CHAPTER 7: REAL NUMBERS

LECTURE NOTES FOR MATH 378 (CSUSM, SPRING 2009). WAYNE AITKEN

1. Introduction

In this chapter we construct the set of real numbers $\mathbb{R}$. There are several ways to introduce the real numbers. Three popular approaches are to introduce $\mathbb{R}$ with (i) new axioms, with (ii) Dedekind cuts of $\mathbb{Q}$, or with (iii) Cauchy sequences in $\mathbb{Q}$. We will use the third approach and construct real numbers as equivalence classes of Cauchy sequences of rational numbers. This approach is chosen since it avoids the need for additional axioms, and gives students practice with the concept of Cauchy sequence.

The main theorem of this chapter is that $\mathbb{R}$ is a complete ordered field. We begin with a discussion of the limitations of $\mathbb{Q}$. Then we discuss infinite sequences. At first the sequences we require are rational sequence. However, much of what we do easily extends to other ordered fields. So instead of proving theorems for $\mathbb{Q}$, we will often prove the result for a general ordered field. With this in mind, throughout this chapter we let F be an ordered field that is either equal to $\mathbb{Q}$, or contains $\mathbb{Q}$.[1]

2. Limitations of $\mathbb{Q}$

In geometry we learn the Pythagorean Theorem which allows us to compute one side of a right triangle assuming we know the lengths of the other two sides. As an application, one easily shows that the diagonal of the unit square has length $\sqrt{2}$. In other words, the length d of the diagonal has the property that $d^2 = 2$. There is a problem: there is no such d in the field $\mathbb{Q}$. This elementary observation shows that $\mathbb{Q}$ does not have all the numbers required to do even basic geometry. This compels us to construct a richer number system $\mathbb{R}$ called the *real numbers*. Any real number that is not in $\mathbb{Q}$ is called an *irrational real number.*

Before constructing $\mathbb{R}$, we prove that there is no $r \in \mathbb{Q}$ such that $r^2 = 2$. There are several proofs of this fact. The proof given here is designed to build on our familiarity with modular arithmetic.

Theorem 1. *There is no $r \in \mathbb{Q}$ with the property that $r^2 = 2$.*

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[1] One can show that $\mathbb{Q}$ can be embedded into any ordered field in a unique way. So, really, we just need to assume that F is an ordered field.

Proof. Suppose such an r exist. By a result of Chapter 6 we can write $r = a/b$ where a and b are relatively prime integers. This implies that a and b cannot both be even. From the assumption $r^2 = 2$ we get the equation $a^2 = 2b^2$. This, in turn, implies that

$$a^2 \equiv 2b^2 \mod 4$$

CASE 1: a and b are both odd. In this case, $a^2 \equiv b^2 \equiv 1 \bmod 4$ by the following lemma. Substituting into $a^2 \equiv 2b^2$ gives $1 \equiv 2$ modulo 4. This is a contradiction.

CASE 2: a is odd, b is even. In this case, $a^2 \equiv 1 \bmod 4$ as before, but $b^2 \equiv 0 \bmod 4$ (see the following lemma). Substituting into $a^2 \equiv 2b^2$ gives $1 \equiv 0$ modulo 4. This is a contradiction.

CASE 3: a is even, b is odd. This case is similar to the previous case: $a^2 \equiv 0 \bmod 4$ but $b^2 \equiv 1 \bmod 4$. Substituting into $a^2 \equiv 2b^2$ gives us $0 \equiv 2$ modulo 4. This is a contradiction.

