CHAPTER 9: CONSTRUCTING THE REAL NUMBERS

MATH 378, CSUSM. SPRING 2016.

1. Introduction

In this chapter we introduce the field of real numbers $\mathbb{R}$. There are several ways to introduce the real numbers. Three popular approaches are to introduce $\mathbb{R}$ with (i) new axioms, with (ii) Dedekind cuts of $\mathbb{Q}$, or with (iii) Cauchy sequences in $\mathbb{Q}$. We will use the third approach and construct real numbers as equivalence classes of Cauchy sequences of rational numbers. This approach is chosen since it avoids the need for additional axioms by building on the previously developed number systems, and it gives students practice with sequences in general and Cauchy sequences in particular.

The main theorem of this chapter is that $\mathbb{R}$, as constructed from Cauchy sequences, is a complete ordered field.

2. The real numbers

Our idea for constructing $\mathbb{R}$ is based on two intuitive principles: (1) every Cauchy sequence in $\mathbb{Q}$ should determine a real number, and (2) equivalent sequences should determine the same real number. The second principle can be reexpressed as the requirement that every Cauchy sequence in an equivalence class $\big[(a_i)\big]$ should determine the same real number. This idea leads us to the idea that real numbers correspond to equivalence classes $\big[(a_i)\big]$ of Cauchy sequences.

Finally we take one more conceptual step: real numbers don't merely *correspond* to equivalence classes of Cauchy sequence, but can be *defined* as equivalence classes of Cauchy sequences. In other words, if we wish to construct the real numbers then they have to be defined somehow, why not define them via this intuitive correspondence?[1]

Definition 1. If (a_i) is a Cauchy sequence in $\mathbb{Q}$, then let $\big[(a_i)\big]$ be the equivalence class containing (a_i) under the equivalence relation $\sim$ on the set of sequences in $\mathbb{Q}$. We call $\big[(a_i)\big]$ a *real number*.

Date: April 13, 2016. Author: Professor W. Aitken (2007-2016).

[1]As mentioned in the introduction, there is another intuitive correspondence with Dedekind cuts of rational numbers. So we could also define real numbers as Dedekind cuts. The Cauchy sequence approach and the Dedekind cut approach lead to isomorphic ordered fields, so from the mathematical point of view it does not matter which approach is followed.

Definition 2. The set of real numbers $\mathbb{R}$ is defined as follows:

$$\mathbb{R} \stackrel{\text{def}}{=} \left\{ \left[(a_i)\right] \;\middle|\; (a_i) \textit{ is a Cauchy sequence in } \mathbb{Q} \right\}.$$

In order to make $\mathbb{R}$ into a field we need to define an addition and multiplication operation on $\mathbb{R}$.

Definition 3. Let $\left[(a_i)\right]$ and $\left[(b_i)\right]$ be real numbers. Then

$$\left[(a_i)_{i\ge n_0}\right] + \left[(b_i)_{i\ge m_0}\right] \stackrel{\text{def}}{=} \left[(a_i + b_i)_{i\ge l_0}\right]$$

and

$$\left[(a_i)_{i\ge n_0}\right] \cdot \left[(b_i)_{i\ge m_0}\right] \stackrel{\text{def}}{=} \left[(a_i b_i)_{i\ge l_0}\right].$$

Here l_0 is the maximum of n_0 and m_0. Our definitions give two binary operations $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

In order to check that these definitions are well-defined we need to verify three facts that are not totally obvious: (i) $(a_i + b_i)$ and $(a_i b_i)$ are Cauchy sequences, (ii) if $(a_i') \sim (a_i)$ then we can replace (a_i) with (a_i') in the definition and the resulting sum and product will give the same real number, and (iii) if $(b_i') \sim (b_i)$ then we can replace (b_i) with (b_i') in the definition and the resulting sum and product will give the same real number.

The remainder of this section will be devoted to verifying the facts needed to confirm that the definition is well-defined.

Lemma 1. *Suppose that $(a_i)_{i\ge n_0}$ and $(b_i)_{i\ge m_0}$ are Cauchy sequences in an ordered field F. Then $(a_i + b_i)_{i\ge l_0}$ and $(a_i b_i)_{i\ge l_0}$ are also Cauchy. Here l_0 is the maximum of n_0 and m_0.*

Proof. First we prove the result for products, and leave the easier sum case to the reader.

Let $\varepsilon > 0$. We must find a suitable N. We know that Cauchy sequences are bounded (Chapter 8). So there is a bound A such that $|a_i| \le A$ for all terms a_i of the first sequence. Likewise, there is a bound B such that $|b_i| \le B$ for all terms b_i of the second sequence. Clearly we can assume that A and B are chosen to be positive.

Let $\varepsilon_1 = \varepsilon/(2B)$. By assumption (a_i) is Cauchy. So there is an integer N_1 such that $i, j \ge N_1$ implies $|a_i - a_j| < \varepsilon_1$. Similarly, if $\varepsilon_2 = \varepsilon/(2A)$, there is an integer N_2 such that $i, j \ge N_2$ implies $|b_i - b_j| < \varepsilon_2$. Let N be the maximum of N_1 and N_2. If $i, j \ge N$, then

$$\begin{aligned} |a_i b_i - a_j b_j| &= |a_i b_i - a_i b_j + a_i b_j - a_j b_j| \\ &\le |a_i b_i - a_i b_j| + |a_i b_j - a_j b_j|. \end{aligned}$$

Observe that

$$|a_i b_i - a_i b_j| = |a_i||b_i - b_j| \le A|b_i - b_j| < A\varepsilon_2 = \varepsilon/2.$$

Similarly,

$$|a_i b_j - a_j b_j| = |a_i - a_j||b_j| \le |a_i - a_j|B < \varepsilon_1 B = \varepsilon/2.$$

Thus $|a_i b_i - a_j b_j| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ as desired. □

Exercise 1. Finish the above proof for the case of $(a_i + b_i)_{i \geq l_0}$. Hint: use $\varepsilon' = \varepsilon/2$ to find N_1 and N_2.

