CHAPTER 9: COMPLEX NUMBERS

LECTURE NOTES FOR MATH 378 (CSUSM, SPRING 2009). WAYNE AITKEN

1. Introduction

Although $\mathbb{R}$ is a complete ordered field, mathematicians do not stop at real numbers. The real numbers are limited in various ways. For example, not every polynomial with real coefficients factors into linear polynomials with real coefficients. This is related to the fact that there are real polynomials such as $x^2 + 1$ or $x^4 + 2x^2 + 5$ that have no real roots. The need to solve polynomial equations led to the use of complex numbers.

As we know from basic algebra, when we work with quadratic equations sometimes the discriminant $b^2 - 4ac$ is negative, and in those cases we need to use complex numbers to find roots. The complex numbers allow us to use the quadratic equation successfully in all circumstances. However, the complex numbers did not arise first from quadratic equations. When a quadratic equation has no real solutions, why look for any solution at all? Just declare the problem unsolvable. This was the tactic used by early algebraists. It was later, in Renaissance Italy when the cubic and quartic equations were investigated, that square roots of negative numbers were first used. In fact, these so-called "imaginary numbers" are needed in the cubic equation even when looking for real solutions. For real solutions, the imaginary parts cancel out by the last step, but complex number arithmetic is required in intermediate computations. You cannot avoid the complex numbers even when your goal is to find real solutions.

At first the complex numbers were viewed as fictitious numbers which were useful sometimes in finding "real" solutions. Later, about 1800, the idea arose of treating complex numbers as points in the plane. This made the complex numbers into tangible, non-fictitious objects. We will follow this approach and define $\mathbb{C}$ as $\mathbb{R} \times \mathbb{R}$. After defining addition and multiplication on $\mathbb{C}$, our goal will be to establish that $\mathbb{C}$ is a field. However, this field cannot be made into an ordered field. Even though it is not an ordered field, we can still define an absolute value on $\mathbb{C}$. We consider some of the properties of this absolute value.

In the second half of this chapter we consider further properties of $\mathbb{C}$. Due to the limitation of this course, we will have to use results, including facts about trigonometry, which we will not prove here. So the standards of rigor are less than in the early chapters of this course. We will consider the polar form of complex numbers, and De Moivre's law. Finally, we will establish

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that every nonzero element $z \in \mathbb{C}$ has n distinct nth roots for every positive integer n. In the next chapter, we will consider the important *fundamental theorem of algebra* concerning complex roots of polynomials.

2. Basic Definitions

Definition 1. Define the set $\mathbb{C}$ of complex numbers as follows:

$$\mathbb{C} \stackrel{\text{def}}{=} \{(x, y) \mid x, y \in \mathbb{R}\}.$$

Remark 1. Recall that in set theory if S is a set then S^2 is defined to be $S \times S$ where $\times$ is the Cartesian product. So, as a set, $\mathbb{C}$ is just $\mathbb{R}^2$. There are differences between $\mathbb{C}$ and $\mathbb{R}^2$ when we start talking about binary operations. For example, the complex numbers $\mathbb{C}$ have a multiplication defined as a true binary operation, but $\mathbb{R}^2$ is typically given only a scalar multiplication.[1]

We now consider addition and a multiplication on $\mathbb{C}$.

Definition 2. Suppose that (x_1, y_1) and (x_2, y_2) are in $\mathbb{C}$. Then

$$(x_1, y_1) + (x_2, y_2) \stackrel{\text{def}}{=} (x_1 + x_2,\ y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) \stackrel{\text{def}}{=} (x_1x_2 - y_1y_2,\ y_1x_2 + x_1y_2).$$

The addition and multiplication on the right hand side of these equations are the addition and multiplication in $\mathbb{R}$ defined in Chapter 7.

Remark 2. Thus $\mathbb{C}$ has two binary operations: addition $+ : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and multiplication $\cdot : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$. This means that the sum and product of two complex numbers is a complex number: $\mathbb{C}$ is *closed* under addition and multiplication.

Exercise 1. Prove the following theorem.

Theorem 1. *Addition and multiplication on* $\mathbb{C}$ *are associative.*

Exercise 2. Prove the following two theorems.

Theorem 2. *Addition and multiplication on* $\mathbb{C}$ *are commutative.*

Theorem 3. *Addition and multiplication on* $\mathbb{C}$ *satisfy the distributive law.*

3. The Canonical Embedding

We want to view the complex numbers as an extension of the real numbers. In other words, we want to think of $\mathbb{R}$ as a subset of $\mathbb{C}$. Our definition (Definition 1) does not define $\mathbb{C}$ in such a way to make $\mathbb{R}$ automatically a subset. In order to regard $\mathbb{R}$ as a subset of $\mathbb{C}$ we need an injective function that embeds $\mathbb{R}$ into $\mathbb{C}$.