So in any case, we get a contradiction. So no such $r \in \mathbb{Q}$ exists. $\square$

Lemma 2. *If $c \in \mathbb{Z}$ is odd then $c^2 \equiv 1$ modulo 4. If $c \in \mathbb{Z}$ is even then $c^2 \equiv 0$ modulo 4.*

Proof. Suppose $c \in \mathbb{Z}$ is odd. By the Quotient-Remainder Theorem and the definition of odd integer, $c = 2q + 1$. Thus

$$c^2 = (2q + 1)^2 = 4q^2 + 4q + 1,$$

but

$$4q^2 + 4q + 1 \equiv 0 + 0 + 1 \equiv 1 \bmod 4.$$

Now suppose $c \in \mathbb{Z}$ is even. Then $c = 2d$ for some $d \in \mathbb{Z}$. So $c^2 = 4d^2$. Since $4 \mid c^2$ the result follows. $\square$

Exercise 1. Adapt the proof of Theorem 1 to show that if $n \equiv 2 \bmod 4$ then there is no $r \in \mathbb{Q}$ such that $r^2 = n$. This shows, for example, that $\sqrt{10}$ is irrational.

Remark 1. One can generalize the above theorem to show that if $n \in \mathbb{Z}$ is not equal to some m^2 (with $m \in \mathbb{N}$) then there is no $r \in \mathbb{Q}$ with $r^2 = n$. This result can, in turn, be generalized to other powers beyond 2.

3. Infinite Sequences and Limits

In this section we consider infinite sequences in an ordered field F. Recall the definition of infinite sequences from Section 9 of Chapter 4. An infinite sequence with values in F is a function whose domain is a set of the form $\{i \in \mathbb{Z} \mid i \geq n_0\}$, and whose codomain is F. We use notation such as $(a_i)_{i \geq n_0}$ to denote such a sequence. We simply write (a_i) when the domain is not important to the discussion.

An important concept associated to sequences is that of a *limit*. What do we mean by the limit of a sequence? Informally, a sequence (a_i) has limit b if the terms of the sequence eventually get and stay arbitrarily close to b.

This informal description is a bit ambiguous and is unsuitable to use in a proof, so we give a more precise definition.

Definition 1. Suppose F is an ordered field, and (a_i) is a sequence in F. We say that $b \in F$ is the *limit* of (a_i) if the following holds: for all positive $\varepsilon \in F$ there is a $k \in \mathbb{N}$ such that if $i \geq k$ then $|a_i - b| < \varepsilon$. We can write this with three quantifiers as follows:

$$(\forall \varepsilon \in F_{>0})(\exists k \in \mathbb{N})(\forall i \in \mathbb{N})(i \geq k \;\Rightarrow\; |a_i - b| < \varepsilon).$$

Here $F_{>0}$ denotes the positive elements of F.

Not all sequences have limits. A sequence that has a limit is said to *converge*. A sequence that does not have a limit is said to *diverge*.

Exercise 2. Use the rules of basic logic to negate the definition of limit. Complete the following sentence: *the sequence* (a_i) *does not have limit* b *means that there is a positive* $\varepsilon \in F$ *such that for all* $k \in \mathbb{N}$ *...*

Exercise 3. Consider the sequence $(i)_{i\in\mathbb{N}}$ in $F = \mathbb{Q}$. In other words, consider the sequence given by the identity function. Show that this sequence does not have a limit. Hint: work with $\varepsilon = 1$, and either use the previous exercise or give a proof by contradiction.

Notice that in the above definition we used the term *the* limit. It sounds like we are treating limits as if they are unique. This is justified by the following theorem.

Theorem 3. *A convergent sequence in an ordered field has a unique limit.*

Proof. Suppose otherwise that (a_i) is a sequence in F with two distinct limits b and c. Let $\varepsilon = |b - c|/2$. Since $b \neq c$, we have that $b - c \neq 0$. Hence $|b - c| > 0$. Since $2 > 0$ we have $2^{-1} > 0$. Thus the product $\varepsilon = |b - c|2^{-1}$ is positive.