Lemma 2. *Let $(a_i), (a'_i), (b_i), (b'_i)$ be sequences with values in an ordered field F. if $(a_i) \sim (a'_i)$ then*

$$(a_i + b_i) \sim (a'_i + b_i),$$

and if $(b_i) \sim (b'_i)$ then

$$(a_i + b_i) \sim (a_i + b'_i).$$

Similarly, if $(a_i) \sim (a'_i)$ then

$$(a_i b_i) \sim (a'_i b_i),$$

and if $(b_i) \sim (b'_i)$ then

$$(a_i b_i) \sim (a_i b'_i).$$

Proof. We leave the (easier) case of sums to the reader. We prove the first statement for products; the second statement is similar. So we assume that $(a_i) \sim (a'_j)$, and prove that $(a_i b_i) \sim (a'_i b_i)$. In other words, for each $\varepsilon > 0$ in F we find a $N \in \mathbb{N}$ such that

$$i \geq N \;\Rightarrow\; |a_i b_i - a'_i b_i| < \varepsilon.$$

To find N, we use the fact that (b_i) is Cauchy, so is bounded (Chapter 8). So there is a $B \in F$ such that $|b_i| \leq B$ for all terms of the sequence. Clearly we can choose B to be positive. Thus B^{-1} is also positive. Let $\varepsilon' = B^{-1}\varepsilon$. Since $(a_i) \sim (a'_j)$, there is a $N' \in \mathbb{N}$ such that

$$i \geq N' \quad\Longrightarrow\quad |a_i - a'_i| < \varepsilon'.$$

Thus, if we define N to be N' and if $i \geq N$ then

$$|a_i b_i - a'_i b_i| = |a_i - a'_i| \cdot |b_i| \;\leq\; |a_i - a'_i| B \;<\; \varepsilon' B.$$

Since $\varepsilon' B = \varepsilon$ we have

$$i \geq N \quad\Longrightarrow\quad |a_i b_i - a'_i b_i| < \varepsilon$$

as desired. □

Exercise 2. Complete the proof by proving the case of $(a_i + b_i) \sim (a'_i + b_i)$. Hint: you do not need boundedness for Cauchy sequences in that case. In fact, it is possible to just use $\varepsilon' = \varepsilon$.

Remark 1. Because of the above lemmas, we now know that addition and multiplication are well-defined operations on $\mathbb{R}$.

3. The Real Numbers $\mathbb{R}$ as a Commutative Ring

Our next step is to prove that t $\mathbb{R}$ is a commutative ring. It is a bit harder to show it is a field, and so we will postpone that for a later section.

Theorem 3. *Addition and multiplication on $\mathbb{R}$ are commutative and associative.*

Exercise 3. Prove the above theorem.

Recall that in Chapter 7 we proved that the constant sequences (c) converge to c. Since they converge, such sequences are Cauchy. So if $c \in \mathbb{Q}$, the constant sequence gives a real number $[(c)] \in \mathbb{R}$. We are particularly interest in $[(0)]$ and $[(1)]$:

Theorem 4. *An additive identity for $\mathbb{R}$ exists and is $[(0)]$. A multiplicative identity for $\mathbb{R}$ exists and is $[(1)]$.*

Remark 2. Identities, if they exist, are unique.[2] Thus we can say "*the* additive identity" and "*the* multiplicative identity" of $\mathbb{R}$.

Proof. Let $x = \big[(a_i)\big]$ be an arbitrary real number. By definition of $+$ in $\mathbb{R}$,

$$x + [(0)] = \big[(a_i)\big] + \big[(0)\big] = \big[(a_i + 0)\big] = \big[(a_i)\big] = x$$

where the next-to-last equality is due to the fact that 0 is the additive identity of $\mathbb{Q}$ (Chapter 6). By the commutative law (Theorem 3) we get $[(0)] + x = x + [(0)] = x$. Thus $[(0)]$ is the additive identity.

The proof that $[(1)]$ is the multiplicative identity is similar. □

We now consider inverses.