[1] However, vector addition in $\mathbb{R}^2$ and addition in $\mathbb{C}$ are really the same operation.

Definition 3. The *canonical embedding* $\mathbb{R} \to \mathbb{C}$ is the function defined by the rule

$$x \mapsto (x, 0).$$

Theorem 4. *The canonical embedding* $\mathbb{R} \to \mathbb{C}$ *is injective.*

Proof. Call the canonical embedding F. To show that $F : \mathbb{R} \to \mathbb{C}$ is injective we must show that, for all $a, b \in \mathbb{R}$, if $F(a) = F(b)$ then $a = b$.

Suppose $F(a) = F(b)$ where $a, b \in \mathbb{R}$ are arbitrary. Then $(a, 0) = (b, 0)$. By the definition of ordered pair (in set theory), this implies that the first coordinates are equal and that the second coordinates are equal. Since the first coordinates are equal, $a = b$ as desired. □

If we identify $x \in \mathbb{R}$ with its image $(x, 0)$ in $\mathbb{C}$, then we can think of $\mathbb{R}$ as a subset of $\mathbb{C}$. So, from now on, if $x \in \mathbb{R}$ then we will think of x and $(x, 0)$ as being the same number.

In particular, $0 \in \mathbb{R}$ can be identified with $(0, 0)$, and $1 \in \mathbb{R}$ can be identified with $(1, 0)$.

Theorem 5. *The number* 0 *is an additive identity for* $\mathbb{C}$ *and* 1 *is a multiplicative identity for* $\mathbb{C}$.

Proof. Identify 0 with $(0, 0)$. We must show that $(0, 0)$ is the additive identity. This follows from Definition 2 and the fact that $\mathbb{R}$ is a ring (Ch. 7):

$$(x, y) + (0, 0) = (x + 0, y + 0) = (x, y)$$

for all $(x, y) \in \mathbb{C}$. Likewise, $(0, 0) + (x, y) = (x, y)$ by the commutative law. Checking that 1 is a multiplicative identity is left as an exercise. □

Exercise 3. Check that 1 is a multiplicative identity for $\mathbb{C}$.

Since we now think of $\mathbb{R}$ as a subset of $\mathbb{C}$ we have to be careful with $+$ and $\cdot$ in $\mathbb{R}$. We defined these operations for $\mathbb{R}$ in one way in Chapter 7, and then defined them for $\mathbb{C}$ in the current chapter. Do we get the same answer for real numbers $a, b \in \mathbb{R}$ as for the corresponding complex numbers $(a, 0)$ and $(b, 0)$? The answer is yes since, using Definition 2,

$$(a, 0) + (b, 0) = (a + b,\ 0 + 0) = (a + b,\ 0)$$

and

$$(a, 0) \cdot (b, 0) = (a \cdot b - 0 \cdot 0,\ a \cdot 0 + 0 \cdot b) = (ab,\ 0).$$

We summarize the above observations as follows.

Theorem 6. *Consider* $\mathbb{R}$ *as a subset of* $\mathbb{C}$. *Then the addition and multiplication operations on* $\mathbb{C}$ *extend the corresponding binary operations on* $\mathbb{R}$.

4. The square root of -1

The complex numbers possesses a number whose square is -1.

Definition 4. Let i be the complex number $(0, 1)$.

Remark 3. Observe that i is not in the image of the canonical embedding $\mathbb{R} \to \mathbb{C}$. In other words, it is not a real number.

We now show the key property of i.

Theorem 7. *The number $i \in \mathbb{C}$ satisfies the equation*

$$i^2 = -1.$$

Exercise 4. Use Definition 2 and the canonical embedding to prove the theorem.

Remark 4. Because of this theorem, we call i the square root of -1. However, the square root of -1 is not unique: $(0, -1)$ is also a square root of -1.

Informal Exercise 5. Show that $i = (0, 1)$ and $(0, -1)$ are the only square roots of -1. Hint: suppose that $(x, y) \cdot (x, y) = (-1, 0)$ and show that $x = 0$ and $y = \pm 1$. You can use the fact that, for real numbers, the only square roots of positive 1 are ± 1. You also know that $x^2 \geq 0$.

Remark 5. The complex numbers $\mathbb{C}$ cannot be an ordered field. To see this, consider $i^2 = -1$. In an ordered field, all squares are nonnegative but -1 is always negative (since 1 must be positive).

