CHAPTER 8: EXPLORING $\mathbb{R}$

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In the previous chapter we discussed the need for a complete ordered field. The field $\mathbb{Q}$ is not complete, so we constructed the real numbers $\mathbb{R}$. In this chapter we investigate some important properties of $\mathbb{R}$ that are a consequence of its completeness, and which fail for $\mathbb{Q}$. For example, every decimal expansion defines a real number, but not always a rational number. Also, for every positive integer n and every nonnegative real number x, there is a unique nonnegative nth root $x^{1/n}$. The existence of such roots often fails for rational numbers. We end the chapter by showing that $\mathbb{Q}$ is countable, but $\mathbb{R}$ is uncountable.

Before we try to establish these results, we first need to develop some useful tools including the result that every bounded monotonic sequence is Cauchy and so converges in $\mathbb{R}$. This result is not only important here, but is important in real analysis and mathematics in general.

1. Monotonic sequences

Monotonic sequences are commonly used in mathematics and are easier to deal with than arbitrary sequences.

Definition 1. Let $(a_i)_{i\geq n_0}$ be a sequence in an ordered field F. The sequence is said to be *upward monontonic* (or *nondecreasing*) if $a_{i+1} \geq a_i$ for all $i \geq n_0$. The sequence is *downward monontonic* (or *nonincreasing*) if $a_{i+1} \leq a_i$ for all $i \geq n_0$. In either case (a_i) is said to be *monotonic*.

The sequence (a_i) is said to be *strictly upward monotonic* or *increasing* if $a_{i+1} > a_i$ for all i in the domain. The sequence (a_i) is *strictly downward monotonic* or *decreasing* if $a_{i+1} < a_i$ for all i in the domain.

The following is a basic consequence of the trichotomy law. It is stated for upward monotonic sequences, but the statement holds, with the obvious modifications, for downward, strictly upward, or strictly downward monotonic sequences.

Lemma 1. *Suppose that $(a_k)_{k\geq n_0}$ is an upward monotonic sequence in an ordered field F. If $j > i \geq n_0$ then $a_j \geq a_i$.*

Proof. Consider the set $S_i = \{u \in \mathbb{Z} \mid u \geq n_0 \text{ and } a_u \geq a_i\}$. By Definition 1, $a_{i+1} \geq a_i$. Also, $i + 1 \geq n_0$ since $i \geq n_0$. Thus $i + 1 \in S_i$ (base case).

Now suppose $k \in S_i$. This implies $a_{k+1} \geq a_k \geq a_i$ (the first inequality by Definition 1, the second since $k \in S_i$). So $k + 1 \in S_i$.

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By induction, every integer $i+1$ or greater is in S_i. In particular, $j \in S_i$. Thus $a_j \geq a_i$ as desired. □

An upward monotonic sequence is automatically bounded from below: if a_{n_0} is the first term of such a sequence then the above lemma shows a_{n_0} is a lower bound. So to say that such a sequence is *bounded* means that it is also bounded from above. Obviously the same idea, but reversed, applies to downward monotonic sequences. This motivates the following:

Definition 2. An upward monotonic sequence $(a_i)_{i \geq n_0}$ in an ordered field F is said to be *bounded* if there is a $B \in F$ such that $a_i \leq B$ for all $i \geq n_0$. Such a B is called an *upper bound*. A downward monotonic sequence $(a_i)_{i \geq n_0}$ in F is *bounded* if there is a $C \in F$ such that $a_i \geq C$ for all $i \geq n_0$. Such a C is called a *lower bound*

2. Useful Tools

We now establish a tool-kit of useful results for later use.

Theorem 2. *Suppose (a_i) and (b_i) are two converging sequences in an ordered field F. Let a be the limit of (a_i) and b be the limit of (b_i). Suppose there is a $k \in \mathbb{N}$ such that $a_i \leq b_i$ for all $i \geq k$. Then $a \leq b$.*

Proof. Suppose otherwise that $a > b$. Let $\epsilon = (a-b)/2$. Note: $\epsilon > 0$. By the convergence hypothesis for (a_i), there is a k_1 such that $i \geq k_1$ implies $|a_i - a| < \epsilon$. Likewise, there is a k_2 such that $i \geq k_2$ implies $|b_i - b| < \epsilon$. The existence of k_1 and k_2 follow from the definition of limit. Let k be as in the hypothesis of the lemma, and let i be the maximum of k, k_1, and k_2.

Since $i \geq k_1$ we have $|a_i - a| < \epsilon$. Thus $-\epsilon < a_i - a < \epsilon$. Since $a_i - a > -\epsilon$ we have $a_i > a - \epsilon$.

Since $i \geq k_2$ we have $|b - b_i| < \epsilon$. Thus $-\epsilon < b_i - b < \epsilon$. Since $b_i - b < \epsilon$ we have $b_i < b + \epsilon$. Thus $-b_i > -b - \epsilon$.

From $a_i > a - \epsilon$ and $-b_i > -b - \epsilon$ and properties of $>$ under addition

$$a_i + (-b_i) > (a - \epsilon) + (-b - \epsilon) = (a - b) - 2\epsilon = (a - b) - (a - b) = 0.$$

(Recall $\epsilon = (a-b)/2$). Since $a_i - b_i > 0$, we have $a_i > b_i$, which contradicts $i \geq k$ and the hypothesis of the theorem that $a_i \leq b_i$. □

Remark 1. This result cannot be generalized to $<$. Without more information, given $a_i < b_i$ for all i we cannot conclude $a < b$ in the limit. Consider, for instance $a_i = 1 - 1/i$ and $b_i = 1 + 1/i$.

Corollary 3. *Suppose (a_i) is a converging sequence in an ordered field F with limit a. Suppose that b is an upper bound of (a_i). Then $a \leq b$.*

Suppose instead that b is a lower bound of (a_i). Then $a \geq b$.