Lemma 5. *If (a_i) is a Cauchy sequence in an ordered field F, then $(-a_i)$ is also Cauchy.*

Proof. Suppose (a_i) is Cauchy. Let $\varepsilon > 0$ be given. In order to show that $(-a_i)$ is Cauchy, we must find an $N \in \mathbb{N}$ such that $\big|(-a_i) - (-a_j)\big| < \varepsilon$ for all $i, j \geq N$. Since (a_i) is Cauchy, there is a $N' \in \mathbb{N}$ such that $|a_i - a_j| < \varepsilon$ for all $i, j \geq N'$. Let $N = N'$. If $i, j \geq N$ then

$$\big|(-a_i) - (-a_j)\big| = \big|(-1)(a_i - a_j)\big| = |-1||a_i - a_j| = |a_i - a_j|.$$

But $|a_i - a_j| < \varepsilon$, so $\big|(-a_i) - (-a_j)\big| < \varepsilon$ as desired. Thus $(-a_i)$ is Cauchy. □

Theorem 6. *Every element of $\mathbb{R}$ has an additive inverse. In fact, if $x = \big[(a_i)\big]$, then $-x = \big[(-a_i)\big]$.*

Proof. Let $x \in \mathbb{R}$. Write x as $\big[(a_i)\big]$ where (a_i) is Cauchy in $\mathbb{Q}$. By Lemma 5 the sequence $(-a_i)$ is also Cauchy, so $y = \big[(-a_i)\big]$ is a real number. We leave it to the reader to show that y is the additive inverse of x. □

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[2] For any binary operation on a set S, one can show that if there is an identity, it must be unique. For example, if 0 and $0'$ are additive identities, $0 = 0 + 0' = 0'$.

Exercise 4. Complete the proof of the above theorem. Hint: show $x + y = y + x = 0$.

As we will see, *multiplicative* inverses are trickier. Fortunately we do not need multiplicative inverses to conclude that $\mathbb{R}$ is a ring:

Theorem 7. *The real numbers $\mathbb{R}$ form a commutative ring.*

Exercise 5. Prove the above. Hint: some steps have been proved above. What laws have not been proved yet?

Now that we know that $\mathbb{R}$ is a commutative ring, we can use all the familiar algebraic manipulations and laws valid in rings.

4. The Canonical Embedding

Now that we have constructed $\mathbb{R}$ we wish to regard $\mathbb{Q}$ as a subset of $\mathbb{R}$. To do so we need to embed $\mathbb{Q}$ into $\mathbb{R}$. This will require an injective map $\mathbb{Q} \to \mathbb{R}$. What we will do is send any $r \in \mathbb{Q}$ to the constant sequence $(r)_{i\in\mathbb{N}}$.

Theorem 8. *Let $b, c \in F$ where F is an ordered field. Suppose $b \neq c$. Then $(b)_{i\in\mathbb{N}}$ and $(c)_{i\in\mathbb{N}}$ are non-equivalent.*

Proof. Observe that both sequences converge. If they were equivalent then they would have to have the same limit (Chapter 7). However, the first sequence converges to b and the second to c. A contradiction. □

Corollary 9. *Let $b, c \in \mathbb{Q}$ be distinct. Then $[(b)] \neq [(c)]$ in $\mathbb{R}$.*

Proof. This follows from basic properties of equivalence classes: two non-equivalent elements give non-equal (and disjoint) equivalence classes. □

Definition 4 (Canonical embedding)**.** We define the *canonical embedding* $\mathbb{Q} \to \mathbb{R}$ by the rule $c \mapsto \big[(c)_{i\in\mathbb{N}}\big]$.

Theorem 10. *The canonical embedding $\mathbb{Q} \to \mathbb{R}$ is injective.*

Exercise 6. Prove the above theorem using Corollary 9.

Once we have a canonical embedding $\mathbb{Q} \to \mathbb{R}$ we can use this to identify elements of $\mathbb{Q}$ with certain elements of $\mathbb{R}$. Thus we can think of $\mathbb{Q}$ as being a subset of $\mathbb{R}$.

We can go further. We can think of $\mathbb{Q}$ as a *subfield* of $\mathbb{R}$ (a concept defined in Chapter 7). To do so, we need to check that the addition and multiplication of $\mathbb{R}$ extends the addition and multiplication of $\mathbb{Q}$ defined in Chapter 6. In other words, when we are working with addition and multiplication on $\mathbb{Q}$ we want to be assured that we get the same result whether we use the addition of $\mathbb{Q}$ (Chapter 6) or the addition of $\mathbb{R}$ (this chapter). This is demonstrated in the following lemma.

Lemma 11. *The definitions of addition and multiplication on $\mathbb{R}$ extend the definitions of addition and multiplication on $\mathbb{Q}$. So $\mathbb{Q}$ is a subfield of $\mathbb{R}$.*

Proof. We give the proof for addition; the proof for multiplication is similar. Let $a, b \in \mathbb{Q}$ be given and let $+_{\mathbb{Q}}$ be the addition defined in Chapter 6. Let $+_{\mathbb{R}}$ be the addition defined in the current chapter. We must show that $a +_{\mathbb{Q}} b$ is identified with the same real number as $a +_{\mathbb{R}} b$ (via the canonical embedding).

This is actually easy once we figure out what is involved. The canonical embedding maps the rational number $a +_{\mathbb{Q}} b$ to the equivalence class of the constant sequence $(a +_{\mathbb{Q}} b)_{i\in\mathbb{N}}$. Since a is identified with the equivalence class of the constant sequence $(a)_{i\in\mathbb{N}}$ and b is identified with the equivalence class of $(b)_{i\in\mathbb{N}}$, the sum $a +_{\mathbb{R}} b$ is equal to the sum $\left[(a)_{i\in\mathbb{N}}\right] +_{\mathbb{R}} \left[(b)_{i\in\mathbb{N}}\right]$. By the definition of $+_{\mathbb{R}}$

$$\left[(a)_{i\in\mathbb{N}}\right] +_{\mathbb{R}} \left[(b)_{i\in\mathbb{N}}\right] = \left[(a +_{\mathbb{Q}} b)_{i\in\mathbb{N}}\right].$$

The result follows. □

Remark 3. Since 0 in $\mathbb{Q}$ is identified with the equivalence class $[(0)]$ of the constant sequence (0), and since $[(0)]$ is the additive identity of $\mathbb{R}$, we usually write 0 for the additive identity of $\mathbb{R}$. This is consistent with the practice of writing 0 for the additive identity of any ring.