5. Standard Form of Complex Numbers

We do not typically write complex numbers as ordered pairs: we like to write $x + yi$ for (x, y). We now establish that (x, y) and $x + yi$ are indeed the same complex number:

Theorem 8. *Let $(x, y) \in \mathbb{C}$ be a complex number. Then*

$$(x, y) \;=\; x + yi.$$

Proof. Observe that

$$\begin{aligned}
x + yi &= (x, 0) + (y, 0)i && \text{(Canonical Embed.)} \\
&= (x, 0) + (y, 0) \cdot (0, 1) && \text{(Def. 4)} \\
&= (x, 0) + (y \cdot 0 - 0 \cdot 1,\ y \cdot 1 + 0 \cdot 0) && \text{(Def. 2)} \\
&= (x, 0) + (0, y) && \text{(Laws in Ch. 7: } \mathbb{R} \text{ is a ring)} \\
&= (x, y) && \text{(Def. 2)}
\end{aligned}$$

$\square$

Remark 6. By the above theorem, we can think of the set $\mathbb{C}$ as follows:

$$\mathbb{C} \;=\; \{x + yi \mid x, y \in \mathbb{R}\}.$$

Theorem 9. *Let $x+yi$ and $v+wi$ be complex numbers where $x, y, v, w \in \mathbb{R}$. Then*
$$x + yi = v + wi \iff x = v \text{ and } y = w.$$

Proof. The ($\Leftarrow$) direction uses the substitution law (of equality).

We wish to prove the ($\Rightarrow$) direction, so suppose $x + yi = v + wi$. By Theorem 8,
$$(x, y) = x + yi \quad \text{and} \quad (v, w) = v + wi.$$
Thus $(x, y) = (v, w)$. By set theory, two ordered pairs are equal if and only if their components are equal. So $x = v$ and $y = w$. □

Remark 7. The above theorems shows that every complex number can be written uniquely in the form $x + yi$ where $x, y \in \mathbb{R}$.

6. Ring properties

We almost have everything we need to establish that $\mathbb{C}$ is a commutative ring: we have commutative, associative, distributive laws, and additive and multiplicative identities. We also need additive inverses:

Theorem 10. *Let $x + yi$ be a complex numbers (where $x, y \in \mathbb{R}$). Then $(-x) + (-y)i$ is the additive inverse of $x + yi$. In other words,*
$$-(x + yi) = (-x) + (-y)i$$
where the inverse on the left denotes additive inverse in $\mathbb{C}$, while the inverses on the right denote additive inverse in $\mathbb{R}$.

Proof. Observe that
$$\begin{aligned}
\big((-x) + (-y)i\big) + (x + yi) &= (-x, -y) + (x, y) \quad \text{(Thm. 8)}\\
&= (-x + x, -y + y) \quad \text{(Def. 2)}\\
&= (0, 0) \qquad \text{(Chapter 7 laws about } \mathbb{R})\\
&= 0 \qquad \text{(Use of canonical embed.)}
\end{aligned}$$
Since $\big((-x) + (-y)i\big) + (x + yi) = 0$, and since addition is commutative, the result follows. □

We now have everything we need for the following:

Theorem 11. *The set of complex numbers $\mathbb{C}$ is a commutative ring.*

Exercise 6. Review the definition of commutative ring, and verify that we have proved everything we need for the above theorem. Cite where each was done.

Remark 8. Now we can use all the laws that hold in general rings. For example, we know that if z is a complex number, then $0 \cdot z = 0$. Of course we could verify this directly, but the point is we do not have to: such a law holds in all rings. Likewise, $-(-z) = z$ for all $z \in \mathbb{C}$ since such an identity is true in all rings. Furthermore, if $z, w \in \mathbb{C}$ then
$$-(zw) = (-z)w = z(-w)$$

since such a law holds in all rings. Also $(-1)z = -z$ and $(-z)(-w) = zw$ and $-(z+w) = (-z)+(-w)$ for all $z, w \in \mathbb{C}$ since such laws holds in all rings.

Remark 9. Theorem 10 says that

$$-(x+yi) = (-x)+(-y)i$$

where the left-hand use of $-$ is additive inverse in $\mathbb{C}$, and the right hand use of $-$ is additive inverse in $\mathbb{R}$. When $y = 0$ we get $-x = -x$ where left-hand use of $-$ is additive inverse in $\mathbb{C}$, and the right hand use of $-$ is additive inverse in $\mathbb{R}$. Thus the two definitions of inverse agree. In other words, the additive inverse of $\mathbb{C}$ extends that of $\mathbb{R}$.

As in any ring, we define $z - w$ as $z + (-w)$. Since both the additive inverse in $\mathbb{C}$ and the addition in $\mathbb{C}$ extends the corresponding operations in $\mathbb{R}$, we can conclude that subtraction in $\mathbb{C}$ extends subtraction in $\mathbb{R}$.