Proof. Apply Theorem 2 to (a_i) and the constant sequence with terms b. □

Theorem 4. *Let (a_i) and (b_i) be sequences with values in an ordered field F. If (a_i) converges to a and (b_i) converges to b with $a, b \in F$, then $(a_i b_i)$ converges to ab.*

Proof. Let $\epsilon > 0$ be in F. We must show that there is a $k \in \mathbb{N}$ such that $|a_i b_i - ab| < \epsilon$ for all $i \geq k$.

Since (a_i) has a limit, it is Cauchy. This means that there is a bound $M_1 \in F$ such that $|a_i| \leq M_1$ for all i in the domain of (a_i) (Chapter 7). Let M be the maximum of $M_1, |b|$, and 1. Thus $|b| \leq M$, $0 < M$, and $|a_i| \leq M$ for all a_i. Let $\epsilon' = \epsilon/(2M)$. Observe that ϵ' is positive. Since (a_i) converges to a, there is a k_1 such that $|a_i - a| < \epsilon'$ for all $i \geq k_1$. Likewise, there is a k_2 such that $|b_i - b| < \epsilon'$ for all $i \geq k_2$. Let k be the maximum of k_1 and k_2. Suppose $i \geq k$. Then

$$\begin{aligned}
|a_i b_i - ab| &= |a_i b_i - a_i b + a_i b - ab| && (F \text{ is a field}) \\
&= |a_i(b_i - b) + b(a_i - a)| && (F \text{ is a field}) \\
&\leq |a_i(b_i - b)| + |b(a_i - a)| && (\text{Ch. 6: triangle inequality}) \\
&= |a_i||b_i - b| + |b||a_i - a| && (\text{Ch. 6: } |xy| = |x||y|) \\
&\leq M|b_i - b| + M|a_i - a| && (\text{bound on } |a_i| \text{ and } |b|) \\
&< M\epsilon' + M\epsilon' && (i \geq k_1 \text{ and } i \geq k_2) \\
&= M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} && (\text{choice of } \epsilon') \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} && (MM^{-1} = 1) \\
&= \left(\frac{1}{2} + \frac{1}{2}\right)\epsilon && (F \text{ is a field}) \\
&= \epsilon && (1 + 1 \stackrel{\text{def}}{=} 2 \text{ in } F).
\end{aligned}$$

□

Corollary 5. *Let (a_i) be a sequence in an ordered field F with limit $a \in F$. Then (a_i^n) has limit a^n for all $n \in \mathbb{N}$.*

Proof. This can be proved by induction using Theorem 4. □

Theorem 6. *Let (a_i) and (b_i) be sequences with values an ordered field F. If (a_i) converges to a and (b_i) converges to b with $a, b \in F$, then the sequence $(a_i + b_i)$ converges to $a + b$.*

Proof. Let $\epsilon > 0$ be in F. We must show that there is a $k \in \mathbb{N}$ such that $|(a_i + b_i) - (a + b)| < \epsilon$ for all $i \geq k$.

Let $\epsilon' = \epsilon/2$. Since (a_i) converges to a, there is a k_1 such that $|a_i - a| < \epsilon'$ for all $i \geq k_1$. Likewise, there is a k_2 such that $|b_i - b| < \epsilon'$ for all $i \geq k_2$.

Let k be the maximum of k_1 and k_2. Suppose $i \geq k$. Then

$$\begin{aligned}
|(a_i + b_i) - (a + b)| &= |(a_i - a) + (b_i - b)| && (F \text{ is a field}) \\
&\leq |a_i - a| + |b_i - b| && (\text{Ch. 6: triangle inequality}) \\
&< \epsilon' + \epsilon' && (i \geq k_1 \text{ and } i \geq k_2) \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} && (\text{choice of } \epsilon') \\
&= \left(\frac{1}{2} + \frac{1}{2}\right)\epsilon && (F \text{ is a field}) \\
&= \epsilon && (1 + 1 \stackrel{\text{def}}{=} 2 \text{ in } F).
\end{aligned}$$

□

Lemma 7. *Let x, y be positive elements of an ordered field F, and let $n \in \mathbb{N}$. Then x^n and y^n are also positive. Furthermore, if $n \geq 1$ then*

$$x^n \leq y^n \iff x \leq y$$

and

$$x^n < y^n \iff x < y.$$

Proof. (Sketch) If $x \in F$ is positive, then x^n can be shown to be positive by induction using closure of the positive subset $P \subseteq F$.

If $x \leq y$, then $x^n \leq y^n$ can be shown by induction (even for $n = 0$). If $x < y$ then $x^n < y^n$ can also be shown by induction (for $n \geq 1$).

Suppose $x^n \leq y^n$. If $y < x$ then $y^n < x^n$ by the previous argument. This is a contradiction to trichotomy. Thus $x \leq y$.

Suppose $x^n < y^n$. If $y \leq x$ then $y^n \leq x^n$ by the earlier argument. This is a contradiction to trichotomy. Thus $x < y$. □

Lemma 8. *Let x, y be nonnegative elements of an ordered field F, and let $n \in \mathbb{N}$. Then x^n and y^n are also nonnegative. Furthermore, if $n \geq 1$,*

$$x^n \leq y^n \iff x \leq y.$$

Proof. (Sketch) The case where x and y are both positive is covered by Lemma 7. The case where one or both is zero is easy to verify directly. □

Lemma 9. *Let x, y be nonnegative elements of an ordered field F. If n is a positive integer then*

$$x^n = y^n \iff x = y.$$

Proof. The direction $\Leftarrow$ follows from properties of equality. The direction $\Rightarrow$ follows from Lemma 8 (applied both ways). □

Lemma 10. *Suppose $x \in \mathbb{R}$. Then there is a unique integer $n \in \mathbb{Z}$ such that $n \leq x < n + 1$.*

Remark 2. In other words, $x = n + y$ where $0 \leq y < 1$. The number y is sometimes called the *fractional part* of x (although it is not always a fraction in the sense of being in $\mathbb{Q}$). We call n the *floor* of x. If $n < x$ then $n + 1$ is called the *ceiling* of x, but if $n = x$ then n is the ceiling of x.

