1. Introduction

The goal of this chapter is to give a formal definition of the notion of a complete ordered field, and explore issues related to this concept. We will construct a complete ordered field $\mathbb{R}$ in the next chapter.

Recall that in Chapter 6 we showed that $\mathbb{Q}$ does not have a square root of 2. Informally this illustrates the incompleteness of $\mathbb{Q}$. One of the goals of this chapter is to show that any complete ordered field possesses a square root of 2, and in fact a square root of any non-negative element. To do so we will prove a basic version of the intermediate value theorem. Since $\mathbb{Q}$ lacks a square root for 2 we conclude that it is, as expected, incomplete in the formal sense.

The Intermediate Value Theorem requires the notion of continuous function. This is one of the key concepts of analysis, and we will just scratch the surface of this concept in this course. We want just enough results about continuity to prove the intermediate value theorem, and apply it to functions such as $f(x) = x^2$ in order to obtain square roots.

2. Completeness

We begin with a formal definition of the concept of complete. There are several equivalent ways to define this concept. For example, the following uses the existence of suprema, but obviously one could form an alternative definition that uses the existence of infima instead.

**Definition 1** (Complete). Let $F$ be an ordered field. We say that $F$ is complete if every nonempty subset $S \subseteq F$ which is bounded from above has a supremum (least upper bound).

The existence of suprema gives us the existence of infima as well:

**Theorem 1.** Suppose $F$ is a complete ordered field. Suppose $S \subseteq F$ is a nonempty subset which is bounded from below. Then $S$ has an infimum (greatest lower bound).

**Proof.** Consider the set $S' = \{-x \mid x \in S\}$ of additive inverses. Observe that $S'$ is nonempty and bounded from above. Since $F$ is complete, the set...
2 MATH 378, CSUSM. SPRING 2016.

\[ S \text{ has a supremum } M. \text{ Let } m = -M. \text{ Then observe that } m \text{ is the infimum of } S. \]

\[ \square \]

Exercise 1. Complete the above proof by giving a detailed justifications the two observations in the proof.

Theorem 2. Suppose \( F \) is a complete ordered field containing \( \mathbb{Q} \) as an ordered subfield. Then \( F \) is an archimedean ordered field.

Proof. From earlier results in Chapter 7 it is enough to show that for all \( y \in F \) there is an \( n \in \mathbb{N} \) such that \( n > y \).

Suppose instead that this condition fails. Then there is a \( y \in F \) such that \( n < y \) for all \( n \in \mathbb{N} \). This implies that \( \mathbb{N} \) is bounded. Since \( F \) is complete, this means that \( \mathbb{N} \) has a supremum \( M \). Since \( M - 1 < M \), the definition of supremum tells us that \( M - 1 \) is not an upper bound for \( \mathbb{N} \), so there must be an integer \( n \in \mathbb{N} \) such that \( M - 1 < n \). In other words, \( M < n + 1 \). Thus \( M \) is also not an upper bound, a contradiction. \( \square \)

3. Continuous Functions

Now we explore the concept of continuity in order to set the stage for the intermediate value theorem. Due to the emphasis we place on sequences in this course, we use the sequential definition of continuity. The next section (optional) gives another very common definition of continuity, and shows that the two definitions are equivalent.

Definition 2 (Continuity). Let \( S \) be a subset of an ordered field \( F \). Then a function \( f: S \to F \) is said to be continuous on \( S \) if for all converging sequences \((a_i)\) with terms and limit in \( S \), the sequence \((f(a_i))\) also converges, and

\[ \lim_{i \to \infty} f(a_i) = f \left( \lim_{i \to \infty} a_i \right). \]

The most basic examples of continuous functions are the identity and constant functions:

Theorem 3. Let \( S \) be a subset of an ordered field \( F \). Any constant function \( x \mapsto c \) with \( c \in F \) is a continuous function \( S \to F \). The identity function \( x \mapsto x \) is a continuous function \( F \to F \).

Exercise 2. Prove the above. Hint: showing this for the identity function is truly trivial. For a constant function, observe that \( f(a_i) \) is a constant sequence. What do you know about limits of constant sequences?

Definition 3. Let \( f \) and \( g \) be functions \( S \to F \) where \( F \) is a field. The function \( f + g \) is defined to be the function \( S \to F \) which sends \( x \in S \) to \( f(x) + g(x) \). The function \( f \cdot g \) is defined to be the function \( S \to F \) which sends \( x \in S \) to \( f(x) \cdot g(x) \).
Theorem 4. Let $S$ be a subset of an ordered field $F$. If $f, g$ are continuous on $S$ then $f + g$ is also continuous on $S$. In other words, the set of continuous functions is closed under addition.

Proof. We prove continuity for $f + g$. The proof for $f \cdot g$ is similar. By the definition of continuity (Definition 2), we take an arbitrary converging sequence $(a_i)$ with limit $a$. We assume that each $a_i$ is in $S$ and that $a$ is in $S$, and we must show that the sequence $((f + g)(a_i))$ converges with limit $(f + g)(a)$.

Observe that, by Definition 3, $(f + g)(a_i) = f(a_i) + g(a_i)$ and $(f + g)(a) = f(a) + g(a)$. Since $f$ and $g$ are continuous on $S$ the sequences $(f(a_i))$ and $(g(a_i))$ both converge. By the limit laws of Chapter 7, the sequence $(f(a_i) + g(a_i))$ must also converge since it is the sum of convergent sequences. In fact,

$$
\lim_{i \to \infty} (f + g)(a_i) = \lim_{i \to \infty} (f(a_i) + g(a_i)) \quad \text{(Sum of functions (Def 4))}
$$

$$
= \lim_{i \to \infty} f(a_i) + \lim_{i \to \infty} g(a_i) \quad \text{(Limit law for + (Ch. 7))}
$$

$$
= f \left( \lim_{i \to \infty} a_i \right) + g \left( \lim_{i \to \infty} a_i \right) \quad \text{($f$ and $g$ continuous)}
$$

$$
= f(a) + g(a) \quad \text{(a is the limit)}
$$

$$
= (f + g)(a) \quad \text{(Sum of functions (Def 4))}
$$

□

Exercise 3. Let $S$ be a subset of an ordered field $F$. If $f, g$ are continuous on $S$ then the function $f \cdot g$ is also continuous on $S$. In other words the set of continuous functions is closed under multiplication.