Similarly, we write 1 for the multiplicative identity of $\mathbb{R}$.

Remark 4. In a similar manner, we see that additive inverse in $\mathbb{R}$ extends additive inverse in $\mathbb{Q}$. This follows from the identity $-[(r)] = [(-r)]$ proved above.

Subtraction in $\mathbb{R}$ extends the subtraction in $\mathbb{Q}$. This follows from the definition of $r - s$ (in any ring) as $r + (-s)$, and the fact that addition and additive inverse in $\mathbb{R}$ extend the operations in $\mathbb{Q}$.

5. Positive Real Numbers

An important part of showing that $\mathbb{R}$ is an ordered field is to define the set of positive real numbers P, and to show that this set has the required properties: closure and trichotomy. We will do this even before proving that $\mathbb{R}$ is a field.

Since a real number can be thought of as $[(a_i)]$ where (a_i) is Cauchy, we might be tempted to say that $x = [(a_i)]$ is positive if each a_i is positive. Warning: this does not work. For example, the sequence $(1/i^2)$ converges to 0, and so is equivalent to the constant sequence (0). Thus $[(1/i^2)]$ is zero even though all its terms are strictly positive.

Furthermore, if a sequence has a finite number of zero or negative terms, and the rest are positive, then the sequence could represent a positive number. Thus there are two ways in which the naive definition of positive is defective. The following definition corrects both deficiencies.

Definition 5 (Positive)**.** A *positive-type Cauchy sequence* in an ordered field F is a Cauchy sequence (a_i) with the following property: there is a positive $d \in F$ and an $N \in \mathbb{N}$ such that $a_i \geq d$ for all $i \geq N$.

A *positive* real number is a real number of the form $[(a_i)]$ where (a_i) is a positive-type Cauchy sequence with values in $\mathbb{Q}$.

Exercise 7. Suppose (a_i) and (b_i) are Cauchy sequences where $(a_i) \sim (b_i)$. Show that (a_i) is positive-type Cauchy if and only if (b_i) is.

Hint: suppose there is a $d > 0$ and a $N \in \mathbb{N}$ such that $a_i \geq d$ for all $i \geq N$. We must find d' and N' that work for (b_i). Let $\varepsilon = d/2$ and choose a N_0 so that $|a_i - b_i| < \varepsilon$ for all $i \geq N_0$. Why does such a N_0 exist? Choose N' as the maximum of N and N_0. What do you think d' should be? Prove that your choice of N' and d' work for (b_i).

Remark 5. The above exercise tells us that if we wish to decide if a real number x is positive, we can take *any* Cauchy sequence from the equivalence class defining x, and check the definition for that particular sequence.

For example, if $r \in \mathbb{Q}$ is thought of as a real number via the canonical embedding, then we can decide if r is positive in $\mathbb{R}$ just by looking at the constant sequence (r). From this we conclude that a rational number r is positive in $\mathbb{R}$ if and only if it is positive in $\mathbb{Q}$. Thus the present definition of positive for $\mathbb{R}$ is compatible with the definition of Chapter 6. (Similarly for negative).

Theorem 12 (Closure)**.** *If $x, y \in \mathbb{R}$ are positive then so is $x + y$ and xy.*

Proof. Let $x = [(a_i)]$ and $y = [(b_i)]$ where (a_i) and (b_i) are positive-type Cauchy sequences of rational numbers. By definition there is a positive $d_1 \in \mathbb{Q}$ and a $N_1 \in \mathbb{N}$ such that $a_i \geq d_1$ for all $i \geq N_1$. Likewise, there is a positive $d_2 \in \mathbb{Q}$ and a $N_2 \in \mathbb{N}$ such that $b_i \geq d_2$ for all $i \geq N_2$. Let $d = d_1 + d_2$. We know that $d > 0$ (Chapter 6). Let N be the maximum of N_1 and N_2. If $i \geq N$, then

$$a_i + b_i \geq d_1 + b_i \geq d_1 + d_2 = d$$

by properties of rational numbers (Chapter 6). Thus $x + y = [(a_i + b_i)]$ is positive.

The proof for xy is similar. □

Exercise 8. Prove the above for the case of multiplication.

We now want to prove a trichotomy law: for all $x \in \mathbb{R}$ exactly one of the following occurs (i) x is positive, (ii) $x = 0$, or (iii) $-x$ is positive. In the third case we also say that x is *negative.*

We divide the proof of this law into lemmas:

Lemma 13. *The real number* 0 *is neither positive nor negative.*

Proof. The real number 0 is defined by the constant sequence $(a_i) = (0)$. Since $a_i = 0$, there can be no $d > 0$ and N such that $a_i \geq d$ for all $i \geq N$. (Since if $a_i \geq d$ and $d > 0$ then $a_i > 0$, a contradiction). So 0 cannot be positive.