By definition of limit, there is a $k_1 \in \mathbb{N}$ such that $|a_i - b| < \varepsilon$ for all $i \geq k_1$. Likewise there is a $k_2 \in \mathbb{N}$ such that $|a_i - c| < \varepsilon$ for all $i \geq k_2$. Let k be the maximum of k_1 and k_2. Then, for $i \geq k$ we have

$$|b - c| = |(b - a_i) + (a_i - c)| \leq |b - a_i| + |a_i - c| < \varepsilon + \varepsilon.$$

Here we used the triangle inequality. Since $2\varepsilon = |b - c|$ we have that

$$|b - c| < |b - c|,$$

a contradiction. $\square$

Informal Exercise 4. Draw a picture of a number line (representing F). Draw b and c in the above proof, and indicate the sets defined by $|x - b| < \varepsilon$ and $|x - c| < \varepsilon$ where ε is as in the above proof. Observe that the sets do not intersect so there can be no a_i simultaneously in both. This explains why we chose $\varepsilon = |b - c|/2$. Note, we could have chosen $\varepsilon = |b - c|/4$, for instance, and gotten a contradiction. However, $\varepsilon = 2|b - c|$ would not work. Why not?

Remark 2. In the above theorem we get to choose ε to be whatever we want since we are *assuming* a limit. If instead you are trying to *prove* a limit, you cannot choose ε, but must allow ε to be an arbitrary positive element of F.

Exercise 5. Let $F = \mathbb{Q}$. Show that $(1/j)_{j\geq 1}$ has limit 0. Use the following fact from Chapter 6: for every $r \in \mathbb{Q}$, there is a $k \in \mathbb{N}$ with $k > r$. The case $r = 1/\varepsilon$ will come up in your solution.

If a sequence converges, then the terms of the sequence get and stay arbitrarily close to each other. This is shown in the following theorem. Such sequences are called *Cauchy sequences*. So an easy way to see that a sequence does not converge is to show that the sequence is not Cauchy: the terms do not get and stay arbitrarily close to each other.

Theorem 4. *Suppose (a_i) is a convergent sequence in an ordered field F. Then for all positive ε in F there is a $k \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$*

$$i, j \geq k \quad\Rightarrow\quad |a_i - a_j| < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be an arbitrary positive element of F. We must find a $k \in \mathbb{N}$ that satisfies the statement of the theorem. Let $\varepsilon' = \varepsilon/2$, and let b be the limit of (a_i) that we assume exists.

By the definition of limit (Definition 1) there is a $k \in \mathbb{N}$ with $|a_i - b| < \varepsilon'$ for all $i \geq k$. So, for $i, j \geq k$ we have

$$|a_i - a_j| = |(a_i - b) + (b - a_j)| \leq |a_i - b| + |b - a_j| < \varepsilon' + \varepsilon'.$$

Here we have used the triangle inequality. Since $2\varepsilon' = \varepsilon$ we have that $|a_i - a_j| < \varepsilon$. Thus k has the desired property. □

The above theorem says that all convergent sequences satisfy the following definition:

Definition 2. Suppose (a_i) is an infinite sequence in an ordered field F. We say that (a_i) is *Cauchy* if the following occurs: for all positive ε in F there is a $k \in \mathbb{N}$ such that for all $i, j \in \mathbb{N}$

$$i, j \geq k \quad\Rightarrow\quad |a_i - a_j| < \varepsilon.$$

Remark 3. We can interpret Theorem 4 through its contrapositive: *if a sequence is not Cauchy, it cannot converge.*

Is this the only way a sequence can fail to converge? In other words, do all Cauchy sequences converge? The answer is no for $F = \mathbb{Q}$, but will turn out to be yes in the real numbers. The problem with $\mathbb{Q}$ is that it has 'holes'. For example, we saw above that there is no $r \in \mathbb{Q}$ with $r^2 = 2$. Define a sequence by the rule $a_i = n_i/10^i$ where n_i is the largest integer such that $a_i^2 < 2$. This sequence will not be convergent in $\mathbb{Q}$, but can be shown to be Cauchy. It will turn out that this sequence is convergent in $\mathbb{R}$, and has limit $\sqrt{2}$.[2]

[2]In fact, the equivalence class of this Cauchy sequence can be used to define $\sqrt{2}$.

Informal Exercise 6. Find the first five terms of (a_i) defined in the above remark. Assume the domain is the set of $i \geq 0$. Hint: punch $\sqrt{2}$ into your calculator.