Definition 4. Let $S$ be a subset of an ordered field $F$. Then let $C(S)$ be the set of continuous functions $S \to F$.

By Definition 4 and the above theorem and exercise, the set $C(S)$ has binary operations $+$ and $\cdot$. Is $C(S)$ a ring? Let us begin with the associative law:

Lemma 5. The operation $+$ is associative on $C(S)$.

Proof. We must show that if $f, g, h \in C(S)$ then $(f + g) + h = f + (g + h)$. To show functions are equal, it is enough to show that they have equal value for an arbitrary $x$ in the domain. So let $x \in S$ be in the domain. Then

$$
((f + g) + h)(x) = (f + g)(x) + h(x) \quad \text{(Sum of functions (Def 4))}
$$

$$
= (f(x) + g(x)) + h(x) \quad \text{(Sum of functions (Def 4))}
$$

$$
= f(x) + (g(x) + h(x)) \quad \text{(Addition in field $F$ is assoc.)}
$$

$$
= f(x) + (g + h)(x) \quad \text{(Sum of functions (Def 4))}
$$

$$
= (f + (g + h))(x) \quad \text{(Sum of functions (Def 4))}
$$
Since \( x \in S \) is arbitrary, \((f + g) + h = f + (g + h)\). □

**Exercise 4.** Show that + for \(C(S)\) is also commutative. Show multiplication for \(C(S)\) is associative and commutative. Show that the distributive law holds for \(C(S)\).

**Theorem 6.** Let \(S\) be a subset of an ordered field \(F\). Then the set of continuous functions \(C(S)\) is a commutative ring.

**Exercise 5.** Complete the proof of the above theorem. What are the 0 and 1 elements in the ring \(C(S)\)?

**Informal Exercise 6.** Assume the existence of \(\mathbb{R}\) (informally at this point). Informally, a continuous function on \([0, 1]\) is a function \([0, 1] \rightarrow \mathbb{R}\) whose graph is a connected curve.

Show that \(C([0, 1])\) is not an integral domain. Do so by sketching the graph of two continuous functions whose product is zero.

**Example.** Let \(S\) be a subset of an ordered field \(F\). If \(f(x) = x\), then we know that \(f \in C(S)\) since it is the identity function (Theorem 3). Thus \(f \cdot f \in C(S)\) by closure under multiplication. However \(f \cdot f\) is just the function \(x \mapsto x^2\). Thus \(g(x) = x^2\) is a continuous function.

By induction, we can similarly show \(g(x) = x^k\) is continuous for all \(k \in \mathbb{N}\).

**Example.** The previous example shows that any monomial function \(g(x) = x^k\) is continuous. Since constant functions are continuous, and since continuous functions are closed under multiplication, the function \(g(x) = cx^k\) is continuous.

Since any polynomial function is the sum of functions of the form \(g(x) = cx^k\), and since continuous functions are closed under sum, we can prove any polynomial function is continuous.

4. **The \(\delta\)-\(\varepsilon\) Definition of Continuity (Optional)**

Definition 2 is sometimes called the definition of sequential continuity. There is another definition of continuity that is often used in analysis called the \(\delta\)-\(\varepsilon\) definition of continuity. In this section we will show that both definitions are equivalent. This fact will not be used in this course, but it is an important fact in analysis.

**Definition 5** (Second Definition of Continuity). Let \(S\) be a subset of an ordered field \(F\). Let \(f: S \rightarrow F\) be a function and let \(a \in S\). We say that \(f\) is continuous at \(a\) in the \(\delta\)-\(\varepsilon\) sense if the following holds. For all \(\varepsilon > 0\) in \(F\) there is a \(\delta > 0\) such that for all \(x \in S\) if \(|x - a| < \delta\) then \(|f(x) - f(a)| < \varepsilon\).

If \(f\) is continuous for all \(a \in S\) then we say that \(f: S \rightarrow F\) is continuous on \(S\) (in the \(\delta\)-\(\varepsilon\) sense).

Notice that we defined continuity at a point. We have not yet done this for the sequential form of continuity. We do so now:
Definition 6. Let \( S \) be a subset of an ordered field \( F \). Let \( f : S \to F \) be a function and let \( a \in S \). We say that \( f \) is continuous at \( a \) in the sequential sense if the following holds. For all sequences \((a_i)\) converging to \( a \) with terms in \( S \), the sequence \((f(a_i))\) converges to \( f(a) \).

By definition \( f \) is continuous in the sequential sense if and only if it is continuous at \( a \) in the sequential sense for all \( a \in S \).

We will now prove the equivalence of the two notions of continuity using two lemmas, one for each direction of the equivalence.

Lemma 7. Let \( S \) be a subset of an ordered field \( F \). Let \( f : S \to F \) be a function and let \( a \in S \). If \( f \) is continuous at \( a \) in the \( \delta \)-\( \varepsilon \) sense, then it is continuous at \( a \) in sequential sense.

Proof. Assume \((a_i)\) is a sequence converging to \( a \) with terms in \( S \). Our goal according to Definition 6 is to show that the sequence \((f(a_i))\) converges to \( f(a) \). To achieve this goal, we use the definition of limit in Chapter 7.

So assume \( \varepsilon > 0 \) is given. Our goal reduces to the following goal: find an \( N \in \mathbb{N} \) such that if \( i \geq N \) then \( |f(a_i) - f(a)| < \varepsilon \).

By assumption, \( f \) is continuous at \( a \) in the \( \delta \)-\( \varepsilon \) sense, so by Definition 5 there is a \( \delta > 0 \) such that \( |f(x) - f(a)| < \varepsilon \) if \( |x - a| < \delta \) and \( x \in S \). Now since \((a_i)\) converges to \( a \), there is an \( N \in \mathbb{N} \) such that if \( i \geq N \) then \( |a_i - a| < \delta \). (We use the definition of limit from Chapter 7, using \( \delta \) for our epsilon).

By the above property of \( \delta \) for \( f \), \( |a_i - a| < \delta \) implies that \( |f(a_i) - f(a)| < \varepsilon \).