We now show that 0 cannot be negative. Suppose 0 is negative. Then -0 is positive (by definition of negative). But we have already shown that $-0 = 0$ is not positive. □

Lemma 14. *Let $x \in \mathbb{R}$. It is not possible for both x and $-x$ to be positive.*

Exercise 9. Prove the above. Hint: suppose not. Write $x = [(a_i)]$. Then $-x = [(-a_i)]$ by Theorem 6. Define a d_1 and N_1 for (a_i) and d_2 and N_2 for $(-a_i)$. Let i be the maximum of N_1 and N_2, and show that a_i is both positive and negative in $\mathbb{Q}$.

Remark 6. Notice how we use a trichotomy law for $\mathbb{Q}$ (Chapter 6) to help prove a trichotomy law for $\mathbb{R}$.

The above two lemmas show part of the trichotomy law: together they show that at least one of the trichotomy conditions hold. We still need to show that at least one condition holds. This follows from the next lemma. The proof is a bit tricky. Part of its trickiness comes from the need to manipulate quantifiers.

Lemma 15. *If $x \neq 0$ is a real number, then either x or $-x$ is positive.*

Proof. Write $x = \big[(a_i)\big]$ where (a_i) is a Cauchy sequence in $\mathbb{Q}$. By Theorem 6, $-x = \big[(-a_i)\big]$.

By assumption $x \neq 0$. In other words, $\big[(a_i)\big] \neq \big[(0)\big]$. Thus we have that (a_i) and the constant sequence (0) are non-equivalent as $\mathbb{Q}$-sequences. When we negate the definition of equivalence we find that there exists a positive $\varepsilon \in \mathbb{Q}$ such that for all $N \in \mathbb{N}$ there is an integer $i \geq N$ with $|a_i - 0| \geq \varepsilon$. Fix such an ε_0 for what follows.

Let $\varepsilon' = \varepsilon_0/2$. Since (a_i) is Cauchy, there is a $N' \in \mathbb{N}$ such that

$$i, j \geq N' \implies |a_i - a_j| < \varepsilon'.$$

Above we determined that for all $N \in \mathbb{N}$ there is an integer $i \geq N$ with $|a_i - 0| \geq \varepsilon_0$. In particular, there is an integer $i_0 \geq N'$ with $|a_{i_0}| \geq \varepsilon_0$. By a result of Chapter 7, this implies that $a_{i_0} \geq \varepsilon_0$ or $a_{i_0} \leq -\varepsilon_0$.

We first consider the case where $a_{i_0} \geq \varepsilon_0$. Suppose $j \geq N'$. Since both $i_0, j \geq N'$ we have $|a_j - a_{i_0}| < \varepsilon'$ by the Cauchy condition. By a result of Chapter 7, $-\varepsilon' < a_j - a_{i_0}$. This gives us $-\varepsilon' + a_{i_0} < a_j$. Therefore,

$$a_j > a_{i_0} - \varepsilon' \geq \varepsilon_0 - \varepsilon' = 2\varepsilon' - \varepsilon' = \varepsilon'.$$

In summary, we have $\varepsilon' > 0$ and $N' \in \mathbb{N}$ such that $j \geq N'$ implies $a_j \geq \varepsilon'$. By Definition 5, we conclude that $x = \big[(a_i)\big]$ is positive.

Next we consider the case where $a_{i_0} \leq -\varepsilon_0$. Suppose $j \geq N'$. Since $i_0, j \geq N'$ we have $|a_j - a_{i_0}| < \varepsilon'$ by the Cauchy condition. By a result of Chapter 7, $a_j - a_{i_0} < \varepsilon'$. This gives us $a_j < \varepsilon' + a_{i_0}$. Thus

$$a_j < \varepsilon' + a_{i_0} \leq \varepsilon' - \varepsilon_0 = \varepsilon' - 2\varepsilon' = -\varepsilon'.$$

Hence, $-a_j \geq \varepsilon'$. In summary, we have $\varepsilon' > 0$ and $N' \in \mathbb{N}$ such that $j \geq N'$ implies $-a_j \geq \varepsilon'$. By definition of positive, we conclude that $-x = \big[(-a_i)\big]$ is positive.

So either x or $-x$ is positive, assuming $x \neq 0$. □

Putting these lemmas together, we conclude the following:

Theorem 16. *For every $x \in \mathbb{R}$ exactly one of the following occurs: (i) x is positive, (ii) $x = 0$, or (iii) $-x$ is positive.*

6. The Real Numbers $\mathbb{R}$ as an Ordered Field

We will now see that every non-zero element of $\mathbb{R}$ has a multiplicative inverse. We first focus on positive elements. Recall that a sequence (a_i) is *positive-type Cauchy* if there is a $N \in \mathbb{N}$ and and $d > 0$ such that $i \geq N$ implies $a_i \geq d$ (Definition 5). In particular, such sequences are allowed to have a finite number of zero (or even negative) terms. This is why we have to change from n_0 to k_0 in the following lemma.

Lemma 17. *Suppose $(a_i)_{i \geq n_0}$ is a positive-type Cauchy sequence in an ordered field F. Then there is an integer k_0 such that $\left(a_i^{-1}\right)_{i \geq k_0}$ is a Cauchy sequence.*

Proof. By definition of positive-type Cauchy sequence (Definition 5), there is a $N \in \mathbb{N}$ and positive $d \in F$ such that $a_i \geq d$ for all $i \geq N$. Let $k_0 = N$. In particular, if $i \geq k_0$ then $a_i \neq 0$, so a_i has a multiplicative inverse in F. By properties of ordered fields (Chapter 7) $a_i^{-1} \leq d^{-1}$, and a_i^{-1} is positive.