Since $\mathbb{C}$ is a ring, we know that $-(z-w) = w - z$, $(z+w) - w = z$, $(z-w)+w = z$, etc.

Remark 10. It is interesting that, now that we know $\mathbb{C}$ is a ring, we can rederive Definition 2. In other words, if we forget Definition 2, we can rederive formulas for addition and multiplication. For addition:

$$\begin{aligned}(x+yi)+(v+wi) &= (x+v)+(yi+wi) \quad \text{(Assoc. and Comm.)}\\ &= (x+v)+(y+w)i \quad \text{(Distributive Law)}\end{aligned}$$

(here the first step combines several uses of the associative and commutative laws). For multiplication:

$$\begin{aligned}(x+yi)(v+wi) &= x(v+wi)+(yi)(v+wi) \quad \text{(Distributive Law)}\\ &= xv + x(wi) + (yi)v + (yi)(wi) \quad \text{(Distributive Law)}\\ &= xv + (xw)i + (yv)i + (yw)i^2 \quad \text{(Assoc./comm. for mult)}\\ &= xv + (xw)i + (yv)i + (yw)(-1) \quad \text{(Thm. 7)}\\ &= xv - yw + (xw)i + (yv)i \quad \text{(Properties of rings)}\\ &= (xv - yw) + (xw + yv)i \quad \text{(Distr. law)}\end{aligned}$$

Another way of saying this is that the formulas for addition and multiplication are a result of the fact that $i^2 = -1$ and that $\mathbb{C}$ is a ring. If someone constructed the complex numbers in another way such $i \in \mathbb{C}$ with $i^2 = -1$, such that $\mathbb{C}$ was a ring, and such that every element was of the form $x + yi$ with $x, y \in \mathbb{R}$, then that person would have the same formulas for addition and multiplication as we do.[2]

[2] A fancy way of saying this is that all rings with these properties are canonically isomorphic.

7. Complex Conjugation

We need to show that $\mathbb{C}$ is not just a ring, but is a field. To do this we need to show that every nonzero element has a multiplicative inverse. We will need complex conjugation to show how to form the multiplicative inverses.

Definition 5. Suppose $z \in \mathbb{C}$ where $z = x + yi$ with $x, y \in \mathbb{R}$. Then

$$\overline{z} \stackrel{\text{def}}{=} x - yi.$$

The complex number $\overline{z}$ is called the *complex conjugate* of z.

Theorem 12. *Let $z \in \mathbb{C}$. Then $\overline{z} = z$ if and only if z is a real number.*

Exercise 7. Show the above theorem. Hint: use Theorem 9 and properties of the real numbers.

Exercise 8. Prove the following two theorems.

Theorem 13. *Let $w, z \in \mathbb{C}$. Then*

$$\overline{z + w} = \overline{z} + \overline{w} \qquad \textit{and} \qquad \overline{zw} = \overline{z}\,\overline{w}.$$

Theorem 14. *If $z \in \mathbb{C}$ then $\overline{\overline{z}} = z$.*

Theorem 15. *If $z \in \mathbb{C}$ then $-\overline{z} = \overline{-z}$.*

Proof. By Theorem 13

$$\overline{-z} + \overline{z} = \overline{(-z) + z} = \overline{0} = 0.$$

Now subtract $\overline{z}$ from both sides. □

Corollary 16. *If $z, w \in \mathbb{C}$ then $\overline{z - w} = \overline{z} - \overline{w}$.*

Proof. This follows from the definition of $z - w$ as $z + (-w)$ together with Theorems 13 and 15. □

Theorem 17. *Let $z \in \mathbb{C}$. If $z = x + yi$ with $x, y \in \mathbb{R}$ then*

$$\overline{z}z = x^2 + y^2.$$

Exercise 9. Prove the above theorem.

Theorem 18. *Let $z \in \mathbb{C}$. Then $z\overline{z}$ is a nonnegative real number. Furthermore, if $z \neq 0$ then $z\overline{z} > 0$.*

Proof. Write z as $x + yi$ with $x, y \in \mathbb{R}$. From Theorem 17 we know that $\overline{z}z = x^2 + y^2$. In particular, $\overline{z}z$ is a real number since $\mathbb{R}$ is closed under addition and multiplication.

If $z = 0$ the result is nonnegative since $0^2 + 0^2 = 0$.

