $\theta = \theta_1 + \theta_2$.

Remark 4. In other words, when you multiply complex numbers, you multiply the radii and add the angles. So multiplication has a very nice geometric interpretation.

Addition also has a geometric interpretation. It is just vector addition.

Proof. For convenience, write $\cos\theta_1$ as c_1, $\cos\theta_2$ as c_2, $\sin\theta_1$ as s_1, and $\sin\theta_2$ as s_2. So

$$\begin{aligned}
zw &= (r_1c_1 + r_1s_1i)(r_2c_2 + r_2s_2i) \\
&= r_1(c_1 + s_1i)r_2(c_2 + s_2i) && \text{(Distr. Law)} \\
&= r_1r_2(c_1 + s_1i)(c_2 + s_2i) && \text{(Comm./Assoc. Laws)} \\
&= r_1r_2(c_1c_2 + c_1s_2i + s_1ic_2 + s_1is_2i) && \text{(Distr. Law)} \\
&= r_1r_2\big(c_1c_2 + c_1s_2i + s_1c_2i - s_1s_2\big) && (i^2 = -1) \\
&= r_1r_2\big((c_1c_2 - s_1s_2) + (c_1s_2 + s_1c_2)i\big) \\
&= r_1r_2\big(\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)i\big). && \text{(Proposition 2)}
\end{aligned}$$

□

Corollary 8 (Inverse formula)**.** *Let $z \in \mathbb{C}$. If $z \neq 0$ is*

$$z = r\cos\theta + r\sin\theta \cdot i$$

in polar coordinates, then

$$z^{-1} = r^{-1}\cos(-\theta) + r^{-1}\sin(-\theta)i$$

and

$$z^{-1} = r^{-1}\cos\theta - r^{-1}\sin\theta \cdot i.$$

Informal Exercise 2. Prove the above corollary.

3. De Moivre's theorem

In Chapter 6 we considered exponentiation in general rings and fields. So if $z \neq 0$ in $\mathbb{C}$ it has an power z^n for all $n \in \mathbb{Z}$. De Moivre's theorem gives a interesting polar form formula for these powers.

Theorem 9 (De Moivre's theorem)**.** *Let $z \in \mathbb{C}$ and $n \in \mathbb{Z}$. Suppose $z \neq 0$. If z is*

$$z = r\cos\theta + r\sin\theta \cdot i$$

in polar form, then

$$z^n = r^n\cos(n\theta) + r^n\sin(n\theta)i.$$

Proof. Fix $z \neq 0$. First we prove

$$z^n = r^n\cos(n\theta) + r^n\sin(n\theta)i.$$

for all nonnegative n using induction.

Observe the result holds for $n = 0$ since $z^0 = 1$ (exponentiation in rings) and

$$r^0\cos(0) + r^0\sin(0)i = 1 + 0i = 1$$

since $r^0 = 1$, $\cos 0 = 1$ and $\sin 0 = 0$.

Now suppose that the result holds for the natural number $n = u$. Then

$$z^{u+1} = z^u z = \big(r^u\cos(u\theta) + r^u\sin(u\theta)i\big)\big(r\cos(\theta) + r\sin(\theta)i\big)$$

By Theorem 7,

$$z^u z = r'\cos\theta' + r'\sin\theta' \cdot i$$

where $r' = r^u r = r^{u+1}$ and $\theta' = u\theta + \theta = (u+1)\theta$. Thus the result holds for $n = u + 1$.

By induction, the theorem holds for all $n \geq 0$. For $n < 0$, let $m = -n$. So the theorem holds for z^m. Now use Corollary 8 to show that $z^n = (z^m)^{-1}$ has radius $(r^m)^{-1}$ and angle $-(m\theta)$. Since $(r^m)^{-1} = r^n$ and $-(m\theta) = n\theta$, the result follows. □

Remark 5. This shows that, in polar coordinates, when one takes the nth power one takes the nth power of the radius and multiplies the angle by n.

4. The complex exponential function

We now consider the complex exponential function $z \mapsto \exp z$ as a function $\mathbb{C} \to \mathbb{C}$.