Our approach will be to assume that all Cauchy sequences in $\mathbb{Q}$ should determine a real number. Non-Cauchy sequences cannot possibly converge, so should not determine real numbers. There is a problem: different sequences can determine the same real number. For example, the sequence defined by the rule $b_i = n_i/2^i$ where n_i is the largest integer such that $b_i^2 < 2$ determines the same real number as the sequence (a_i) discussed above (in fact, they both determine $\sqrt{2}$: the sequence (a_i) is related to the decimal expansion of $\sqrt{2}$ and (b_i) is related to the base 2 expansion of $\sqrt{2}$). How do we tell if two sequences determine the same number? There is an equivalence relation that solves the problem:

Definition 3. Suppose that (a_i) and (b_i) are two sequences in F. We write $(a_i) \sim (b_i)$ if the following occurs: for all positive $\varepsilon \in F$ there is a $k \in \mathbb{N}$ such that, for all $i \in \mathbb{N}$,

$$i \geq k \quad \Rightarrow \quad |a_i - b_i| < \varepsilon.$$

Remark 4. Informally the above definition says that the terms of the two sequences get and stay arbitrarily close to each other.

Theorem 5. *The relation $\sim$ is reflexive on the set of all sequences in F.*

Proof. We need to show $(a_i) \sim (a_i)$ for any given sequence $(a_i)_{i \geq n_0}$ in F. Let ε be an arbitrary positive element of F. We must find a k such that

$$i \geq k \quad \Rightarrow \quad |a_i - a_i| < \varepsilon.$$

Let us propose $k = n_0$. If $i \geq k$ then $|a_i - a_i| < \varepsilon$ since $|a_i - a_i| = 0$ and ε is positive. So k has the desired property. □

Theorem 6. *The relation $\sim$ is transitive on the set of all sequences in F.*

Proof. Suppose $(a_i) \sim (b_i)$ and $(b_i) \sim (c_i)$. We need to show $(a_i) \sim (c_i)$. Let ε be an arbitrary positive element of F. We must find a $k \in \mathbb{N}$ such that

$$i \geq k \quad \Rightarrow \quad |a_i - c_i| < \epsilon.$$

To find k we need to use the fact that $(a_i) \sim (b_i)$ and $(b_i) \sim (c_i)$. Let $\varepsilon' = \varepsilon/2$. Since $(a_i) \sim (b_i)$ there is a $k_1 \in \mathbb{N}$ such that

$$i \geq k_1 \quad \Rightarrow \quad |a_i - b_i| < \varepsilon'.$$

Likewise, there is a $k_2 \in \mathbb{N}$ such that

$$i \geq k_2 \quad \Rightarrow \quad |b_i - c_i| < \varepsilon'.$$

Let us propose for k the larger of k_1 or k_2. If $i \geq k$ then

$$|a_i - c_i| = |(a_i - b_i) + (b_i - c_i)| \leq |a_i - b_i| + |b_i - c_i| < \varepsilon' + \varepsilon'.$$

Here we use the triangle inequality and the fact that $i \geq k_1$ and $i \geq k_2$. Since $\varepsilon = 2\varepsilon'$,

$$|a_i - c_i| < \varepsilon.$$

Thus k has the desired property. □

Theorem 7. *The relation $\sim$ is symmetric on the set of all sequences in F.*

Exercise 7. Prove the above.

Corollary 8. *The relation $\sim$ is an equivalence relation on the set of sequences of (a_i) in F where F is any given ordered field.*

Remark 5. When we restrict $\sim$ just to Cauchy sequences, it is still an equivalence relation. In fact, any equivalence relation on a set restricts to an equivalence relation on any subset.

Exercise 8. Show that if $(a_i) \sim (b_i)$ and if (a_i) has a limit, then (b_i) converges and has the same limit as (a_i).