In summary, for this choice of \( N \), if \( i \geq N \), then \( |a_i - a| < \delta \). This implies in turn that \( |f(a_i) - f(a)| < \varepsilon \). So this choice of \( N \) achieves our goal. \( \square \)

Lemma 8. Let \( S \) be a subset of an archimedean ordered field \( F \). Assume \( f : S \to F \) is a function and that \( a \in S \). If \( f \) is continuous at \( a \) in the sequential sense, then it is continuous at \( a \) in the \( \delta \)-\( \varepsilon \) sense.

Proof. We prove the contrapositive. So we suppose that \( f \) is not continuous at \( a \) in the \( \delta \)-\( \varepsilon \) sense. Our goal is to show that \( f \) is not continuous at \( a \) in the sequential sense.

We negate the definition of continuity at \( a \) in the \( \delta \)-\( \varepsilon \) sense. This means that there exists an \( \varepsilon_0 > 0 \) such that for all \( \delta > 0 \) there is an \( x \in S \) such that \( |x - a| < \delta \) but \( |f(x) - f(a)| \geq \varepsilon \).

We use this to define a sequence \((a_i)\). For any positive \( k \in \mathbb{N} \), let \( \delta = 1/k \). By the above property there is an element \( a_k \) such that \( |a_k - a| < 1/k \) but \( |f(a_k) - f(a)| \geq \varepsilon_0 \).

The first claim is that \((a_i)\) has limit \( a \). To see this, let \( \varepsilon > 0 \) be given. (This \( \varepsilon \) is independent of the \( \varepsilon_0 \) above). Since \( F \) is archimedean, there is an \( N \in \mathbb{N} \) such that \( 1/N < \varepsilon \) (see Chapter 7). If \( i \geq N \) then

\[
|a_i - a| < \frac{1}{i} \leq \frac{1}{N} < \varepsilon.
\]

This \((a_i)\) converges to \( a \) as desired.
The second claim is that \((f(a_i))\) does not converge to \(f(a)\). This follows from the fact that \(|f(a_i) - f(a)| \geq \varepsilon_0\). In other words, for this particular epsilon value, there is no \(N'\) such that \(i \geq N'\) implies \(|f(a_i) - f(a)| < \varepsilon_0\).

If we combine the two claims, we see that \(f\) cannot be continuous at \(a\) in the sense of Definition 6.

We now combine the above lemmas to give the following:

**Theorem 9.** Let \(S\) be a subset of an archimedean ordered field \(F\). Assume \(f : S \to F\) is a function on \(S\). Then \(f\) is continuous at \(a \in S\) according to the sequential definition if and only if it is continuous at \(a \in S\) according to the \(\delta-\varepsilon\) definition. Therefore, \(f\) is continuous on \(S\) according to the sequential definition if and only if it is continuous on \(S\) according to the \(\delta-\varepsilon\) definition.

5. **Intermediate Value Theorem**

One of the key foundational results in real analysis is the intermediate value theorem. It can be proved from the completeness property.

First we remind the reader of a result proved in Chapter 7, which we restate here for the convenience of the reader.

**Theorem 10.** Let \(S\) be a nonempty subset of an archimedean ordered field \(F\). If \(M\) is the supremum of \(S\) then \(M\) is the limit of a sequence \((a_i)\) of elements \(a_i \in S\) and is the limit of a sequence \((b_i)\) of elements \(b_i \notin S\). If \(a < M\) is given we can assume that each \(a_i\) is in the interval \([a,M]\). Similarly, if \(b > M\) is given we can assume that each \(b_i\) is in the interval \([M,b]\).

This gives us the main tool that we need to prove the following. (In the following, \(C\) is between \(A\) and \(B\). This is intended to include the case where \(C\) is \(A\) or \(B\) itself.)

**Theorem 11** (Intermediate Value Theorem). Let \(F\) be a complete ordered field that contains \(\mathbb{Q}\) as an ordered subfield. Let \([a,b]\) be a closed interval in \(F\) where \(a < b\) are elements of \(F\). Suppose \(f : [a,b] \to F\) is continuous. If \(C \in F\) is any value between \(A = f(a)\) and \(B = f(b)\) then there is an element \(c \in [a,b]\) such that \(f(c) = C\).

**Proof.** Without loss of generality we can assume \(A < C < B\) (the case where \(B < C < A\) is similar, and the case where \(C\) is \(A\) or \(B\) is trivial). Define the following set

\[
S \overset{\text{def}}{=} \{u \in [a,b] \mid f(u) \leq C\}.
\]

Observe that \(S\) is nonempty since \(a \in S\), and that \(b\) is an upper bound for \(S\). Since \(F\) is complete, the set \(S\) has a supremum, call it \(c\). We claim that \(c\) satisfies the conclusion of the theorem.

We begin by showing that \(f(c) \leq C\). By Theorem 10 there is a sequence \((a_i)\) of elements in \(S\) that converges to \(c\). Since \(a_i \in S\) we have \(f(a_i) \leq C\). Since \(f\) is continuous, the sequence \((f(a_i))\) converges to \(f(c)\). Thus

\[
f(c) = \lim_{i \to \infty} f(a_i) \leq C.
\]
Next we show that $c$ is in the interval $[a, b)$. (For the sake of the theorem, we only need $c \in [a, b]$, but we need $c < b$ below in the proof.) Observe that $a \leq c$ since $a \in S$ and $c$ is an upper bound for $S$. Since $b$ is an upper bound of $S$, we have $c \leq b$ since $c$ is the least upper bound of $S$. Finally, $b \neq c$ since $f(b) = B > C$ and $f(c) \leq C$. So $a \leq c < b$.

Finally we show $f(c) \geq C$, which with the above will yield the result. By Theorem 10 there is a sequence $(b_i)$ of elements not in $S$ converging to $c$, and since $c < b$ we can also assume that each $b_i \in [c, b]$. In particular, each $b_i$ is in the interval $[a, b]$ since $[c, b]$ is a subset of $[a, b]$. Since $b_i \not\in S$ we have $f(b_i) \geq C$. (Actually $f(b_i) > C$, but we just need $\geq$). Since $f$ is continuous, the sequence $(f(b_i))$ converges to $f(c)$. Thus

$$f(c) = \lim_{i \to \infty} f(b_i) \geq C.$$

□

As an application of this theorem, we show that square roots exists in complete ordered fields.