Proof. First consider the case where $x \in \mathbb{Q}$. So $x = a/b$ where $a, b \in \mathbb{Z}$ and where $b > 0$ (Chapter 6). Write $a = bq + r$ for some $0 \le r < b$ (Quotient-Remainder Theorem, Chapter 4). Multiplying by b^{-1} gives $a/b = q + r/b$ and $0 \le r/b < 1$. Adding q to $0 \le r/b < 1$ gives $q \le a/b < q + 1$. This established existence for $x \in \mathbb{Q}$.

Now for the general case. Given $x \in \mathbb{R}$ and $\epsilon > 0$ real, we know that there is a rational number u such that $|x - u| < \epsilon$ (Chapter 7). In particular, we can find an $u \in \mathbb{Q}$ such that $|x - u| < 1/2$. Let $m \in \mathbb{Z}$ be such that $m \le u < m + 1$. Such an m exists by the first case of the theorem.

Since $|x - u| < 1/2$, we have $-1/2 < x - u < 1/2$. Adding u gives $u - 1/2 < x < u + 1/2$. Now $m - 1 \le u - 1 < u - 1/2$ (since $m \le u$ and $1/2 < 1$). Similarly $u + 1/2 < u + 1 \le m + 2$. Thus $m - 1 < x < m + 2$.

Among $m - 1, m, m + 1$ choose the largest one less than or equal to x. Let n be this integer. Then $n \le x < n + 1$. This gives existence for $x \in \mathbb{R}$.

For uniqueness, suppose $x \in \mathbb{R}$, and $n_1, n_2 \in \mathbb{N}$ satisfy $n_1 \le x < n_1 + 1$ and $n_2 \le x < n_2 + 1$. So $n_1 < n_2 + 1$ and $n_2 < n_1 + 1$. This implies $n_1 - 1 < n_2 < n_1 + 1$. Thus $n_2 = n_1$. This gives uniqueness. □

3. Bounded monotonic sequences are Cauchy

We now establish one of the most useful facts about $\mathbb{R}$: every bounded monotonic sequence converges. We do so by proving that monotonic sequences in $\mathbb{R}$ are Cauchy. Since $\mathbb{R}$ is complete, this means that such sequences converge (Chapter 7).

To motivate the proof, suppose that $(a_i)_{i \ge n_0}$ is monotonic. Assume (a_i) is upward monotonic, but the following discussion can be adapted to downward monotonic sequences. Since (a_i) is upward monotonic, if $j > i \ge n_0$ then $a_j \ge a_i$ (Lemma 1). In particular, $a_j - a_i \ge 0$, so $|a_j - a_i| = a_j - a_i$.

Suppose, for arguments sake, that (a_i) is not Cauchy. Using basic logic to negate the definition of Cauchy sequence we conclude that there is an $\epsilon > 0$ in $\mathbb{R}$ such that, for all $k \in \mathbb{N}$, there are $i, j \ge k$ with $|a_j - a_i| \ge \epsilon$. We can choose i, j so that $j > i$, hence $|a_j - a_i| = a_j - a_i$ as above. Thus $a_j \ge a_i + \epsilon$. Since we can take i, j larger than any $k \in \mathbb{N}$, such an inequality happens an infinite number of times. This suggests that (a_i) is not bounded:

Lemma 11. *Let $(a_i)_{i \ge n_0}$ be a upward monotonic sequence in $\mathbb{R}$. If (a_i) is not Cauchy, then it is not bounded.*

Proof. Let ϵ be as in the above discussion. By a density result of Chapter 7, there is a rational number a/n such that $0 < a/n < \epsilon$ where a and n are positive integers. Since $0 < 1 \le a$ we have $0 < 1/n \le a/n$. So $0 < 1/n < \epsilon$.

As in the discussion preceding the lemma, for all $k \in \mathbb{N}$, there are $i, j \ge k$ with $j > i$ and $a_j \ge a_i + \epsilon$. In particular $a_j \ge a_i + 1/n$.

By Lemma 10, there is an integer m such that $m \le a_{n_0}$ where a_{n_0} is the first term of the sequence. Let

$$S = \left\{ l \in \mathbb{N} \;\middle|\; m - 1 + \frac{l}{n} \text{ is not an upper bound of } (a_i) \right\}.$$

Our first goal is to prove that $S = \mathbb{N}$ by induction. Observe $0 \in S$ since $m - 1 < a_{n_0}$.

Now suppose that $u \in S$. We wish to show $u + 1 \in S$. By definition of S, we have that $m - 1 + u/n$ is not an upper bound of (a_i). Thus there is a term a_k with $m-1+u/n < a_k$. As discussed above, there are $j > i \geq k$ such that $a_j \geq a_i + 1/n$. But (a_i) is upward monotonic, so $a_i \geq a_k$ (Lemma 1). Since $a_k > m - 1 + u/n$ we get

$$a_j \ \geq \ a_i + \frac{1}{n} \ \geq \ a_k + \frac{1}{n} \ > \ m - 1 + \frac{u}{n} + \frac{1}{n} \ = \ m - 1 + \frac{u+1}{n}.$$

Thus $m - 1 + (u + 1)/n$ is not an upper bound. So $u + 1 \in S$ as desired.

By the induction axiom, $S = \mathbb{N}$. In other words, for all $l \in \mathbb{N}$, the number $m - 1 + l/n$ is not an upper bound. Informally, as l gets larger, $m - 1 + l/n$ gets larger as well. This seems to suggest that there can be no upper bound at all. We now prove this rigorously.