We wish to show that $\left(a_i^{-1}\right)_{i \geq k_0}$ is Cauchy. So let $\varepsilon \in F$ be positive; we want an N such that if $i, j \geq N$ then $\left|a_i^{-1} - a_j^{-1}\right| < \varepsilon$.

Let $\varepsilon' = \varepsilon d^2$. Since d and ε are positive, so is ε'. Since (a_i) is Cauchy, there is a N' such that $|a_i - a_j| < \varepsilon'$ for all $i, j \geq N'$. Let N be the maximum of N' and k_0. If $i, j \geq N$ then

$$
\begin{aligned}
|a_i^{-1} - a_j^{-1}| &= \left|(a_j - a_i)a_i^{-1}a_j^{-1}\right| \quad (F \text{ is a field}) \\
&= |a_j - a_i|\left|a_i^{-1}\right|\left|a_j^{-1}\right| \quad (F \text{ is an ordered field}) \\
&= |a_j - a_i|\, a_i^{-1}a_j^{-1} \quad (i, j \geq k_0, \text{ so } a_i, a_j \geq d > 0) \\
&\leq |a_j - a_i|\, d^{-1}d^{-1} \quad (i, j \geq k_0, \text{ so } a_i, a_j \geq d > 0) \\
&< \varepsilon' d^{-2} = \varepsilon. \qquad (i, j \geq N')
\end{aligned}
$$

So $|a_i^{-1} - a_j^{-1}| < \varepsilon$ as desired. We conclude that $\left(a_i^{-1}\right)_{i \geq k_0}$ is Cauchy. □

Lemma 18. *Let $x \in \mathbb{R}$. If x is positive, then x has a multiplicative inverse.*

Proof. Write $x = [(a_i)]$ where (a_i) is a positive-type Cauchy sequence of rational numbers. By the previous lemma, there is a k_0 such that $\left(a_i^{-1}\right)_{i \geq k_0}$ is Cauchy. Thus

$$y = \left[(a_i^{-1})_{i \geq k_0}\right]$$

is a real number. By definition of multiplication in $\mathbb{R}$,

$$xy = [(a_i)] \left[(a_i^{-1})\right] = \left[(a_i a_i^{-1})\right] = [(1)].$$

Thus $xy = 1$. By the commutative law for multiplication, $yx = xy = 1$. We have shown that x has a multiplicative inverse. □

Theorem 19. *Let $x \in \mathbb{R}$. If $x \neq 0$, then x has a multiplicative inverse.*

Exercise 10. Prove the above theorem. Hint: if x is positive, use the lemma. If $-x$ is positive, try $-y$ where y is the multiplicative inverse of $-x$. Use the fact that $x(-y) = (-x)y$ (true in any ring).

We now come to one of the main theorems of this chapter.

Theorem 20. *The real numbers $\mathbb{R}$ are an ordered field.*

Proof. We know that $\mathbb{R}$ is a commutative ring by Theorem 7. We know that $0 \neq 1$ by Corollary 9. Multiplicative inverses exist by Theorem 19. We conclude that $\mathbb{R}$ is a field.

To show that $\mathbb{R}$ is an ordered field we need to check that (i) the positive elements are closed under addition and multiplication, and (ii) the positive elements satisfy the trichotomy law. Both these were done in the previous section (Theorem 12 and Theorem 16). □

Remark 7. Now that we know that $\mathbb{R}$ is an ordered field, we can use all the definitions and results about ordered fields F from Chapter 7 including facts about $<$ and absolute values. You should review these definitions and results from Chapter 7.

Remark 8. As mentioned above, positivity defined for $\mathbb{R}$ is compatible with the earlier concept of positivity defined for $\mathbb{Q}$. Since $x < y$ means $y - x$ is positive, if follows that inequality in $\mathbb{R}$ is compatible with inequality in $\mathbb{Q}$. In other words, we can show that if $x, y \in \mathbb{Q}$ then $x < y$ holds for $\mathbb{Q}$ if and only if it holds for $\mathbb{R}$.

7. Relationship between $\mathbb{R}$ and $\mathbb{Q}$

In this section we will consider a few useful results relating $\mathbb{R}$ and $\mathbb{Q}$. For example, we will see that Cauchy sequences of rational numbers always converge to real numbers, and that all real numbers are limits of rational sequences. We will also see that $\mathbb{R}$ is an archimdean ordered field. Recall from Chapter 7 that this implies that $\mathbb{Q}$ is dense in $\mathbb{R}$; in other words, between any two distinct real numbers we can always find a rational number.

We begin with a lemma that can be used to establish $x \leq y$ where $x, y \in \mathbb{R}$.

Lemma 21. *Suppose (a_i) and (b_i) are two Cauchy sequences of rational numbers, and let $x = [(a_i)]$ and $y = [(b_i)]$ be the corresponding real numbers. If there is a $k \in \mathbb{N}$ such that $a_i \leq b_i$ for all $i \geq k$, then $x \leq y$.*

Remark 9. If we have $a_i < b_i$ instead, we cannot necessarily conclude that $x < y$. Without extra information, we can only conclude that $x \leq y$.