**Corollary 12.** Suppose $C \in F$ where $F$ is a complete ordered field that contains $\mathbb{Q}$ as an ordered subfield. Then if $C \geq 0$ there is an element $c \in F$ such that $c^2 = C$.

**Proof.** If $C < 1$, let $b = 1$, but if $C \geq 1$, let $b = C$. Consider the function $f: F \to F$ defined by $x \mapsto x^2$. This function is continuous since it is the product of the identity function with itself, and continuous functions are closed under products. So the restriction $f|_{[0,b]}$ to $[0,b]$ is also continuous.

Claim: $f(0) \leq C \leq f(b)$. To see $f(0) \leq C$, observe that $C \geq 0$ and $f(0) = 0$. To show $C \leq f(b) = b^2$ we divide into two cases. First consider the case where $C < 1$. Then $b = 1$. So $f(b) = 1^2 = 1$. Thus $C \leq f(b)$ as desired. Next assume that $C \geq 1$. Then

$$C = C \cdot 1 \leq C \cdot C = b \cdot b = f(b).$$

The hypotheses of the Intermediate Value Theorem are satisfied. By the Intermediate Value Theorem, there is a $c \in [0,b]$ such that $f(c) = C$. So $c^2 = C$ as desired. □

**Exercise 7.** Suppose $C \geq 0$ in a complete ordered field $F$. Show that there is $c \in F$ such that $c^3 = C$. What if $C < 0$?

**Corollary 13.** The field $\mathbb{Q}$ is not complete.

**Proof.** In Chapter 6 we proved that there is no $r \in \mathbb{Q}$ such that $r^2 = 2$. However, if $\mathbb{Q}$ were complete, then there would be such an $r$ by the previous corollary. □

### 6. Cauchy Sequences

If a sequence converges, then the terms of the sequence get and stay arbitrarily close to each other. This is shown in the following theorem.
Such sequences are called \textit{Cauchy sequences}. Cauchy sequences will play a key role in our construction of \( \mathbb{R} \). In addition, they play an important role in analysis quite generally. Informally, a Cauchy sequence is a sequence that seems like it “ought to converge”. It might not actually converge in incomplete ordered fields, though. For example, not every Cauchy sequence in \( \mathbb{Q} \) converges in \( \mathbb{Q} \). However, in a complete ordered field, every Cauchy sequences will converge.

\textbf{Theorem 14.} Suppose \((a_i)\) is a convergent sequence in an ordered field \( F \). Then for all positive \( \varepsilon \) in \( F \) there is a \( N \in \mathbb{N} \) such that for all \( i, j \in \mathbb{N} \)

\[ i, j \geq N \implies |a_i - a_j| < \varepsilon. \]

\textit{Proof.} Since we assume that \((a_i)\) converges, it has a limit. Let \( b \) be the limit of the sequence \((a_i)\).

Let \( \varepsilon > 0 \) be an arbitrary positive element of \( F \). We must find a \( N \in \mathbb{N} \) that satisfies the statement of the theorem.

Let \( \varepsilon' = \varepsilon/2 \). By the definition of limit there is a \( N \in \mathbb{N} \) with \( |a_i - b| < \varepsilon' \) for all \( i \geq N \). So, for \( i, j \geq N \) we have

\[ |a_i - a_j| = |(a_i - b) + (b - a_j)| \leq |a_i - b| + |b - a_j| < \varepsilon' + \varepsilon'. \]

Here we have used the triangle inequality. Since \( 2\varepsilon' = \varepsilon \) we have that \( |a_i - a_j| < \varepsilon \). Thus \( N \) has the desired property. \( \square \)

The above theorem says that all convergent sequences satisfy the following definition:

\textbf{Definition 7.} Suppose \((a_i)\) is an infinite sequence in an ordered field \( F \). We say that \((a_i)\) is \textit{Cauchy} if the following occurs: for all positive \( \varepsilon \) in \( F \) there is a \( N \in \mathbb{N} \) such that for all \( i, j \in \mathbb{N} \)

\[ i, j \geq N \implies |a_i - a_j| < \varepsilon. \]

\textit{Remark 1.} We can interpret Theorem 14 through its contrapositive: if a sequence is not Cauchy, it cannot converge.

Is the converse true? In other words, do all Cauchy sequences converge? The answer is no for \( F = \mathbb{Q} \). The problem with \( \mathbb{Q} \) is that it has ‘holes’. For example, we saw that there is no \( r \in \mathbb{Q} \) with \( r^2 = 2 \). Define a sequence by the rule \( a_i = n_i/10^i \) where \( n_i \) is the largest integer such that \( a_i^2 < 2 \). This sequence will not be convergent in \( \mathbb{Q} \), but can be shown to be Cauchy. It will turn out that this sequence is convergent in \( \mathbb{R} \), and has limit \( \sqrt{2} \).

\textit{Informal Exercise 8.} Find the first five terms of \((a_i)\) defined in the above remark. Assume the index set is the set of \( i \geq 0 \). Hint: punch \( \sqrt{2} \) into your calculator.

Our approach in the next chapter will be to assume that all Cauchy sequences in \( \mathbb{Q} \) should determine a real number. Non-Cauchy sequences cannot possibly converge, so should not determine real numbers. There is a problem: different sequences can determine the same real number. For
example, the sequence defined by the rule \( b_i = n_i / 2^i \) where \( n_i \) is the largest integer such that \( b_i^2 < 2 \) determines the same real number as the sequence \((a_i)\) discussed above (in fact, they both determine \( \sqrt{2} \): the sequence \((a_i)\) is related to the decimal expansion of \( \sqrt{2} \) and \((b_i)\) is related to the base 2 expansion of \( \sqrt{2} \)). How do we tell if two sequences determine the same number? We can use the equivalence relation defined in Chapter 7: two Cauchy sequence determine the same real number if and only if \((a_i) \sim (b_i)\).

We will make this approach more precise in Chapter 9. When we do, we will need the following.

**Theorem 15.** Let \( F \) be an archimedean ordered field. If \((a_i) \sim (b_i)\) and if \((a_i)\) is Cauchy, then \((b_i)\) is Cauchy.