Suppose that there is an upper bound B for (a_i). By Lemma 10 we can find an integer C such that $B < C$. So C must also be an upper bound. In particular, $m - 1 + l/n < C$ for all $l \in \mathbb{N}$, since $m - 1 + l/n$ is not an upper bound. Thus $nm - n + l < nC$ for all $l \in \mathbb{N}$. Hence $l < nC - nm + n$ for all $l \in \mathbb{N}$. But this clearly fails in the case where $l = nC - nm + n$. From this contradiction, we conclude that no such bound B exists. □

Lemma 12. *Let $(a_i)_{i \geq n_0}$ be a downward monotonic sequence in $\mathbb{R}$. If (a_i) is not Cauchy, then it is not bounded.*

Proof. Either modify the proof of the previous lemma in the obvious way, or apply the previous lemma to the upward monotonic sequence $(-a_i)$ to get a contradiction. □

Here is the main theorem of the section.

Theorem 13. *Every bounded monotonic sequence in $\mathbb{R}$ is Cauchy.*

Proof. Suppose there is a bounded monotonic sequence that is not Cauchy. If it is upward monotonic, Lemma 11 gives a contradiction. If it is downward monotonic, Lemma 12 gives a contradiction. □

Corollary 14. *Every bounded monotonic sequence in $\mathbb{R}$ converges.*

Proof. Since $\mathbb{R}$ is complete, every Cauchy sequence converges. □

We end this section with three lemmas, the first and third are consequences of the above theorem.

Lemma 15. *If $x \in \mathbb{R}$ is such that $x > 1$ then the sequence $(x^i)_{i \geq 0}$ of powers is an unbounded strictly upward monotonic sequence.*

Proof. Observe that every term of the sequence is positive (simple proof by induction). Also observe that for all $i \in \mathbb{N}$,

$$x^{i+1} = x^i x > x^i 1 = x^i$$

(properties of ordered fields). Thus (x_i) is strictly upward monotonic.

Observe that, for all $i \in \mathbb{N}$,

$$x^{i+1} - x^i = x^i(x-1) > 1(x-1) = x-1.$$

This implies that (x^i) is not Cauchy (take $\epsilon = x - 1$ to get a counter-example). Thus the sequence (x^i) is unbounded (Theorem 13). □

Lemma 16. *If (x_i) is an unbounded upward monotonic sequence of positive elements of an ordered field F, then $\left(x_i^{-1}\right)$ converges to 0.*

Proof. Let $\epsilon > 0$ be given. Since (x_i) is unbounded, ϵ^{-1} cannot be an upper bound of (x_i). So there is a $k \in \mathbb{N}$ such that $x_k > \epsilon^{-1}$. Hence $x_k^{-1} < \epsilon$. If $i \geq k$ then $x_i \geq x_k$ by Lemma 1. So $\left|x_i^{-1} - 0\right| = x_i^{-1} \leq x_k^{-1} < \epsilon$. We conclude that $\left(x_i^{-1}\right)$ converges to 0. □

Lemma 17. *Suppose (a_i) is a series in $\mathbb{R}$, and suppose that $b_i = a_i + 1/B^i$ where $B > 1$ is a fixed real number. Then $(a_i) \sim (b_i)$.*

Proof. Let $\epsilon > 0$ be a positive real number. By Lemma 15, $\left(B^i\right)$ is an unbounded upward monotonic sequence. By Lemma 16, $\left(1/B^i\right)$ converges to 0. So there is a $k \in \mathbb{N}$ such that, for all $i \geq k$,

$$\left|1/B^i - 0\right| < \epsilon.$$

So, for all $i \geq k$, $|b_i - a_i| < \epsilon$. Thus $(a_i) \sim (b_i)$. □

4. Decimal Sequences

Decimal sequences give an example of the usefulness of Corollary 14. It is common to think informally of a real number as something that can be written as an infinite decimal, such as $3.14159\ldots$ or $1.41421\ldots$. Even rational numbers can be written in this way: $3/2 = 1.5000\ldots$ or $2/3 = 0.666666\ldots$. Our goal in this section and the following two sections is to formally justify this view of real numbers. For convenience, in what follows we typically restrict our attention to non-negative real numbers.

Definition 3. Suppose $n \in \mathbb{N}$ and $(d_i)_{i \geq 1}$ is a sequence where $d_i \in \{0, \ldots, 9\}$ for all $i \geq 1$. Then the sequence (s_i) whose ith term is

$$s_i = n + \sum_{j=1}^{i} \frac{d_j}{10^j}$$

is called a *decimal sequence*. It can be thought of as a $\mathbb{Q}$ or an $\mathbb{R}$ sequence.

Let N be the base 10 numeral representing n, and let D_i be the standard digit symbol representing d_i. Then the notation

$$N.D_1 D_2 D_3 D_4 \ldots$$

is used to represent *the limit* of the above decimal sequence (s_i). This assumes, of course, that (s_i) has a limit. This will require proof. We will show that such a sequence is monotonic and bounded, so has a limit in $\mathbb{R}$.

Example. The notation $3.22222\ldots$ denotes the limit of (s_i) where

$$s_i = 3 + \sum_{j=1}^{i} \frac{2}{10^j}.$$

So $3.22222\ldots$ is the limit of the sequence with terms 3, 3.2, 3.22, 3.222, ... If one allows facts about geometric series, one can show that $\sum_{j=1}^{i} \frac{2}{10^j}$ has limit $2/9$, so (s_i) has limit $3 + 2/9$. Thus $3.22222\ldots$ is $29/9$.

Remark 3. The sequence (s_i) above is an example of a type of sequence called a *series.* Series are sequences defined in terms of summation. Each s_i is called a *partial sum* of the series, and the limit, if it exists, is called the *value* of the series.

Remark 4. There is nothing sacred about the number 10. We can easily replace 10 with another positive integer B in Definition 3, and insist that $d_i \in \{0, \ldots, B-1\}$. This would give us base B expansions of real numbers.

Now we establish that any decimal sequence is bounded and monotonic.