Remark 6. A similar argument gives the following: if $(a_i) \sim (b_i)$ and if (a_i) is Cauchy, then (b_i) is Cauchy. However, in this case we choose $\varepsilon' = \varepsilon/3$ instead of $\varepsilon' = \varepsilon/2$. The key step of the proof is

$$|b_i - b_j| = |(b_i - a_i) + (a_i - a_j) + (a_j - b_j)| < \varepsilon' + \varepsilon' + \varepsilon'.$$

Remark 7. Suppose (a_i) and (b_i) have the property that $a_i = b_i$ for sufficiently large i. In other words, suppose that there is a k such that $a_i = b_i$ for all $i \geq k$. Then $(a_i) \sim (b_i)$. This is easily proved from the definition. As a consequence, if one is Cauchy then both are, and if one converges then both do with the same limit.

In particular, if we take a sequence (a_i) and change a finite number of terms, then the resulting sequence is equivalent to (a_i). Likewise, if we change the domain of $(a_i)_{i \geq n_0}$ by replacing n_0 with a larger integer, then the resulting sequence is equivalent.

Remark 8. Several of the definitions and proofs given in this section are similar in nature. Be very careful to learn the subtle differences between them.

4. Boundedness Lemmas

In this section we will consider lemmas concerning bounds on Cauchy sequences. These lemmas will be needed in later sections.

Lemma 9. *Every Cauchy sequence $(a_i)_{i \geq n_0}$ in an ordered field F is bounded from above and below. In other words, there is a $B \in F$ such that $a_i \leq B$ for all $i \geq n_0$, and there is an element $b \in F$ such that $a_i \geq b$ for all $i \geq n_0$.*

Proof. Since (a_i) is Cauchy, there is a $k \in \mathbb{N}$ such that $|a_i - a_j| < 1$ for all $i, j \geq k$ (choose $\varepsilon = 1$). Let A be the maximum of $a_{n_0}, \ldots, a_k$, and let $B = A + 1$. We will show that B is an upper bound for (a_i).

First consider the case where $i \leq k$. In this case

$$a_i \leq A < A + 1.$$

Since $B = A + 1$, we have $a_i \leq B$ as desired.

Next consider the case where $i > k$. Since $i, k \geq k$, we have $|a_i - a_k| < 1$. Thus $-1 < a_i - a_k < 1$. So $a_i < a_k + 1$. Since $a_k + 1 \leq A + 1 = B$, we get $a_i \leq B$ as desired.

The proof of the existence of b is similar. (Subtract one from a min). □

Lemma 10. *If $(a_i)_{i \geq n_0}$ is a Cauchy sequence in an ordered field F, there is a bound $B \in F$ such that $|a_i| \leq B$ for all $i \geq n_0$.*

Proof. Let b and B be as in the previous lemma. We can replace B with anything larger, and b with anything less, and the previous lemma still holds. If $B < -b$ replace B with $-b$. On the other hand, if $-b \leq B$, then $-B \leq b$, and we can replace b with $-B$. In either case, for the new bounds, $b = -B$.

For each $i \geq n_0$, we have $-B \leq a_i \leq B$. Thus $|a_i| \leq B$. □

5. The real numbers

Our idea for constructing $\mathbb{R}$ is based on the notion that every Cauchy sequence in $\mathbb{Q}$ should determine a real number, and that equivalent sequences should determine the same real number. In other words, every Cauchy sequence in an equivalence class $\big[(a_i)\big]$ should determine the same real number. This idea leads us to the idea of defining real numbers as equivalence classes of Cauchy sequences of rational numbers.

Definition 4. If (a_i) is a Cauchy sequence in $\mathbb{Q}$, then let $\big[(a_i)\big]$ be the equivalence class containing (a_i) under the equivalence relation $\sim$ on the set of all Cauchy sequences in $\mathbb{Q}$. We call $\big[(a_i)\big]$ a *real number*.

Definition 5. The set of real numbers $\mathbb{R}$ is defined as follows:

$$\mathbb{R} \stackrel{\text{def}}{=} \left\{ \big[(a_i)\big] \;\middle|\; (a_i) \textit{ is a Cauchy sequence in } \mathbb{Q} \right\}.$$

Remark 9. It might seem like a strange transition to go from *every equivalence class of Cauchy sequences in $\mathbb{Q}$ should determine a real number*, to the bolder claim that *every equivalence class of Cauchy sequences in $\mathbb{Q}$ is a real number*. In fact, some people accepted the former, but are a bit uncomfortable about the latter. However, the latter allows us to avoid having to use extra axioms about the existence of new types of objects.