**Exercise 9.** Prove the above theorem. The proof is similar to the proof that if \((a_i) \sim (b_i)\) and if \((a_i)\) converges, then \((b_i)\) converges. You might wish to choose \( \varepsilon' = \varepsilon / 3 \). The key step of the proof is

\[
|b_i - b_j| = |(b_i - a_i) + (a_i - a_j) + (a_j - b_j)| < \varepsilon' + \varepsilon' + \varepsilon'.
\]

We conclude with a lemma that shows that every Cauchy sequence is bounded.

**Lemma 16.** If \((a_i)_{i \geq n_0}\) is a Cauchy sequence in an ordered field \( F \), then there is a bound \( B \in F \) such that \(|a_i| \leq B\) for all \( i \geq n_0 \).

**Proof.** First we show that \( a_i \leq B_1 \) for some positive upper bound \( B_1 \).

Since \((a_i)\) is Cauchy, there is a \( N \in \mathbb{N} \) such that \(|a_i - a_j| < 1\) for all \( i, j \geq N \) (choose \( \varepsilon = 1 \)). Let \( A \) be the maximum of \( 0, a_{n_0}, \ldots, a_N \), and let \( B_1 = A + 1 \). Since \( A \geq 0 \) we have \( B_1 \) positive. We will show that \( B_1 \) is in fact an upper bound for \((a_i)\).

First consider the case where \( i \leq N \). In this case

\[
a_i \leq A < A + 1.
\]

Since \( B_1 = A + 1 \), we have \( a_i \leq B_1 \) as desired.

Next consider the case where \( i > N \). Since \( i, N \geq N \), we have the inequality \(|a_i - a_N| < 1\). Thus \(-1 < a_i - a_N < 1\). So \( a_i < a_N + 1 \). Since \( a_N + 1 \leq A + 1 = B_1 \), we get \( a_i \leq B_1 \) as desired.

The proof of the existence of a negative lower bound is similar. (Subtract one from a minimum). Write the lower bound as \(-B_2\) where \( B_2 \) is positive. So we get

\[
-B_2 \leq a_i \leq B_1
\]

for all \( i \geq n_0 \). Let \( B \) be the maximum of \( B_1 \) and \( B_2 \). Then

\[
-B \leq a_i \leq B.
\]

So \(|a_i| \leq B\) as desired.
7. Cauchy Criterion

Our goal is to prove the following theorem. We will need this later to show that our construction of $\mathbb{R}$ gives a complete ordered field. The converse is true as well, and will be proved in a later section.

**Theorem 17** (Cauchy Criterion). Let $F$ be an archimedean ordered field. If every Cauchy sequence converges then $F$ is complete.

The proof of this uses the notion of an $\varepsilon$-almost-supremum. We first build a Cauchy sequence out of such “almost-Sups”, and the limit can be shown to be the actual supremum.

Recall the definition (from Chapter 7): let $S$ be a nonempty subset of an ordered field $F$, and let $\varepsilon > 0$ be in $F$. An $\varepsilon$-almost-supremum $A$ of $S$ is an upper bound of $S$ such that there is an $x \in S$ in the interval $(A - \varepsilon, A]$. In Chapter 7 we proved the following (which we restate for convenience):

**Theorem 18.** Let $S$ be a nonempty subset of an archimedean ordered field $F$, and let $\varepsilon > 0$ be in $F$. If $S$ is bounded from above, then $S$ has an $\varepsilon$-almost-supremum.

Using this theorem we can prove the Cauchy Criterion:

**Proof of Theorem 17.** Let $S$ be a nonempty subset of $F$ with an upper bound. Our goal is to show that $S$ has a supremum. This will show $F$ is complete as desired.

For each positive integer $n \in \mathbb{N}$, let $A_n$ be an $1/n$-almost-supremum of $S$. This exists by Theorem 18 (proved in Chapter 7).

Claim: $(A_i)$ is a Cauchy sequence. Let $\varepsilon > 0$ be given. To prove the claim we need to find an $N$ such that if $i, j \geq N$ then $|A_i - A_j| < \varepsilon$. Since $F$ is an archimedean ordered field there is an $N$ such that $1/N \leq \varepsilon$. Now suppose $i, j \geq N$. Without loss of generality, suppose $A_i \geq A_j$. Since $A_i$ is an $1/i$-almost-supremum, $A_i - 1/i$ is not an upper bound of $S$. Since $A_j$ is an upper bound of $S$, we have $A_i - 1/i < A_j$. Hence

$$|A_i - A_j| = A_i - A_j < 1/i \leq 1/N \leq \varepsilon.$$ 

Thus $(A_i)$ is Cauchy. By the assumption of the theorem, $(A_i)$ has a limit, call it $A$.

Claim: $A$ is an upper bound of $S$. To see this we must show $x \leq A$ for all $x \in S$. Suppose otherwise, that $x > A$ for some $x \in S$. Let $\varepsilon = x - A$. Since $A$ is the limit of $(A_i)$, there is an $N$ such that $i \geq N$ implies that $|A_i - A| < \varepsilon$. In particular, $|A_N - A| < \varepsilon = x - A$. Since $A_N$ is an upper bound of $S$, $A < x \leq A_N$. Since $A_N > A$, we have $|A_N - A| = A_N - A$. Thus $A_N - A < x - A$. So $A_N < x$, a contradiction to the fact that $A_N$ is an upper bound.

Finally we show that $A$ is actually the supremum of $S$ by supposing otherwise and deriving a contradiction. So suppose otherwise that there is another upper bound $A'$ with $A' < A$. Let $\varepsilon = A - A'$. Observe that $A - \varepsilon$
is $A'$, hence $A - \varepsilon$ is an upper bound. For any positive $i$, the difference $A_i - 1/i$ is not an upper bound since $A_i$ is a $1/i$-almost supremum. Thus

$$A_i - 1/i < A - \varepsilon.$$ 

Since $F$ is an archimedean ordered field, there is a positive integer $N$ such that $1/N \leq \varepsilon/2$ (Chapter 7). If $i \geq N$, then $1/i \leq 1/N \leq \varepsilon/2$. So for such $i$,

$$A_i - \varepsilon/2 \leq A_i - 1/i < A - \varepsilon.$$

In other words, $A - A_i > \varepsilon/2$. This implies that $|A_i - A| > \varepsilon/2$ for all $i \geq N$. This shows that $A_i$ cannot have limit $A$ (See the definition of limit, Chapter 7), a contradiction. □

8. Bounded Monotonic Sequences Converge

Monotonic sequences are commonly used in mathematics and are often easier to deal with than arbitrary sequences. One of the most useful basic facts about $\mathbb{R}$ is that every bounded monotonic sequences converges. We will show that this follows from the completeness property.