Theorem 18. *Suppose $n \in \mathbb{N}$ and $(d_i)_{i \geq 1}$ is a sequence where $d_i \in \{0, \ldots, 9\}$ for all $i \geq 1$. Then the sequence (s_i) whose ith term is given by*

$$s_i = n + \sum_{j=1}^{i} \frac{d_j}{10^j}$$

is upward monotonic with upper bound $n+1$. In particular, it is is bounded and monotonic.

Proof. (sketch) By induction one can show, for all $i \geq 1$, that $10^{-i} > 0$ and

$$\sum_{j=1}^{i} \frac{d_j}{10^j} \quad \leq \quad \sum_{j=1}^{i} \frac{9}{10^j}.$$

So, by Lemma 19 (below),

$$\sum_{j=1}^{i} \frac{d_j}{10^j} \quad < \quad 1.$$

Adding n gives $s_i \leq n+1$. So $n+1$ is an upper bound for the sequence.

Since $d_{i+1} \geq 0$,

$$s_{i+1} = s_i + \frac{d_{i+1}}{10^{i+1}} \geq s_i$$

for all i. Thus (s_i) is upward monotonic (Definition 1). □

The following is a special case of the formula for geometric series.

Lemma 19. *For all i,*

$$\sum_{j=1}^{i} \frac{9}{10^j} = 1 - \frac{1}{10^i}.$$

Proof. Let $s = \sum_{j=1}^{i} \frac{9}{10^j}$. By properties of summations and powers,

$$\frac{s}{10} = \frac{1}{10}\sum_{j=1}^{i} \frac{9}{10^j} = \sum_{j=1}^{i} \frac{9}{10^{j+1}} = \sum_{j=2}^{i+1} \frac{9}{10^j} = \sum_{j=2}^{i} \frac{9}{10^j} + \frac{1}{10}\frac{9}{10^i}.$$

This is similar to the expression for s. In fact, we can write s as follows:

$$s = \frac{9}{10} + \sum_{j=2}^{i} \frac{9}{10^j}.$$

The sum term cancels when we take the difference:

$$s - \frac{s}{10} = \frac{9}{10} - \frac{1}{10}\frac{9}{10^i} = \frac{9}{10}\left(1 - \frac{1}{10^i}\right).$$

Multiply both sides by 10, then divide by 9. The result follows. ☐

Corollary 20. *Suppose $n \in \mathbb{N}$ and $(d_i)_{i\geq 1}$ is a sequence where $d_i \in \{0, \ldots, 9\}$ for all $i \geq 1$. Then the sequence (s_i) whose ith term is given by*

$$s_i = n + \sum_{j=1}^{i} \frac{d_j}{10^j}$$

converges to a real number $\mathbb{R}$. More specifically, it converges to a real number x with $n \leq x \leq n + 1$.

Proof. By Theorem 18, (s_i) is bounded and monotonic. By Corollary 14 it converges to a real number x. By induction, we can show $n \leq s_i$ for each i, and by Theorem 18, $s_i \leq n + 1$ for each i. By Corollary 3, we have $n \leq x \leq n + 1$. ☐

Remark 5. This shows that every decimal expansion, for example $3.14159\ldots$, defines a real number between 3 and 4.

5. Decimal Expansions

In this section we consider the converse to the problem in the previous section. We establish that every non-negative real number is the limit of a decimal sequence. The decimal sequence giving x as its limit is called the *decimal expansion* of x. For negative real numbers x, we typically specify the x in terms of the decimal expansion of $|x|$.

Theorem 21. *Every non-negative real number is the limit of a decimal expansion.*

Proof. Let x be a non-negative real number. We divide the proof into three steps. First we define a sequence (a_i) recursively. Next we show that (a_i) is a decimal sequence. Finally we show that (a_i) converges to x.

By Lemma 10, there is an integer n such that $n \leq x < n + 1$. We define a_0 to be n.

Now suppose that a_i has been defined. Consider $y = 10^{i+1}(x - a_i)$. By Lemma 10, there is an integer d such that $d \leq y < d + 1$. We define $a_{i+1} = a_i + d/10^{i+1}$. From $d \leq y < d+1$, we get $a_{i+1} \leq x < a_{i+1} + 1/10^{i+1}$.

Now we need to show that this sequence is a decimal sequence. In particular, let d_{i+1} be the integer used in the definition of a_{i+1}:

$$a_{i+1} = a_i + \frac{d_{i+1}}{10^{i+1}}.$$

We need to show that $0 \leq d_i \leq 9$ for all positive i.

Above we showed that $a_{i+1} \leq x < a_{i+1} + 1/10^{i+1}$ for all $i \geq 0$. This just means $a_i \leq x < a_i + 1/10^i$ for all $i \geq 1$. The inequality holds also for $i = 0$ based on the choice of a_0. This tells us that $0 \leq x - a_i < 1/10^i$. So $0 \leq y < 10$. This implies $0 \leq d_{i+1} \leq 9$ for all $i \geq 0$. This just means that, $0 \leq d_i \leq 9$ for all $i \geq 1$.

Next we show that

$$a_i = a_0 + \sum_{j=1}^{i} \frac{d_j}{10^j}.$$

This can be shown easily by induction. We have established that (a_i) is a decimal sequence.

Finally, we show that (a_i) converges to x. By Corollary 20, (a_i) has a real limit, call it s. Above we established

$$a_i \leq x < a_i + 1/10^i$$

for all i. By Corollary 3, $s \leq x$ since $a_i \leq x$. By Lemma 17, the sequence $(a_i + 1/10^i)$ is equivalent to (a_i). Thus $(a_i + 1/10^i)$ also has limit s (Chapter 7). Since $x < a_i + 1/10^i$, we have $x \leq s$ (Corollary 3). Thus $s \leq x$ and $x \leq s$. This implies that $s = x$. Thus (a_i) converges to x. □

6. Uniqueness

In this section we will consider the question of uniqueness of a decimal expansion. The results are presented without proof, and their proofs are left as a challenge for the curious reader.