In order to make $\mathbb{R}$ into a field we need to define an addition and multiplication operation on $\mathbb{R}$.

Definition 6. Let $\big[(a_i)\big]$ and $\big[(b_i)\big]$ be real numbers. Then

$$\big[(a_i)_{i \geq n_0}\big] + \big[(b_i)_{i \geq m_0}\big] \stackrel{\text{def}}{=} \big[(a_i + b_i)_{i \geq l_0}\big]$$

and

$$\big[(a_i)_{i \geq n_0}\big] \cdot \big[(b_i)_{i \geq m_0}\big] \stackrel{\text{def}}{=} \big[(a_i b_i)_{i \geq l_0}\big].$$

Here l_0 is the maximum of n_0 and m_0. Our definitions give two binary operations $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

In order to check that these definitions are well-defined we need to verify facts that are not totally obvious: (i) $(a_i + b_i)$ and $(a_i b_i)$ are Cauchy, (ii) if $(a'_i) \sim (a_i)$ then we can replace a_i with a'_i in the definition and the result will be the same real number, and (iii) a similar invariance when $(b_i) \sim (b'_i)$.

The remainder of this section will be devoted to verifying all the non-obvious facts needed to confirm that the definition is well-defined.

Lemma 11. *Suppose that $(a_i)_{i \ge n_0}$ and $(b_i)_{i \ge m_0}$ are Cauchy sequences in F. Then $(a_i + b_i)_{i \ge l_0}$ and $(a_i b_i)_{i \ge l_0}$ are also Cauchy. Here l_0 is the maximum of n_0 and m_0.*

Proof. For the first claim, suppose ε is a positive element of F. We must find a suitable k for the sequence $(a_i + b_i)$.

Let $\varepsilon' = \varepsilon/2$. By assumption, there is an integer k_1 such that $i, j \ge k_1$ implies $|a_i - a_j| < \varepsilon'$. Similarly, there is an integer k_2 such that $i, j \ge k_2$ implies $|b_i - b_j| < \varepsilon'$. Let k be the larger of k_1 and k_2. If $i, j \ge k$, then

$$|(a_i + b_i) - (a_j + b_j)| = |(a_i - a_j) + (b_i - b_j)| \le |a_i - a_j| + |b_i - b_j|.$$

Since $|a_i - a_j| + |b_i - b_j| < \varepsilon' + \varepsilon'$, we get $|(a_i + b_i) - (a_j + b_j)| \le \varepsilon$ as desired.

For the second claim, suppose ε is positive. We must find a suitable k for $(a_i b_i)$. By Lemma 10, there is a bound A such that $|a_i| \le A$ for all terms a_i of the first sequence. Likewise, there is a bound B such that $|b_i| \le B$ for all terms b_i of the second sequence. Clearly we can assume that A and B are chosen to be positive.

Let $\varepsilon_1 = \varepsilon/(2B)$. By assumption (a_i) is Cauchy. So there is an integer k_1 such that $i, j \ge k_1$ implies $|a_i - a_j| < \varepsilon_1$. Similarly, if $\varepsilon_2 = \varepsilon/(2A)$, there is an integer k_2 such that $i, j \ge k_2$ implies $|b_i - b_j| < \varepsilon_2$. Let k be the larger of k_1 and k_2. If $i, j \ge k$, then

$$\begin{aligned} |a_i b_i - a_j b_j| &= |a_i b_i - a_i b_j + a_i b_j - a_j b_j| \\ &\le |a_i b_i - a_i b_j| + |a_i b_j - a_j b_j|. \end{aligned}$$

Observe that

$$|a_i b_i - a_i b_j| = |a_i||b_i - b_j| \le A|b_i - b_j| < A\varepsilon_2 = \varepsilon/2.$$