Proof. Suppose that $a_i \leq b_i$ for all $i \geq k$. We wish to show $x \leq y$. Suppose instead that $x > y$. So $x - y$ is positive. Now $x - y$ is given by $[(a_i - b_i)]$. So $(a_i - b_i)$ is positive-type Cauchy sequence since $x - y$ is positive. In other words, there is a $N \in \mathbb{N}$ and a positive $d \in \mathbb{Q}$ such that $a_i - b_i \geq d$ for all $i \geq N$. Since $d > 0$ this means that $a_i > b_i$ for all $i \geq N$.

Let i be the maximum of k and N. Then we have both that $a_i \leq b_i$ and $a_i > b_i$, contradicting trichotomy in $\mathbb{Q}$. □

Corollary 22. *Suppose $x \in \mathbb{R}$ is given by $x = [(a_i)]$. Suppose that b is a rational number. If there is a $k \in \mathbb{N}$ such that $a_i \leq b$ for all $i \geq k$, then $x \leq b$ (where here we are thinking of b as a real number). If, instead, there is a $k \in \mathbb{N}$ such that $a_i \geq b$ for all $i \geq k$, then $x \geq b$.*

Proof. For the first statement, apply Lemma 21 to the sequences (a_i) and (b). For the second statement, switch the order and apply Lemma 21 again. □

Theorem 23. *Suppose $y > 0$ is a real number. Then there is a positive integer n such that $1/n \leq y$.*

Proof. Write y as $[(a_i)]$ where (a_i) is a positive-type Cauchy sequence of rational numbers. By Definition 5, there is a $N \in \mathbb{N}$ and a positive $d \in \mathbb{Q}$ such that $a_i \geq d$ for all $i \geq N$. Write $d = m/n$ where $m, n \in \mathbb{N}$ are positive. Thus $a_i \geq d \geq 1/n$ for all $i \geq N$.

Since $1/n \leq a_i$ for all $i \geq N$, we get $1/n \leq y$ by the above Corollary. □

Theorem 24. *The real numbers $\mathbb{R}$ form an archimedean ordered field.*

Proof. This follows from the previous theorem (by a result in Chapter 7). □

Corollary 25. *The field $\mathbb{Q}$ is dense in $\mathbb{R}$. In other words, if $x, y \in \mathbb{R}$ are such that $x < y$, there is a $r \in \mathbb{Q}$ with $x < r < y$.*

Exercise 11. Which theorem in Chapter 7 yields the above corollary?

The following theorem says that if a Cauchy sequence of rational numbers represents a certain real number, then the Cauchy sequence converges to the real number.

Theorem 26. *Let (a_i) be a Cauchy sequence of rational numbers. Then (a_i) considered as a sequence of real numbers converges to the real number x where $x = [(a_i)]$.*

Proof. Let ε be an arbitrary positive real number. We must find a $N \in \mathbb{N}$ such that $|a_i - x| < \varepsilon$ for all $i \geq N$. It seems like we should be able to use the definition of Cauchy sequence to find such a N. There is a slight problem: (a_i) is a Cauchy sequence in $\mathbb{Q}$, but ε is an arbitrary positive element of $\mathbb{R}$.

We solve the problem by choosing a positive integer n such that $1/n \leq \varepsilon$ (Theorem 24). Let $\varepsilon' = 1/n$, and note that ε' is a positive element of $\mathbb{Q}$ such that $\varepsilon' < \varepsilon$. By definition of Cauchy sequence in $\mathbb{Q}$, we have an $N \in \mathbb{N}$ such that $|a_i - a_j| < \varepsilon'$ for all $i, j \geq N$. We will show that this N has the desired property for convergence. In other words, that $|a_i - x| < \varepsilon$ for all $i \geq N$. (We will actually show $|a_i - x| \leq \varepsilon'$, which is good enough).

So fix $i \geq N$. Recall that a_i is thought of as both a rational number and a real number via the canonical embedding $\mathbb{Q} \to \mathbb{R}$. More precisely, a_i as a real number is defined by $[(c_j)]$ where (c_j) is the constant sequence whose terms are all equal to the rational number a_i. Recall also that $x = [(a_j)]$.

Assume temporarily that $j \geq N$. By our choice of N we have $|c_j - a_j| < \varepsilon'$ since $c_j = a_i$. By properties of absolute values (in $F = \mathbb{Q}$),

$$-\varepsilon' < c_j - a_j < \varepsilon'.$$

The above holds for all $j \geq N$, so we can use Corollary 22 to conclude that

$$-\varepsilon' \leq [(c_j - a_j)] \leq \varepsilon'.$$

In other words (using properties of absolute values in $F = \mathbb{R}$),

$$\big|[(c_j - a_j)]\big| \leq \varepsilon'.$$

This implies that

$$|a_i - x| = \big|[(c_j)] - [(a_j)]\big| = \big|[(c - a_j)]\big| \leq \varepsilon' < \varepsilon.$$

This completes the proof that (a_i) converges to x. □

Corollary 27. *Every Cauchy sequence of rational numbers converges to some real number.*

Proof. Let (a_i) be a Cauchy sequence of rational numbers. Let $x = [(a_i)]$. By Theorem 26, (a_i) has limit x. □

Remark 10. Let (a_i) be a sequence of rational numbers. There is some ambiguity of what *Cauchy* means for (a_i) when we embed $\mathbb{Q}$ into $\mathbb{R}$. We can mean the Cauchy condition holds for all positive ε in $F = \mathbb{Q}$. Call this $\mathbb{Q}$-Cauchy. Or we can mean that the Cauchy condition holds for all positive ε in $F = \mathbb{R}$. Call this $\mathbb{R}$-Cauchy.