Some numbers have two distinct decimal expansion, but in other cases the expansions are unique. For example, $0.13999999\ldots$ and $0.1400000\ldots$ are two representations for the same real number $7/50$. However, $0.22222222\ldots$ is the unique expansion of $2/9$.

Definition 4. A decimal sequence whose ith term is

$$s_i = n + \sum_{j=1}^{i} \frac{d_j}{10^j}$$

is called a "nine-sequence" if there is a k such that $d_i = 9$ for all $i \geq k$.

Theorem 22. *Suppose the sequence with ith term*

$$s_i = d_0 + \sum_{j=1}^{i} \frac{d_j}{10^j}$$

is a "nine-sequence", and let k be the least positive integer such that $d_i = 9$ for all $i \geq k$. Then $(s_i) \sim (s'_i)$ where

$$s'_i = d'_0 + \sum_{j=1}^{i} \frac{d'_j}{10^j}$$

is defined as follows: $d'_i = 0$ if $i \geq k$, and $d'_i = d_i$ if $i < k-1$. Finally $d'_{k-1} = 1 + d_{k-1}$. (We are not assuming that $d_0 \leq 9$).

Remark 6. The above shows that a "nine-sequence" can be replaced by a "non-nine-sequence" representing the same real number.

Theorem 23. *Every non-negative real number has a unique "non-nine-sequence" decimal expansion.*

7. Existence of Roots

In this section, we will establish that every nonnegative $x \in \mathbb{R}$ has a unique nonnegative nth root in $\mathbb{R}$.

Definition 5. Let F be a field, and let n be a positive integer. If $y^n = x$ where $x, y \in F$, then we say that y is an nth root of x.

In the special case where $n = 2$, then y is called a *square root*. In the special case where $n = 3$, then y is called a *cube root*.

Lemma 24. *Let $x \in \mathbb{R}$ be nonnegative and let n be a postive integer. There is a bounded upward monotonic sequence $(y_i)_{i \in \mathbb{N}}$ of nonnegative real numbers such that, for all $i \in \mathbb{N}$,*

$$y_i^n \leq x < \left(y_i + \frac{1}{2^i} \right)^n.$$

Proof. We define the sequence recursively. We need to first choose y_0. Consider W the set of all natural numbers k such that $k^n > x$. This set is not empty. To see this, select an integer m such that $m > x + 1$ (Lemma 10). Observe that $m > 1$. By Lemma 15, $m^n > m > x$. Hence $m \in W$.

Since it is not empty, W has a least element (well-foundedness, Chapter 2). Let m_0 be the least element of W. Observe that $m_0 \neq 0$ since 0 is clearly not in W ($0^n = 0$), so $m_0 - 1 \geq 0$. Also $m_0^n > x$. Let $y_0 = m_0 - 1$. By the choice of y_0,

$$y_0^n \leq x < \left(y_0 + \frac{1}{2^0} \right)^n.$$

Suppose that y_i has been defined. We must define y_{i+1}. Consider the real number $a = y_i + 1/2^{i+1}$; it is half-way between y_i and $y_i + 1/2^i$. If $a^n \leq x$

then define $y_{i+1} = a$. Otherwise, define $y_{i+1} = y_i$. Thus, by recursion, (y_i) has been defined for all $i \in \mathbb{N}$.

Observe that if

$$y_i^n \leq x < \left(y_i + \frac{1}{2^i}\right)^n$$

then

$$y_{i+1}^n \leq x < \left(y_{i+1} + \frac{1}{2^{i+1}}\right)^n.$$

One can now prove, by induction, that for all $i \in \mathbb{N}$

$$y_i^n \leq x < \left(y_i + \frac{1}{2^i}\right)^n.$$

Also, $y_{i+1} \geq y_i$, by definition of y_{i+1}, so (y_i) is upward monotonic.

Finally, we need to show that (y_i) is bounded. Let m_0 be as above. Since $y_i^n \leq x$ for all i, and $x < m_0^n$ we have that $y_i^n \leq m_0^n$. By Lemma 8, $y_i \leq m_0$. Thus m_0 is an upper bound for (y_i). □

Theorem 25. *Let $x \in \mathbb{R}$ be nonnegative and let $n \in \mathbb{N}$ be postive. Then x has a unique nonnegative nth root y. If x is positive then so is y*

Proof. By Lemma 24 there is a bounded monotonic sequence $(y_i)_{i \in \mathbb{N}}$ of nonnegative real numbers such that, for all $i \in \mathbb{N}$,

$$y_i^n \leq x < \left(y_i + \frac{1}{2^i}\right)^n.$$

By Corollary 14, (y_i) converges to a real number y. By Corollary 3, $y \geq 0$ since each $y_i \geq 0$. We now show that $y^n = x$.

By Corollary 5, (y_i^n) converges to y^n. Since $y_i^n \leq x$, we have $y^n \leq x$ (Corollary 3). We also need to show $y^n \geq x$. To do this we focus on the larger sequence $(y_i + 1/2^i)$. By Lemma 17,

$$(y_i) \sim \left(y_i + \frac{1}{2^i}\right).$$

Thus $(y_i + 1/2^i)$ also converges to y (Chapter 7). By Corollary 5, the sequence $\left((y_i + 1/2^i)^n\right)$ converges to y^n. By Corollary 3, $x \leq y^n$.

Thus $y^n = x$ as claimed. We now show uniqueness. Suppose y' is another nonnegative square root of x. Then

$$(y')^n = x = y^n.$$

By Lemma 9, $y' = y$. Thus the nonnegative square root is unique.

Finally, observe that if $y = 0$ then $x = y^n = 0$ (simple proof by induction). Thus if x is positive then y must also be positive. □

Definition 6. If x is a nonnegative real number, then $\sqrt{x}$ is defined to be the unique nonnegative square root of x.