**Definition 8.** Let $(a_i)_{i \geq n_0}$ be a sequence in an ordered field $F$. The sequence is said to be increasing if $a_{i+1} \geq a_i$ for all $i \geq n_0$. The sequence is decreasing if $a_{i+1} \leq a_i$ for all $i \geq n_0$. In either case $(a_i)$ is said to be monotonic. (Observe that a constant sequence is considered both increasing and decreasing).

The sequence $(a_i)_{i \geq n_0}$ is said to be strictly increasing if $a_{i+1} > a_i$ for all $i \geq n_0$. The sequence $(a_i)_{i \geq n_0}$ is strictly decreasing if $a_{i+1} < a_i$ for all $i \geq n_0$. In either case $(a_i)$ is said to be strictly monotonic. (Observe that a constant sequence is monotonic, but not strictly monotonic).

The following is a simple consequences of the definition. It is stated for increasing sequences, but the statement holds, with the obvious modifications, for decreasing sequence. There are obvious versions for strictly monotonic sequences as well.

**Lemma 19.** Suppose that $(a_k)_{k \geq n_0}$ is a monotonically increasing sequence in an ordered field $F$. If $j \geq i \geq n_0$ then $a_j \geq a_i$.

*Proof.* Fix $i \geq n_0$, and consider the set $S_i = \{ u \in \mathbb{Z} \mid u \geq n_0 \text{ and } a_u \geq a_i \}$ we will show by induction that all $j \geq i$ are in $S_i$.

For the base case, observe that $a_i \geq a_i$ (reflexive). Thus $i \in S_i$.

Now suppose $k \in S_i$. This implies $a_{k+1} \geq a_k \geq a_i$ (the first inequality by Definition 8, the second since $k \in S_i$). So $k + 1 \in S_i$.

By induction, $S_i$ contains all $j \geq i$. In particular $a_j \geq a_i$ if $j \geq i$. □

An increasing sequence is automatically bounded from below: if $a_{n_0}$ is the first term of such a sequence then the above lemma shows $a_{n_0}$ is a lower bound. So to say that such a sequence is bounded really means that it is also bounded from above. Obviously the same idea, but reversed, applies to decreasing sequences.
Theorem 20. Let \((a_i)\) be a bounded monotonic sequence in a complete field \(F\). Then \((a_i)\) converges.

Proof. We assume that \((a_i)_{i \geq n_0}\) is an increasing sequence. The decreasing case is similar. Since \(F\) is complete, and the set \(\{a_i \mid i \geq n_0\}\) is bounded, this set has a supremum \(B\). We will show that \(B\) is in fact the limit.

Let \(\varepsilon > 0\) be given. By a result in Chapter 7, the interval \((B - \varepsilon, B]\) must contain an element of \(\{a_i \mid i \geq n_0\}\). In other words, there is an \(N \in \mathbb{N}\) such that \(B - \varepsilon < a_N \leq B\). We will show that \(N\) has the desired property (as in the definition of limit). So suppose \(i \geq N\). Then \(a_N \leq a_i\) since the sequence is increasing. But \(B\) is an upper bound for the sequence. So \(a_i \leq B\).

Thus \(B - \varepsilon < a_N \leq a_i \leq B\). Hence \(|B - a_i| = (B - a_i)\) since \(a_i \leq B\), and \(B - a_i \leq \varepsilon\) since \(B - \varepsilon < a_i\).

Therefore, \(|B - a_i| < \varepsilon\) as desired. \(\square\)

Exercise 10 (Optional). Use \(\varepsilon\)-almost suprema (and infima) from Chapter 7 to show that if \(F\) is an archimedean ordered field, then every bounded monotonic sequence is Cauchy. (Even if \(F\) is not complete). Hint: it is enough to do the increasing case. The proof is similar to that of the previous theorem. For each \(n \in \mathbb{N}\), let \(B_n\) be an \(1/n\)-almost supremum. Now given an arbitrary \(\varepsilon > 0\), choose \(1/n < \varepsilon\). Show that you can choose an \(N \in \mathbb{N}\) such that \(a_N \in (B_n - 1/n, B_n]\). Show that \(i, j \geq N\) implies \(|a_i - a_j| < 1/n < \varepsilon\).

9. Accumulation Points (optional)

Consider the sequence defined by the equation \(a_k = (-1)^k\). It is obviously not a Cauchy sequences, so it cannot converge. (Recall that all convergent sequences must be Cauchy). However, it does have an infinite number of terms equal to 1 and an infinite number of terms equal to \(-1\). So 1 and \(-1\) are in some sense limits in a more general sense. A similar phenomenon occurs for the sequence defined by \(b_k = 1/k + (-1)^k\); this sequence seems to have values that “accumulate” near both 1 and \(-1\). This leads to an important concept which we now define:

Definition 9. Let \((a_i)\) be a sequence in an ordered field \(F\). We say that \(b \in F\) is an accumulation point of \((a_i)\) if the following occurs: for all \(\varepsilon > 0\) and \(k \in \mathbb{N}\) there is an integer \(i \geq k\) such that \(|a_i - b| < \varepsilon\).

Remark 2. We can rephrase the above condition to an equivalence condition: for all \(\varepsilon > 0\) there are an infinite number of \(i \in \mathbb{N}\) such that \(|a_i - b| < \varepsilon\).

Exercise 11. Let \(F\) be an archimedean ordered field. Show that 1 is an accumulation point of the sequence defined by \(a_k = (-1)^k\) and of the sequence defined by \(b_k = 1/k + (-1)^k\).