Similarly,

$$|a_i b_j - a_j b_j| = |a_i - a_j||b_j| \le |a_i - a_j|B < \varepsilon_1 B = \varepsilon/2.$$

Thus $|a_i b_i - a_j b_j| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ as desired. $\square$

Lemma 12. *Given sequences with values in F, if $(a_i) \sim (a'_i)$ then*

$$(a_i + b_i) \sim (a'_i + b_i),$$

and if $(b_i) \sim (b'_i)$ then

$$(a_i + b_i) \sim (a_i + b'_i).$$

Exercise 9. Prove the above. Hint: you may wish to look at Lemma 13 for guidance. However, Lemma 13 is more difficult since it requires the use of Lemma 10.

Lemma 13. *Given Cauchy sequences with values in F, if $(a_i) \sim (a'_i)$ then*

$$(a_i b_i) \sim (a'_i b_i),$$

and if $(b_i) \sim (b'_i)$ then

$$(a_i b_i) \sim (a_i b'_i).$$

This is valid for any ordered field F.

Proof. We prove the first statement; the second statement is similar. So suppose that $(a_i) \sim (a'_j)$. We must prove $(a_i b_i) \sim (a_i b'_i)$. In other words, for each positive ϵ in F we must find a k such that

$$i \geq k \;\Rightarrow\; |a_i b_i - a'_i b_i| < \epsilon.$$

By Lemma 10, there is a $B \in F$ such that $|b_i| \leq B$ for all i in the domain of (b_i). Clearly we can choose B to be positive. Thus B^{-1} is also positive. Let $\epsilon' = B^{-1}\epsilon$. Since $(a_i) \sim (a'_j)$, there is a $k \in \mathbb{N}$ such that

$$i \geq k \;\Rightarrow\; |a_i - a'_i| < \epsilon'.$$

Thus, if $i \geq k$ then

$$|a_i b_i - a'_i b_i| = |a_i - a'_i| \cdot |b_i| \leq |a_i - a'_i| B < \epsilon' B.$$

Since $\epsilon' B = \epsilon$ we have

$$i \geq k \;\Rightarrow\; |a_i b_i - a'_i b_i| < \epsilon$$

as desired. □

6. The Real Numbers $\mathbb{R}$ as a Commutative Ring

It is fairly easy to show that $\mathbb{R}$ is a commutative ring. It is a bit harder to show it is a field. We start with showing it is a commutative ring.

Theorem 14. *Addition and multiplication on $\mathbb{R}$ are commutative and associative.*

Exercise 10. Prove the above theorem.

Definition 7. Let F be an ordered field. If $c \in F$ then the *constant sequence* determined by c is the sequence $(a_i)_{i\in\mathbb{N}}$ defined by the rule $i \mapsto c$. In other words, $a_i = c$ for all $i \in \mathbb{N}$. We sometimes write $(c)_{i\in\mathbb{N}}$ or just (c) to denote this constant sequence.

Theorem 15. *If $(c)_{i\in\mathbb{N}}$ is a constant sequence in F, then it has limit c. In particular, such sequences are Cauchy.*

Exercise 11. Prove the above theorem. Hint: for each $\varepsilon > 0$, let $k = 0$.

Theorem 16. *An additive identity for $\mathbb{R}$ exists and is $[(0)]$. A multiplicative identity for $\mathbb{R}$ exists and is $[(1)]$.*

Remark 10. Identities, if they exist, are unique.[3] Thus we can say *the* additive identity and *the* multiplicative identity for $\mathbb{R}$.

Proof. Let $x = \big[(a_i)\big]$ be an arbitrary real number. By definition of $+$ in $\mathbb{R}$,

$$x + [(0)] = \big[(a_i)\big] + \big[(0)\big] = \big[(a_i + 0)\big] = \big[(a_i)\big] = x$$

where the next to last equality is due to the fact that 0 is the additive identity of $\mathbb{Q}$ (Chapter 5). By the commutative law (Theorem 14) we get $[(0)] + x = x + [(0)] = x$. Thus $[(0)]$ is the additive identity.