Definition 1 (Complex exponential function)**.** Let $z \in \mathbb{C}$. If $z = x + yi$ where $x, y \in \mathbb{R}$ then

$$\exp z \stackrel{\text{def}}{=} \exp(x)\,(\cos y + i \sin y).$$

Here $\exp(x)$ is the value of the real exponential function, which we take as given.

Remark 6. We often write $\exp z$ as e^z. The following results will help justify this notation.

Theorem 10. *When we restrict the complex exponential function to $\mathbb{R}$ then the values agree with the real exponential function. Thus the complex exponential function is an extension of the real exponential function to the larger domain $\mathbb{C}$.*

Proof. Observe that

$$\exp(x + 0i) = \exp(x)\,(\cos 0 + \sin 0 \cdot i) = \exp(x) \cdot (1 + 0 \cdot i) = \exp x.$$

□

Corollary 11. *If z is real, then the value e^z of the complex exponential function is real. Furthermore*

$$e^0 = 1.$$

Remark 7. If $y \in \mathbb{R}$,

$$e^{iy} = \cos y + i \sin y.$$

To see this, take $x = 0$ and observe $\exp x = e^0 = 1$. This formula for e^{iy} implies

$$e^z = e^x e^{iy}$$

as expected.

Informal Exercise 3. Let $z = x + yi$ where $x, y \in \mathbb{R}$. Show that $|e^z| = e^x$. Conclude then that $e^z \neq 0$.

Theorem 12. *Let $z \in \mathbb{C}$. If z is*

$$z = r\cos\theta + ir\sin\theta$$

in polar form, then

$$z = re^{\theta i}.$$

Proof. By definition, $e^{i\theta} = \cos\theta + i\sin\theta$. See Remark 7. □

Corollary 13. *Every complex number can be written in the form*

$$z = re^{\theta i}$$

where $r = |z|$.

Informal Exercise 4. Suppose z, w are complex numbers. Then show that

$$e^{z+w} = e^z e^w.$$

Hint: Use Theorem 7, and known properties of e^x when x is real.

Remark 8. A special case of this is when $w = -z$. Since $e^0 = 1$, we conclude that $e^z e^{-z} = 1$. In other words, e^{-z} is the multiplicative inverse of e^z.

Theorem 14. *Suppose z is a complex number, and n is an integer. Then*

$$(e^z)^n = e^{nz}.$$

Proof. This follows by induction using Informal Exercise 4. The case of negative n has to be established separately using the above remark. □

Informal Exercise 5. Show that $e^{\pi i} = -1$. This is Euler's formula, and is considered by many to be one of the most amazing formulas in mathematics since it puts e, π, i into one simple formula. Observe that $e^{\pi i} + 1 = 0$ combines $e, \pi, i, 0, 1$, and involves addition, multiplication, and exponentiation.

5. *N*th roots of complex numbers

In Chapter 11 we established that for every positive integer n and for every nonnegative x in $\mathbb{R}$ there exists a nth root of x in $\mathbb{R}$. When n is odd, then we can extend this result for negative x as well. However, we cannot hope to have nth roots in $\mathbb{R}$ when n is even and $x < 0$.

In this section we establish that nth roots exist for any positive integer n and any $z \in \mathbb{C}$. In fact, if $z \neq 0$ there are exactly n distinct nth roots evenly distributed on a circle in the complex plane.

First we establish existence of nth roots:

Theorem 15 (Existence of roots)**.** *Suppose z is a complex number written in the form $z = re^{\theta i}$. Then $r^{1/n}e^{\theta i/n}$ is an nth root of z.*

Proof. This follows from Theorem 14 and other properties of exponentiation. □

Informal Exercise 6. Sketch the cube root of -1 in the complex plane. Use the formula from Theorem 15.

In order to classify nth roots of a complex number $z \neq 0$, it is convenient to start with $z = 1$.

Definition 2 (Roots of unity)**.** Let n be a fixed positive integer. Every complex number of the form $e^{2k\pi i/n}$ with $k \in \mathbb{Z}$ is called an *nth root of unity.*

Theorem 16. *If z is an nth root of unity then $z^n = 1$. Furthermore, there are n distinct nth roots of unity.*

Proof. Observe that $e^{2\pi i} = 1$. Thus

$$\left(e^{2k\pi i/n}\right)^n = e^{2k\pi i} = \left(e^{2\pi i}\right)^k = 1^k = 1.$$

Thus every nth root of unity is an nth root of 1.