Exercise 12. If a sequence has a limit, show that its limit is the unique accumulation point.
Exercise 13. From the previous exercise we see that a convergent sequence has exactly one accumulation point. Is the converse true? In other words, if a sequence has exactly one accumulation point, does it follow that the sequence converges?

The following will be useful later in showing the converse of Theorem 17.

Theorem 21. Let \((a_i)\) be a Cauchy sequence in an ordered field \(F\). Then \((a_i)\) converges if and only if it has an accumulation point. In this case, the limit is the accumulation point. In particular, a Cauchy sequence has at most one accumulation point.

Proof. If \((a_i)\) converges, then its limit is the unique accumulation point by Exercise 12.

Conversely, suppose \((a_i)\) has an accumulation point \(a\). Our goal is to show that \(a\) is the limit of \((a_i)\). By the definition of limit, for any given \(\varepsilon > 0\) we must show that there is an \(N \in \mathbb{N}\) such that \(|a_i - a| < \varepsilon\) for all \(i \geq N\).

Let \(\varepsilon' = \varepsilon/2\). Since \((a_i)\) is a Cauchy sequence, there is an \(N\) such that \(|a_i - a_j| < \varepsilon'\) for all \(i, j \geq N\).

By the definition of accumulation point, there is an \(j_0 \geq N\) such that \(|a_{j_0} - a| < \varepsilon'\). Now assume that \(i \geq N\). Then
\[
|a_i - a| \leq |a_i - a_{j_0}| + |a_{j_0} - a| < \varepsilon' + \varepsilon' = \varepsilon.
\]

Exercise 14. Show that if \((a_i) \sim (b_i)\) are equivalent sequences, then they have the same accumulation points.

10. Lim Infs and Lim Sups (optional)

We will see that in any complete ordered field, bounded sequence (bounded in both direction) must have accumulation points. In this case there is a greatest accumulation point call the superior limit and a least accumulation point called the inferior limit.

The superior limit, often called lim sup, is formed by seeing how the bounds to the sequence change as a larger and larger number of terms are removed from the beginning of the sequence. These bounds will tend to go down, or stay the same, as more terms are removed. What happens in the long term, as measured by the infimum of these bounds, is the superior limit. The inferior limit, often called lim inf, is formed in a similar way except with lower bounds. The following definition makes this idea precise.

Definition 10. Suppose \((a_i)_{i \geq n_0}\) is a bounded sequence in a complete ordered field \(F\). In other words, suppose there is a bound \(B\) such that \(|a_i| \leq B\) for all \(i \geq n_0\). For each \(k \geq n_0\) consider the following set
\[
S_k = \{a_i \mid i \geq k\}.
\]

Observe that \(S_k\) is a nonempty set with upper bound \(B\) and lower bound \(-B\). Let \(M_k\) be the supremum of \(S_k\) and let \(m_k\) be the infimum of \(S_k\). These
exist since \( F \) is complete. Observe that \(-B \leq m_k \leq M_k \leq B\), so the sets \( \{ m_k \mid i \geq k \} \) and \( \{ M_k \mid i \geq k \} \) are themselves bounded by \(-B\) and \( B\). The superior limit is defined as follows:

\[
\limsup_{i \to \infty} a_i \defeq \inf \{ M_k \mid k \geq n_0 \}.
\]

The inferior limit is defined as follows:

\[
\liminf_{i \to \infty} a_i \defeq \sup \{ m_k \mid k \geq n_0 \}.
\]

These both exist since \( F \) is complete.

Remark 3. Often lim sups and lim infs are defined even for unbounded sequences. For example, if a sequence \( (a_i) \) has no upper bound then the lim sup of \( (a_i) \) is sometimes said to be \( \infty \). Similarly, if \( (a_i) \) has no lower bound then the lim inf of \( (a_i) \) would be \(-\infty\). In this course, however, we will stick to bounded sequences where lim sup and lim inf are elements of \( F \) (assuming \( F \) is complete).

The above definition is a bit tricky, so we present another characterization of lim sups and lim infs that is easier to use in many situations. First a lemma.

**Lemma 22.** Let \( (a_i) \) be a bounded sequence in a complete ordered field \( F \). Suppose \( M \in F \) is such that

\[
M > \limsup_{i \to \infty} a_i.
\]

Then all but a finite number of terms of \( (a_i) \) are strictly smaller than \( M \). In other words, there is an \( N \in \mathbb{N} \) such that if \( i \geq N \) then \( a_i < M \).

**Proof.** Let \( X = \limsup_{i \to \infty} a_i \). By definition \( X \) is the greatest lower bound of the \( M_k \) where \( M_k \) are as in the above definition. Since \( M > X \), we see that \( M \) is not a lower bound of the set of \( M_k \). Thus there is an \( N \in \mathbb{N} \) such that \( M_N < M \). Since \( M_N \) is the least upper bound of \( S_N = \{ a_i \mid i \geq N \} \), it follows that \( a_i \leq M_N \) for all \( i \geq N \). Since \( M_N < M \) we have \( a_i < M \) for all \( i \geq N \). \( \square \)

Here is the promised alternate characterization of lim sups:

**Theorem 23.** Let \( (a_i) \) be a bounded sequence in a complete ordered field \( F \). Then \( \limsup_{i \to \infty} a_i \) is the minimal element \( X \in F \) with the following property: For all \( M > X \) there is an \( N \in \mathbb{N} \) such that if \( i \geq N \) then \( a_i < M \). In other words, \( \limsup_{i \to \infty} a_i \) is the smallest element of \( F \) such that any larger element is a strict upper bound for all but a finite number of terms of \( (a_i) \).