Remark 7. Observe that we have now established that $\sqrt{2}$ is a real number. As we observed in Chapter 7, it is not in $\mathbb{Q}$. Thus $\mathbb{Q}$ is a proper subset of $\mathbb{R}$. This implies that $\sqrt{2}$ is irrational:

Definition 7. An irrational real number is an element of $\mathbb{R}$ that is not in $\mathbb{Q}$.

Definition 8. If x is a nonnegative real number and if $n \in \mathbb{N}$ is a positive integer, then $x^{1/n}$ is defined to be the unique nonnegative nth root of x.

8. Fractional Powers

Here we give a few properties of fractional powers.

Theorem 26. *Let $x, y \in \mathbb{R}$ be nonnegative, and let $n \in \mathbb{N}$ positive. Then*

$$(xy)^{1/n} = x^{1/n}y^{1/n}.$$

Proof. Let $v = x^{1/n}$ and $w = y^{1/n}$. By Definition 8, $v^n = x$ and $w^n = y$, and v and w are nonnegative. By closure properties, vw is nonnegative, and by properties of commutative rings,

$$(vw)^n = v^n w^n = xy.$$

Thus vw is the nonnegative nth root of xy. So $(xy)^{1/n} = vw = x^{1/n}y^{1/n}$. □

Definition 9. Suppose x is a nonnegative real number, and p/q is a positive rational number with p, q positive integers. Then

$$x^{p/q} \overset{\text{def}}{=} (x^p)^{1/q}.$$

Lemma 27. *The above definition is well-defined: it does not depend on the choice of numerator and denominator used to represent the given rational number.*

Proof. Suppose that $p/q = r/s$ where p, q, r, s are positive integers. We must show that

$$(x^p)^{1/q} = (x^r)^{1/s}.$$

Let $v = (x^p)^{1/q}$ and $w = (x^r)^{1/s}$. Observe that

$$v^{qs} = x^{ps} \qquad \text{and} \qquad w^{qs} = x^{rq}.$$

Since $p/q = r/s$, we have $ps = qr$. Thus $v^{qs} = w^{qs}$. By Lemma 9, $v = w$. □

Theorem 28. *Suppose x is a non-negative real number, and p/q is a positive rational number (with p, q positive integers). Then*

$$x^{p/q} = (x^p)^{1/q} = \left(x^{1/q}\right)^p$$

Proof. The first equality is true by definition. To establish $(x^p)^{1/q} = \left(x^{1/q}\right)^p$, raise both sides to the same power q. Both sides simplify to give the same answer, namely x^p. Now use Lemma 9. □

Remark 8. For example, you can computer $8^{2/3}$ in two ways. The first method start with $8^2 = 64$. Then you take the cube root, which is 4. In the second method you take the cube root of 8. This is 2. Next square it. This gives 4. Of course, both methods give the same answer.

9. Uniqueness of Roots

Throughout this section, let n be a positive integer. We will study nth roots where we allow negative real numbers.

First consider the case where n is even.

Theorem 29. *Suppose n is a positive even integer, and $x \in \mathbb{R}$.*
If $x > 0$, then x has exactly two nth roots: $x^{1/n}$ and $-x^{1/n}$.
If $x = 0$ then x has exactly one nth root. That root is 0.
If $x < 0$ then x has no nth roots.

Proof. (Sketch) Write n as $2m$.

First suppose that x is positive. Then $x^{1/n}$ is a positive nth root. Consider the negative real number $-x^{1/n}$. Then

$$\left(-x^{1/n}\right)^n = (-1)^n \left(x^{1/n}\right)^n = \left((-1)^2\right)^m x = 1^m \cdot x = x.$$

So $-x^{1/n}$ is a second nth root. Suppose y is a third nth root. Observe that y cannot be positive by the uniqueness claim of Theorem **??**. $y \neq 0$ since $x \neq 0$. So y is negative. This implies that $-y$ is positive. Observe that $y^n = (-y)^n$. Thus $-y$ is a positive nth root. This implies that $y = -x^{1/n}$. So there is no distinct third nth root.

We leave the case of $x = 0$ to the reader.

Finally, suppose $x < 0$. If $y \in \mathbb{R}$, then y^2 is nonnegative. So $y^n = \left(y^2\right)^m$ is non-negative. So $y^n \neq x$. Thus x has no nth roots. □

Theorem 30. *Suppose n is an odd integer, and $x \in \mathbb{R}$. Then x has a unique nth root.*
If $x < 0$ then the unique nth root of x is $-|x|^{1/n}$.

Proof. (sketch) First show, as a lemma, that if $y < 0$ then y^n is negative. This shows that if $x \geq 0$, then x cannot have any negative nth roots. But we know x has a unique non-negative square root. Thus the theorem holds for $x \geq 0$.

If $x < 0$, then the nth power of $-|x|^{1/n}$ is equal to $-|x|$. But $-|x| = x$ in this case. So $-|x|^{1/n}$ is an nth root. Suppose y is another nth root. Then $(-y)^n = -x = |x|$. By the uniqueness claim for non-negative reals demonstrated above, $-y = |x|^{1/n}$. Thus $y = -|x|^{1/n}$. So the nth root is unique. □

10. Countability

In this chapter we have considered several differences between $\mathbb{Q}$ and $\mathbb{R}$. We consider one more difference: $\mathbb{Q}$ is countable, but $\mathbb{R}$ is not.

Definition 10. A set S is *countable* if there is a surjection $\mathbb{N} \to S$ or if S is empty. If S is nonempty and no such surjection exists, then S is said to be *uncountable*.

Example. Every finite set is countable. The set $\mathbb{N}$ is countably infinite (consider the identity map $\mathbb{N} \to \mathbb{N}$). The set $\mathbb{Z}$ is countably infinite as well: consider the bijection $\mathbb{N} \to \mathbb{Z}$ that sends every even number $2k \in \mathbb{N}$ to k and sends every odd $2k-1 \in \mathbb{N}$ to $-k$.