In the above theorem and corollary we are thinking of $\mathbb{Q}$-Cauchy. We proved that any $\mathbb{Q}$-Cauchy sequence gives a convergent sequence in $\mathbb{R}$. But convergent sequences are automatically Cauchy (Chapter 8). Thus any $\mathbb{Q}$-Cauchy sequence is an $\mathbb{R}$-Cauchy sequence .

Conversely, any $\mathbb{R}$-Cauchy sequence whose terms are in $\mathbb{Q}$ is a $\mathbb{Q}$-Cauchy sequence. (If a condition holds for all $\varepsilon > 0$ in $\mathbb{R}$ then it will hold for all $\varepsilon > 0$ in $\mathbb{Q}$ since $\mathbb{Q} \subseteq \mathbb{R}$). We conclude that if (a_i) is a sequence of rational numbers, there is no difference between being $\mathbb{Q}$-Cauchy or $\mathbb{R}$-Cauchy.

Corollary 28. *Every real number is the limit of a sequence of rational numbers*

Proof. Let $x = [(a_i)]$ be a real number. By Theorem 26, (a_i) has limit x. □

Corollary 29. *If $x \in \mathbb{R}$ and if $\varepsilon \in \mathbb{R}$ is positive, then there is a rational number $r \in \mathbb{Q}$ with $|x - r| < \varepsilon$.*

Proof. Since x is the limit of a sequence (a_i) of rational numbers, there is a $N \in \mathbb{N}$ such that $|a_i - x| < \varepsilon$ for all $i \geq N$. Let $r = a_N$. □

Remark 11. The proceeding two corollaries can also be proved as consequences of the archimdean property of $\mathbb{R}$: they hold for all archimedean ordered fields.

8. $\mathbb{R}$ is Complete

As we have discussed before, $\mathbb{Q}$ has "holes". For example, $\mathbb{Q}$ is missing a square root for 2. Because of this, $\mathbb{Q}$ has Cauchy sequences that do not converge. We will now show that the real numbers $\mathbb{R}$ do not have Cauchy sequences that fail to converge. So $\mathbb{R}$ is complete, and does not have "holes".

We begin by showing that every Cauchy sequence of real numbers converges. We already know, from the previous section, that every Cauchy sequence of rational numbers converges in $\mathbb{R}$. But this is not enough for our current needs. We need to extend the result to Cauchy sequences with terms in $\mathbb{R}$. We begin with a lemma.

Lemma 30. *If (a_i) is a sequence of real numbers, then there is a sequence (b_i) of rational numbers such that $(a_i) \sim (b_i)$. (Equivalence is taken with $F = \mathbb{R}$).*

Proof. For each a_i, we know by Corollary 29 that there is a rational number b_i such that $|a_i - b_i| < 1/i$. Consider the sequence (b_i) formed from such rational numbers.[3] We must show that $(a_i) \sim (b_i)$.

Let $\varepsilon \in \mathbb{R}$ be an arbitrary positive real number. We must find a $N \in \mathbb{N}$ such that $|a_i - b_i| < \varepsilon$ for all $i \geq N$. By Theorem 23 we can find a positive n such that $1/n < \varepsilon$. Let $N = n$. If $i \geq N$ then

$$\begin{aligned} |a_i - b_i| &< 1/i && \text{(choice of } b_i\text{)} \\ &\leq 1/n && (i \geq N \text{ and } N = n) \\ &< \varepsilon && \text{(choice of } n\text{)} \end{aligned}$$

Thus $(a_i) \sim (b_i)$ as desired. □

Theorem 31. *Every Cauchy sequences in $\mathbb{R}$ converges.*

Proof. Let (a_i) be a Cauchy sequence of real numbers. By Lemma 30 there is a sequence (b_i) of rational numbers such that $(b_i) \sim (a_i)$.

In Chapter 8 we proved that if two sequences are equivalent, and if one is Cauchy, then the other is. Since (a_i) is Cauchy, we conclude that (b_i) is also a Cauchy sequence. By Corollary 27 we conclude that (b_i) has a limit.

In Chapter 7 we proved that if two sequences are equivalent, and if one has a limit, then the other does as well. Since (b_i) has a limit, we conclude that (a_i) must have a limit. □

[3] In order to avoid using the axiom of choice, we can select b_i to have the smallest possible denominator, and among fractions with the smallest possible denominator we choose the smallest possible numerator.

Now for the main theorem.

Theorem 32. *The field $\mathbb{R}$ is a complete ordered field.*

Proof. In Chapter 8 we proved that if an ordered field is archimedean and if every Cauchy sequences converges in that field, then that field must be complete. So, since $\mathbb{R}$ is archimedean, and since every Cauchy sequence in $\mathbb{R}$ converges, $\mathbb{R}$ is a complete ordered field. □

Dedekind Cuts (Optional)

Another approach to constructing the real numbers is to use Dedekind cuts.[4]

Uniqueness (Optional)

Any two complete ordered fields are isomorphic as ordered fields.[5] One can define $\mathbb{R}$ to be any complete ordered field, and use the fact that any two constructions are isomorphic to conclude that this definition is well-defined.

[4]Definitions and discussion will be provided in a future draft.
[5]Definition of isomorphism, and discussion will be provided in a future draft.