If $z \neq 0$ then either x or y (or both) is nonzero. Suppose, for example, that x is nonzero. Since the product of two positive numbers is positive, and the product of two negative numbers is also positive, we have $x^2 > 0$

regardless of whether x is positive or negative (ordered fields, Chapter 6). Also $y^2 \geq 0$ (if $y > 0$ it follows as for x, if $y = 0$ then $y^2 = 0$). Thus

$$x^2 + y^2 > 0 + y^2 \geq 0 + 0 = 0$$

by properties of ordered fields (Chapter 6). Thus $\overline{z}z > 0$. A similar argument shows the result if $y \neq 0$. □

8. Field properties

The complex numbers form a field. To see this we need to to check that $1 \neq 0$ which is obvious (since $1 \neq 0$ in the subset $\mathbb{R}$), and that every nonzero element has a multiplicative inverse. Suppose that $z \in \mathbb{C}$ is nonzero. Then $\overline{z}z$ is a positive real number by Theorem 18. Since $\mathbb{R}$ is a field, and since $\overline{z}z \neq 0$, the multiplicative inverse $(\overline{z}z)^{-1}$ exists in $\mathbb{R}$ (and hence in $\mathbb{C}$). Consider

$$w \stackrel{\text{def}}{=} (\overline{z}z)^{-1}\,\overline{z}.$$

Then

$$wz = (\overline{z}z)^{-1}\,\overline{z}z = 1.$$

So z has a multiplicative inverse. We can use this fact to prove the following.

Theorem 19. *The set of complex numbers $\mathbb{C}$ is a field.*

Exercise 10. Review the definition of *field*, and verify that we have proved everything we need for the above theorem.

Informal Exercise 11. Use the above formula for w to find the multiplicative inverse of $z = 2 + i$. Write your answer in the form $a + bi$ with $a, b \in \mathbb{R}$.

Informal Exercise 12. Find the multiplicative inverses of i and $3i$.

Informal Exercise 13. Convert

$$z = \frac{7 + 2i}{2 + 3i}$$

to the form $x + yi$ with $x, y \in \mathbb{R}$.

Exercise 14. Let $z \in \mathbb{C}$ be nonzero. Show that $\overline{z}^{-1} = \overline{z^{-1}}$. In addition, let $w \in \mathbb{C}$. Show that

$$\overline{\left(\frac{w}{z}\right)} = \frac{\overline{w}}{\overline{z}}.$$

9. Absolute Values

In Chapter 8 we showed that every nonnegative real number x has a unique nonnegative square root $\sqrt{x}$. The square root is used in the definition of absolute value in $\mathbb{C}$.

Definition 6 (Absolute Value)**.** Let $z \in \mathbb{C}$. Then the *absolute value* $|z|$ of z is defined as follows:

$$|z| \stackrel{\text{def}}{=} \sqrt{z\overline{z}}.$$

Theorem 20. *If $z, w \in \mathbb{C}$ then*

$$|zw| = |z| \cdot |w|.$$

Proof. Observe that

$$\begin{aligned}(|z| \cdot |w|)^2 &= |z|^2|w|^2 && \text{(Expon. Law, Ch. 4)} \\ &= z\overline{z}\, w\overline{w} && \text{(Def 6)} \\ &= zw\,\overline{z}\,\overline{w} && \text{(Comm./Assoc. Laws)} \\ &= zw\,\overline{zw} && \text{(Thm. 13)} \\ &= |zw|^2. && \text{(Def 6)}\end{aligned}$$

By a result of Chapter 8, this implies that $|z| \cdot |w| = |zw|$. □

Theorem 21. *If $z \in \mathbb{C}$ then $|-z| = |z|$.*

Proof. By Theorem 20,

$$|-z|\,|-z| = |(-z)(-z)| = |zz| = |z||z|.$$

Thus $|-z|^2 = |z|^2$. So $|-z| = |z|$ by a result of Chapter 8. □

Theorem 22. *Suppose $z \in \mathbb{C}$. Then $|z| = 0$ if and only if $z = 0$.*

Exercise 15. Prove the above theorem.

Theorem 23. *Suppose $z = x + yi$ where $z \in \mathbb{C}$ and $x, y \in \mathbb{R}$. Then*

$$|x| \le |z| \qquad \text{and} \qquad |y| \le |z|.$$

Proof. (sketch) Since $x^2 \ge 0$, we have $x^2 + y^2 \ge x^2$. Observe that $|x|^2 = x^2$ and $|z|^2 = x^2 + y^2$. Thus $|z|^2 \ge |x|^2$. So $|z| \ge |x|$.

A similar argument shows $|z| \ge |y|$. □

Now we wish to show the triangle inequality.