If $0 \le k_1 < k_2 < n$ then the angles of the corresponding roots of unity satisfy $0 \le 2k_1\pi/n < 2k_2\pi/n < 2\pi$. By uniqueness of angle (Proposition 4) in the interval $[0, 2\pi)$, $e^{2k_1\pi i/n}$ and $e^{2k_2\pi i/n}$ are distinct. Thus there are at least n distinct nth roots of unity.

For general $k \in \mathbb{Z}$ the remainder upon dividing by n gives rise to the same root of unity. This can be seen as follows. Let $k = qn + r$ with $0 \le r < n$. Then

$$e^{2k\pi i/n} = e^{2(qn+r)\pi i/n} = \left(e^{2\pi i}\right)^q e^{2r\pi i/n} = 1^q\, e^{2r\pi i/n}.$$

So there are no more nth roots of unity beyond the n discussed above. □

The converse is true: if z satisfies $z^n = 1$ then it is an nth root of unity:

Theorem 17. *Let n be a positive integer. Suppose $z \in \mathbb{C}$ is such that $z^n = 1$. Then z is an nth root of unity.*

Proof. Observe that $1 = |z^n| = |z|^n$. By uniqueness of nonnegative nth roots in $\mathbb{R}$, this implies that $|z| = 1$. Write z as $re^{i\theta}$ where r is the radius and $\theta \in \mathbb{R}$. Observe that $r = 1$ since $|z| = 1$. Also

$$1 = z^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

This implies $\cos(n\theta) = 1$. This means that $n\theta = 2\pi k$ for some $k \in \mathbb{Z}$. The result follows. □

Corollary 18. *Let n be a positive integer. There are exactly n solutions to the equation $z^n = 1$ in $\mathbb{C}$. These are the nth roots of unity.*

Theorem 19. *Suppose z is a complex number written in the form $z = re^{\theta i}$. Let n be a positive integer, and let ζ be an nth root of unity. Then $r^{1/n}e^{\theta i/n}\zeta$ is an nth root of z. If $z \neq 0$ then every nth root of z can be written in this way.*

Proof. Observe that

$$\begin{aligned}\left(r^{1/n}e^{\theta i/n}\zeta\right)^n &= \left(r^{1/n}\right)^n \left(e^{\theta i/n}\right)^n \zeta^n \\ &= r \cdot e^{\theta i} \cdot 1 \\ &= z.\end{aligned}$$

Thus $r^{1/n}e^{\theta i/n}\zeta$ is an nth root of z.

This gives one root, now suppose w is any nth root of z where $z \neq 0$. Then $w \neq 0$. Observe that if $u = r^{1/n}e^{\theta i/n}w^{-1}$ then $u^n = 1$. By Theorem 17, we see that u it is an nth root of unity. Hence $\zeta' \stackrel{\text{def}}{=} u^{-1}$ is also a root of unity, and $w = r^{1/n}e^{\theta i/n}\zeta'$. □

Theorem 20 (Nth roots)**.** *Let n be a positive integer. Then every complex number $z \neq 0$ has exactly n distinct nth roots.*

Proof. There are at most n distinct nth roots of z. This follows from the previous theorem and the fact that there are only n distinct nth roots of unity.

Suppose ζ and ζ' are roots of unity such that

$$r^{1/n} e^{\theta i/n} \zeta = r^{1/n} e^{\theta i/n} \zeta'.$$

By cancelling, we get $\zeta = \zeta'$. So distinct nth roots of unity give distinct nth roots of z. So z has exactly n such roots. □

Remark 9. Of course 0 has 0 for an nth root. In fact, 0 is the only nth root of 0.

Informal Exercise 7. Find the general formula for the three cube roots of -1. The formula should be in terms of the polar form of these roots. Use polar coordinates to plot these three roots in the complex plane. Describe and plot the four 4th roots of $1+i$. Describe and plot the five fifth roots of unity. Describe and plot the 6 distinct 6th roots of 2^6.