**Proof.** The above lemma shows that \( X = \limsup_{i \to \infty} a_i \) has the desired property. Now we need to show that no smaller element \( Y \) has the property. Suppose \( Y < X \) has the property. Let \( Z \) be chosen so that \( Y < Z < X \). By the property assumed for \( Y \) there is an \( N \in \mathbb{N} \) such that if \( i \geq N \) then \( a_i < Z \). This implies that \( Z \) is an upper bound of the set \( S_N = \{ a_i \mid i \geq N \} \).
So \( M_N \leq Z \) where \( M_N \) is the least upper bound (supremum) of \( S_N \). Since \( X \) is the infimum of the set of \( M_k \) we have \( X \leq M_N \). So

\[
X \leq M_N \leq Z < X
\]
a contradiction. \( \square \)

For \( \lim \inf \)s we have the following:

**Theorem 24.** Let \( (a_i) \) be a bounded sequence in a complete ordered field \( F \). Then \( \lim \inf_{i \to \infty} a_i \) is the maximal element \( x \in F \) with the following property: For all \( m < x \) there is an \( N \in \mathbb{N} \) such that if \( i \geq N \) then \( a_i > m \). In other words, \( \lim \inf_{i \to \infty} a_i \) is the largest element of \( F \) such that any smaller element is a strict lower bound for all but a finite number of terms of \( (a_i) \).

**Exercise 15.** How would you change the proof of Lemma 22 and Theorem 23 in order to prove Theorem 24?

Now we characterize \( \lim \sup \)s and \( \lim \inf \)s in terms of accumulation points. This is perhaps the simplest way of describing them.

**Theorem 25.** Suppose \( (a_i) \) is a bounded sequence in a complete ordered field \( F \). Then \( \lim \sup_{i \to \infty} a_i \) is an accumulation point of \( (a_i) \). In fact, it is the greatest accumulation point of \( (a_i) \).

**Proof.** Let \( X = \lim \sup_{i \to \infty} a_i \). We will first show that \( X \) is an accumulation point. We assume we are given \( \varepsilon > 0 \) and \( k \in \mathbb{N} \). Our goal, as in the definition of accumulation point, is to show that there is an \( i > k \) such that \( |a_i - X| < \varepsilon \).

Observe that \( X + \varepsilon > X \). So by Theorem 23 there is an \( N \in \mathbb{N} \) such that if \( i \geq N \) then \( a_i < X + \varepsilon \). This implies that \( X - \varepsilon < a_i \). In other words, there is an \( i \geq p \) such that \( X - \varepsilon < a_i \). Since \( i \geq p \geq N \) we have \( a_i < X + \varepsilon \) as well. Since

\[
X - \varepsilon < a_i < X + \varepsilon
\]
we have \( |a_i - X| < \varepsilon \). Observe also that \( i \geq p \geq k \). This concludes the proof that \( X = \lim \sup_{i \to \infty} a_i \) is an accumulation point.

Now suppose that \( Y > X \). We must show that \( Y \) is not an accumulation point. Let \( Z \in F \) be such that \( Y > Z > X \). By Theorem 23, there is an \( N \in \mathbb{N} \) such that if \( i \geq N \) then \( a_i < Z \). Claim: this implies that \( Y \) is not an accumulation point \( (a_i) \). To see this suppose that \( Y \) is an accumulation point. Then for \( \varepsilon = Y - Z \) and \( k = N \) there is an \( i_0 \geq k \) such that \( |a_{i_0} - Y| < \varepsilon \). In other words, \( i_0 \geq N \) and \( |a_{i_0} - Y| < Y - Z \). However, we know that \( a_{i_0} < Z < Y \) since \( i_0 \geq N \), so

\[
Y - a_{i_0} = |a_{i_0} - Y| < Y - Z.
\]
This implies $a_{i_0} > Z$, which contradicts an earlier inequality. \hfill \square

The proof of the following is similar to the proof of the above theorem, so we omit it.

**Theorem 26.** Suppose $(a_i)$ is a bounded sequence in a complete ordered field $F$. Then $\liminf_{i \to \infty} a_i$ is an accumulation point of $(a_i)$. In fact, it is the least accumulation point of $(a_i)$.

**Corollary 27.** Suppose $(a_i)$ is a bounded sequence in a complete ordered field $F$. Then

$$\liminf_{i \to \infty} a_i \leq \limsup_{i \to \infty} a_i.$$

Furthermore $(a_i)$ converges if and only if equality hold. In this case,

$$\lim_{i \to \infty} a_i = \liminf_{i \to \infty} a_i = \limsup_{i \to \infty} a_i.$$

**Proof.** Since $\lim_{i \to \infty} a_i$ is the least accumulation point and $\limsup_{i \to \infty} a_i$ is the greatest accumulation point, we have

$$\liminf_{i \to \infty} a_i \leq \limsup_{i \to \infty} a_i.$$

By Exercise 12, if the sequence $(a_i)$ has a limit, that limit is the unique accumulation point, so

$$\lim_{i \to \infty} a_i = \liminf_{i \to \infty} a_i = \limsup_{i \to \infty} a_i.$$

Finally, suppose $\liminf_{i \to \infty} a_i = \limsup_{i \to \infty} a_i$. We will show that $X = \lim_{i \to \infty} \limsup_{i \to \infty} a_i$ is the limit of $(a_i)$. Let $\varepsilon > 0$ be given. We will find an $N \in \mathbb{N}$ such that $|a_i - X| < \varepsilon$ for all $i \geq N$. To do this, we first use Theorem 23 to obtain an $N_1 \in \mathbb{N}$ such that if $i \geq N_1$ then $a_i < X + \varepsilon$. By Theorem 24 there is an $N_2 \in \mathbb{N}$ such that if $i \geq N_2$ then $a_i > X - \varepsilon$. Thus if $i \geq N$ where $N$ is the maximum of $N_1$ and $N_2$, then

$$X - \varepsilon < a_i < X + \varepsilon.$$

In particular, $|a_i - X| < \varepsilon$ as desired. \hfill \square

**Remark 4.** The above shows that for bounded sequences, convergent is equivalent to the existence of exactly one accumulation point.

### 11. Cauchy Sequences Converge (Optional)

Earlier we showed that if $F$ is an archimedean ordered field such that every Cauchy sequences converges, then $F$ is complete. Now we show the converse.

**Theorem 28.** If $F$ is a complete ordered field, then every Cauchy sequence converges.
Proof. Let \((a_i)\) be a Cauchy sequence in a complete ordered field \(F\). By Lemma 16 the sequence \((a_i)\) is bounded. By Theorem 25 the limsup yields an accumulation point for \((a_i)\). So, by Theorem 21, \((a_i)\) converges. 

Corollary 29. Let \(F\) be an archimedean ordered field. Then \(F\) is complete if and only if every Cauchy sequence converges.