Lemma 24. *If $z \in \mathbb{C}$ then*

$$|z + 1| \le |z| + 1.$$

Proof. (sketch) Let $z = x + yi$ where $x, y \in \mathbb{R}$. Then $z + 1 = (x + 1) + yi$. Thus

$$|z + 1|^2 = (x + 1)^2 + y^2 = x^2 + 2x + 1 + y^2 = (x^2 + y^2) + 2x + 1.$$

If $x \ge 0$ then $x \le |z|$ by Theorem 23. If $x < 0$ then $x \le |z|$ since $|z| \ge 0$. In either case $x \le |z|$. So

$$\begin{aligned}|z + 1|^2 &= (x^2 + y^2) + 2x + 1 \\ &= |z|^2 + 2x + 1 \\ &\le |z|^2 + 2|z| + 1 \\ &= (|z| + 1)^2.\end{aligned}$$

By a result in Chapter 8, this implies $|z + 1| \le |z| + 1$. □

Theorem 25 (Triangle Inequality in $\mathbb{C}$)**.** *Let $z, w \in \mathbb{C}$. Then*

$$|z + w| \leq |z| + |w|.$$

Proof. (sketch) If $w = 0$ then the result is clear. So assume $w \neq 0$. Let $u = zw^{-1}$. By Lemma 24,

$$|u + 1| \leq |u| + 1.$$

Multiply both sides by $|w|$: So

$$|u + 1||w| \leq (|u| + 1)|w| = |uw| + |w| = \left|zw^{-1}w\right| + |w| = |z| + |w|.$$

However,

$$|u + 1||w| = |(u + 1)w| = |uw + w| = \left|zw^{-1}w + w\right| = |z + w|.$$

So

$$|z + w| \leq |z| + |w|.$$

$\square$

10. Polar Coordinates

In order to study the complex numbers in more depth, we need to use the sine and cosine functions. Developing trigonometry formally would take us too far afield, so we will take some basic facts as given. Consequently, for the rest of the chapter all the concepts and results will be informal. In contrast to our usual methodology, we draw on basic mathematical knowledge developed outside this course.

From trigonometry, we know that if (x, y) is a point in $\mathbb{R}^2$, then there are real numbers r and θ such that

$$(x, y) = (r \cos \theta, r \sin \theta).$$

Furthermore, if we require that r be nonnegative, then r is unique. In this case r is called the *radius* of (x, y). If we restrict θ in the range $0 \leq \theta < 2\pi$ (or some other half-open interval of width 2π), and if $(x, y) \neq (0, 0)$, then θ is also unique. We call θ the *angle* of (x, y). In what follows, we will assume that $r \geq 0$, but we will not restrict θ unless specifically indicated. So r is unique, but the angle θ is not necessarily unique.

If we view $\mathbb{R}^2$ as being $\mathbb{C}$, then (x, y) is usually written $x + yi$, so we have the formula

$$z = x + yi = r \cos \theta + r \sin \theta \cdot i.$$

This expression is called the *polar coordinates form of z*.

Theorem 26. *If $z = r \cos \theta + r \sin \theta \cdot i$. Then $r = |z|$.*

Proof. Observe

$$|z|^2 = (r \cos \theta)^2 + (r \sin \theta)^2 = r^2(\cos^2 \theta + \sin^2 \theta) = r^2.$$

The last step uses a standard trigonometric identity. $\square$

Remark 11. The 2 in the expression $\cos^2\theta$ does not refer to iteration. It is used to denote the product $\cos\theta \cdot \cos\theta$. Likewise for $\sin^2\theta$.

Theorem 27. *Let $z, w \in \mathbb{C}$. If z, w are written in polar coordinates as*

$$z = r_1\cos\theta_1 + r_1\sin\theta_1 \cdot i \qquad w = r_2\cos\theta_2 + r_2\sin\theta_2 \cdot i$$

then the product can be written as

$$zw = r\cos\theta + r\sin\theta \cdot i$$

where $r = r_1r_2$ and $\theta = \theta_1 + \theta_2$.

Remark 12. In other words, when you multiply complex numbers, you multiply the radii and add the angles. So multiplication has a very nice geometric interpretation.

Addition also has a geometric interpretation. It is just vector addition.