Theorem 31. *The set $\mathbb{Q}$ is countable.*

Proof. (Informal) Draw set $T = \{(x, y) \mid x, y \in \mathbb{Z} \text{ where } y > 0\}$ in the plane. Draw a path that starts from $(0, 1)$ and winds its way through all the points of T. Define a surjection $f : \mathbb{N} \to \mathbb{Q}$ as follows: $n \mapsto a/b$ where (a, b) is the nth point hit by the path. Here we start with $n = 0$ and the path begins at the point $(0, 1)$. Thus $f(0) = 0/1$, and $f(1)$ is a/b where (a, b) is the next point visited by the path. Since every point of T is visited by the path, and every element of $\mathbb{Q}$ is of the form a/b with $(a, b) \in T$, the function f is surjective. □

Theorem 32. *The set $\mathbb{R}$ is uncountable.*

Proof. (informal) Suppose otherwise, that a surjective $f : \mathbb{N} \to \mathbb{R}$ exists. We can think of f as providing a list of real numbers. The goal is to construct a decimal expansion of a new number not on the list.

Let n be a positive integer greater than $f(0)$. For each $i \geq 1$ let d_i be the 10^{-i} digit of the decimal expansion of $f(i)$. (Choose the decimal expansion to be a "non-nine sequence"). Define d'_i to be 5 if $d_i \neq 5$, but choose $d'_i = 7$ if $d_i = 5$. Consider the decimal sequence defined by

$$s_i = n + \sum_{j=1}^{i} \frac{d'_j}{10^j}.$$

This defines a real number x (by taking the limit of this monotonic and bounded sequence).

Observe that $x \neq f(0)$ since $x \geq n > f(0)$. Observe that $x \neq f(i)$ if $i \geq 1$ since the decimal expansion of x and $f(i)$ differ in the 10^{-i} position: the first has coefficient d'_i and the second d_i. (We are considering only "non-nine sequences" so decimal expansions are unique). Thus x is not in the image of f, a contradiction. □

Appendix: The Least Upper Bound Property

There is another property of $\mathbb{R}$ that is often used. We will not need to use it in this course, but due to its general importance we will discuss it in this appendix. It is called the *least upper bound property of* $\mathbb{R}$.

Theorem 33. *Suppose that S is a nonempty subset of $\mathbb{R}$. Suppose also that S has an upper bound (a real number B such that $s \leq B$ for all $s \in S$). Then S has a least upper bound.*

Remark 9. Observe that this property is false in $\mathbb{Q}$. For example, let S be the set of positive rational numbers r such that $r^2 < 2$. There is no least upper bound of S in $\mathbb{Q}$.

Remark 10. If the least upper bound of a nonempty subset of $\mathbb{R}$ exists, it must be unique.

Proof. Let W be the set of all integers that are upper bounds of S. This set is not empty. To see this, select an upper bound B of S. There is an integer m such that $m \geq B$ (Lemma 10). So $m \in W$.

The set W has a lower bound. To see this, choose any $s \in S$. There is an integer l such that $l \leq s$ (Lemma 10). Since $s \leq B$ for every upper bound B of S, we have $l \leq B$. Thus l is a lower bound for W.

By a property of $\mathbb{Z}$ (Chapter 4), there is a smallest integer m_0 of W since W is nonempty and has a lower bound.

We now define a sequence (y_i) recursively. Let $y_0 = m_0 - 1$. By choice of m_0, the integer y_0 is not an upper bound of S, but $y_0 + 1$ is an upper bound of S.

Suppose that y_i has been defined. We must define y_{i+1}. Consider the real number $a = y_i + 1/2^{i+1}$; it is half-way between y_i and $y_i + 1/2^i$. If a is not an upper bound of S, then define $y_{i+1} = a$. Otherwise, define $y_{i+1} = y_i$. By recursion, (y_i) is defined for all $i \in \mathbb{N}$.

Suppose i has the property that y_i is not an upper bound of S, but $y_i+1/2^i$ is an upper bound of S. The choice of y_{i+1} is such that it is not an upper bound of S but $y_{i+1}+1/2^{i+1}$ is an upper bound. By induction, we can show y_i is not an upper bound of S, but $y_i + 1/2^i$ is an upper bound for all $i \in \mathbb{N}$. Also, $y_{i+1} \geq y_i$, by definition of y_{i+1}, so (y_i) is upward monotonic.

Any upper bound of S is an upper bound of (y_i). Thus (y_i) is bounded and monotonic. By Corollary 14, (y_i) converges to a real number y. Our goal is to show y is the least upper bound of S.

First we show that y is an upper bound. Suppose not. Then there is an $s \in S$ such that $y < s$. By Lemma 15, $\left(2^i\right)$ is an unbounded upward monotonic sequence, and so by Lemma 16 $\left(1/2^i\right)$ converges to 0. So there is a $k \in \mathbb{N}$ such that, for all $i \geq k$,

$$\left|1/2^i - 0\right| < (s - y)$$

(choose $\epsilon = s - y$). In particular, $1/2^k < s - y$. Now $s \leq y_k + 1/2^k$ since $y_k + 1/2^k$ is an upper bound of S. So $s \leq y_k + 1/2^k < y_k + s - y$. Thus $y < y_k$ which contradicts the fact that (y_i) is upward monotonic. So y is an upper bound for S.

Now suppose z is a smaller upper bound for S. Since (y_k) converges to y, there is a $k \in \mathbb{N}$ such that, if $i \geq k$ then $y - y_i = |y - y_i| < (y - z)$ (choosing $\epsilon = y - z$). In particular, $y - y_k < y - z$. Thus $z < y_k$. However, y_k is not an upper bound of S, so z cannot be, a contradiction. Thus y is the least upper bound. $\square$