Proof. For convenience, write $\cos\theta_1$ as c_1, $\cos\theta_2$ as c_2, $\sin\theta_1$ as s_1, and $\sin\theta_2$ as s_2. So

$$\begin{aligned}
zw &= (r_1c_1 + r_1s_1i)(r_2c_2 + r_2s_2i) \\
&= r_1(c_1 + s_1i)r_2(c_2 + s_2i) \qquad \text{(Distr. Law)} \\
&= r_1r_2(c_1 + s_1i)(c_2 + s_2i) \qquad \text{(Comm./Assoc. Laws)} \\
&= r_1r_2(c_1c_2 + c_1s_2i + s_1ic_2 + s_1is_2i) \qquad \text{(Distr. Law)} \\
&= r_1r_2\big(c_1c_2 + c_1s_2i + s_1c_2i - s_1s_2\big) \qquad (i^2 = -1) \\
&= r_1r_2\big((c_1c_2 - s_1s_2) + (c_1s_2 + s_1c_2)i\big) \\
&= r_1r_2\big(\cos(\theta_1 + \theta_2) + \sin(\theta_1 + \theta_2)i\big) \qquad \text{(Trig. identities)}
\end{aligned}$$

□

Corollary 28. *Let $z \in \mathbb{C}$. If z is*

$$z = r\cos\theta + r\sin\theta \cdot i$$

in polar coordinates, then

$$z^{-1} = r^{-1}\cos(-\theta) + r^{-1}\sin(-\theta)i$$

and

$$z^{-1} = r^{-1}\cos\theta - r^{-1}\sin\theta \cdot i.$$

Informal Exercise 16. Prove the above corollary.

11. De Moivre's Theorem

Theorem 29. *Let $z \in \mathbb{C}$ and $n \in \mathbb{Z}$. Suppose $z \neq 0$. If z is*

$$z = r\cos\theta + r\sin\theta \cdot i$$

in polar coordinates, then

$$z^n = r^n\cos(n\theta) + r^n\sin(n\theta)i.$$

Proof. Fix $z \neq 0$. Let S be the set of all $n \in \mathbb{N}$ for which

$$z^n = r^n \cos(n\theta) + r^n \sin(n\theta) i.$$

We begin by showing that $S = \mathbb{N}$. Then we will consider negative powers.

Observe that $0 \in S$ since $z^0 = r^0 = \cos(0) = 1$ and $\sin(0) = 0$.

Now suppose that $u \in S$. Then

$$z^{u+1} = z^u z = \big(r^u \cos(u\theta) + r^u \sin(u\theta) i\big)\big(r\cos(\theta) + r\sin(\theta) i\big)$$

By Theorem 27,

$$z^u z = r' \cos\theta' + r' \sin\theta' \cdot i$$

where $r' = r^u r$ and $\theta' = u\theta + \theta = (u+1)\theta$. Thus $u + 1 \in S$.

By the induction axiom, $S = \mathbb{N}$. So the theorem holds for all $n \geq 0$.

For $n < 0$, let $m = -n$. So the theorem holds for z^m by the above argument. Now use Corollary 28 to show that $z^n = (z^m)^{-1}$ has radius $(r^m)^{-1}$ and angle $-(m\theta)$. Since $(r^m)^{-1} = r^n$ and $-(m\theta) = n\theta$, the result follows. $\square$

Remark 13. This shows that, in polar coordinates, when one takes the nth power one takes the nth power of the radius and multiplies the angle by n.

12. Exponential Function

We now consider the complex exponential function $z \mapsto e^z$ as a function $\mathbb{C} \to \mathbb{C}$. Familiarity with e^x for $x \in \mathbb{R}$ is assumed.

Definition 7. Let $z \in \mathbb{C}$. If $z = x + yi$ where $x, y \in \mathbb{R}$ then

$$e^z \overset{\text{def}}{=} e^x(\cos y + i \sin y).$$

Remark 14. In particular, if $y \in \mathbb{R}$,

$$e^{iy} = \cos y + i \sin y.$$

To see this, take $x = 0$ and observe $e^x = e^0 = 1$. This formula for e^{iy} implies

$$e^z = e^x e^{iy}$$

as expected.

Informal Exercise 17. Let $z = x + yi$ where $x, y \in \mathbb{R}$. Show that $|e^z| = e^x$.

Theorem 30. *Let $z \in \mathbb{C}$. If z is*

$$z = r\cos\theta + r\sin\theta \cdot i$$

in polar coordinates, then

$$z = re^{\theta i}.$$

Proof. By definition, $e^{i\theta} = \cos\theta + i\sin\theta$. See Remark 14. $\square$

Corollary 31. *Every complex number can be written in the form*

$$z = re^{\theta i}$$

where $r = |z|$.

Informal Exercise 18. Suppose z, w are complex numbers. Then show that
$$e^{z+w} = e^z e^w.$$
Hint: Use Theorem 27, and known properties of e^x when x is real.

Remark 15. A special case of this is when $w = -z$. Since $e^0 = 1$, we conclude that $e^z e^{-z} = 1$. In other words, e^{-z} is the multiplicative inverse of e^z.

Theorem 32. *Suppose z is a complex number, and n is an integer. Then*
$$(e^z)^n = e^{nz}.$$

Proof. (sketch) This follows by induction using Informal Exercise 18. The case of negative n has to be established separately using the above remark. □

Informal Exercise 19. Show that $e^{\pi i} = -1$. This is considered by many to be one of the most amazing formula in mathematics since it unites e, π, i. Observe that $e^{\pi i} + 1 = 0$ unites $e, \pi, i, 0, 1$.

13. nTH ROOTS OF REAL NUMBERS

In Chapter 8 we established that for every positive integer n and for every nonnegative x in $\mathbb{R}$ there exists a nth root of x in $\mathbb{R}$. When n is odd, then we can extend this result for negative x as well. However, we cannot hope to have nth roots in $\mathbb{R}$ when n is even and $x < 0$.

In this section we establish that nth roots exist for any positive integer n and any $z \in \mathbb{C}$. In fact, if $z \neq 0$ there are exactly n distinct nth roots evenly distributed on a circle in the complex plane.

First we establish existence of nth roots:

Theorem 33. *Suppose z is a complex number written in the form $z = re^{\theta i}$. Then $r^{1/n} e^{\theta i/n}$ is an nth root of z.*

Proof. (sketch) This follows from Theorem 32. □

Corollary 34. *Every complex number has an nth root.*

Informal Exercise 20. Sketch a cube root of -1 in the complex plane.

In order to come up with all roots of a complex number, it is convenient to start with $z = 1$.

Definition 8. Let n be a fixed positive integer. Every complex number of the form $e^{2k\pi i/n}$ is called an *nth root of unity.*

Theorem 35. *If z is an nth root of unity then $z^n = 1$. Furthermore, there are n distinct nth roots of unity.*

Proof. (sketch) Observe that $e^{2\pi i} = 1$. Thus
$$\left(e^{2k\pi i/n}\right)^n = e^{2k\pi i} = \left(e^{2\pi i}\right)^k = 1^k = 1.$$
Thus every nth root of unity is an nth root of 1.

If $0 \leq k_1 < k_2 < n$ then the angles of the corresponding roots of unity satisfy $0 \leq 2k_1\pi/n < 2k_2\pi/n < 2\pi$. Thus $e^{2k_1\pi i/n}$ and $e^{2k_2\pi i/n}$ are on distinct rays emanating from the origin, so they must be distinct. Thus there are at least n distinct nth roots of unity.

If $k \geq n$, then the remainder upon dividing by n gives rise to the same root of unity. This can be seen as follows. Let $k = qn + r$. Then

$$e^{2k\pi i/n} = e^{2(qn+r)\pi i/n} = \left(e^{2\pi i}\right)^q e^{2r\pi i/n} = 1^q\, e^{2r\pi i/n}.$$

So there are no more nth roots of unity beyond the n discussed above. □

Remark 16. One can show, conversely, that every solution of $z^n = 1$ is an nth root of unity. This can be seen by showing that $z^n = 1$ implies that $|z| = 1$ and the same angle as one of the roots of unity.

Theorem 36. *Suppose z is a complex number written in the form $z = re^{\theta i}$. Let n be a positive integer, and let ζ be an nth root of unity. Then $r^{1/n}e^{\theta i/n}\zeta$ is an nth root of z. If $z \neq 0$ then every nth root of z can be written in this way.*

Proof. (sketch) Observe that

$$\begin{aligned} \left(r^{1/n}e^{\theta i/n}\zeta\right)^n &= \left(r^{1/n}\right)^n \left(e^{\theta i/n}\right)^n \zeta^n \\ &= r \cdot e^{\theta i} \cdot 1 \\ &= re^{\theta i} \\ &= z. \end{aligned}$$

Thus $r^{1/n}e^{\theta i/n}\zeta$ is an nth root of z.

This gives one root, now suppose w is any nth root of z where $z \neq 0$. Then $w \neq 0$. One can check if $u = r^{1/n}e^{\theta i/n}w^{-1}$ then $u^n = 1$. As remarked above, this means that u it is an nth root of unity. Hence $\zeta' \overset{\text{def}}{=} u^{-1}$ is also a root of unity, and $w = r^{1/n}e^{\theta i/n}\zeta'$. □

Theorem 37. *Let n be a positive integer. Then every complex number $z \neq 0$ has exactly n distinct nth roots.*

Proof. This follows from the previous theorem and the fact that there are exactly n distinct nth roots of unity. □

Remark 17. Of course 0 has 0 for an nth root. In fact, 0 is the only nth root of 0. Between this observation and the previous theorem we know that *every complex number has at least one nth root.*

Informal Exercise 21. Sketch the three cube roots of -1, and the four 4th roots of $1 + i$. Sketch the five fifth roots of unity. Sketch the 6 distinct 6th roots of 2^6.