We leave the proof that $[(1)]$ is the multiplicative identity to the reader. $\square$

We now consider inverses.

Lemma 17. *If (a_i) is a Cauchy sequence in an ordered field F, then $(-a_i)$ is also Cauchy.*

Proof. Suppose (a_i) is Cauchy. Let ϵ be an arbitrary positive element of F. In order to show that $(-a_i)$ is Cauchy, we must find a $k \in \mathbb{N}$ such that $\big|(-a_i) - (-a_j)\big| < \epsilon$ for all $i, j \ge k$. Since (a_i) is Cauchy, there is a $k \in \mathbb{N}$ such that $|a_i - a_j| < \epsilon$ for all $i, j \ge k$. Using this k, if $i, j \ge k$ then

$$\big|(-a_i) - (-a_j)\big| = \big|(-1)(a_i - a_j)\big| = |-1||a_i - a_j| = |a_i - a_j|.$$

But $|a_i - a_j| < \epsilon$, so $\big|(-a_i) - (-a_j)\big| < \epsilon$ as desired. Thus $(-a_i)$ is Cauchy. $\square$

Theorem 18. *Every element of $\mathbb{R}$ has an additive inverse.*

Proof. Let $x = \big[(a_i)\big]$ be a real number. Here (a_i) is a Cauchy sequence. By Lemma 17 the sequence $(-a_i)$ is also Cauchy, so $y = \big[(-a_i)\big]$ is a real number. We leave it to the reader to show that y is the additive inverse of x. $\square$

Exercise 12. Complete the proof of the above theorem. Hint: show $x + y = y + x = 0$.

As we we will see, *multiplicative* inverses are trickier. Fortunately we do not need multiplicative inverses for the following.

Theorem 19. *The real numbers $\mathbb{R}$ form a commutative ring.*

Exercise 13. Prove the above. Hint: what about the distributive law? What about the other laws?

Now that we know that $\mathbb{R}$ is a commutative ring, we can use all the familiar algebraic manipulations and laws valid in rings.

[3] For any binary operation on a set S, one can show that if there is an identity, it must be unique. For example, if 0 and $0'$ are additive identities, $0 = 0 + 0' = 0'$.

7. The Canonical Embedding

Now that we have constructed $\mathbb{R}$ we wish to regard $\mathbb{Q}$ as a subset of $\mathbb{R}$. To do so we need to embed $\mathbb{Q}$ into $\mathbb{R}$. This will require an injective map $\mathbb{Q} \to \mathbb{R}$. What we will do is send any $r \in \mathbb{Q}$ to the constant sequence $(r)_{i\in\mathbb{N}}$.

Theorem 20. *Let (a_i) be a sequence in an ordered field F, and let $b \in F$. Then (a_i) has limit b if and only if $(a_i) \sim (b)$ where (b) is the constant sequence with value b.*

Proof. Write the constant sequence (b) as $(b_i)_{i\in\mathbb{N}}$ where b_i is defined to be b. We will show $(a_i) \sim (b_i)$ if and only if (a_i) has limit b.

First suppose $(a_i) \sim (b_i)$. Let ϵ be an arbitrary positive element of F. In order to show that (a_i) has limit b we must find a $k \in \mathbb{N}$ such that $|a_i - b| < \epsilon$ for all $i \geq k$. We do know that $(a_i) \sim (b_i)$, so, by definition, there is a $k \in \mathbb{N}$ such that $|a_i - b_i| < \epsilon$ for all $i \geq k$. However, $b_i = b$, so, using the k mentioned above, if $i \geq k$ then

$$|a_i - b| = |a_i - b_i| < \epsilon.$$

Thus b is the limit of (a_i) as desired.

Now suppose (a_i) has limit b. Let ϵ be an arbitrary positive element of F. In order to show that $(a_i) \sim (b_i)$, we must find a $k \in \mathbb{N}$ such that $|a_i - b_i| < \epsilon$ for all $i \geq k$. We do know that (a_i) has limit b, so, by definition, there is a $k \in \mathbb{N}$ such that $|a_i - b| < \epsilon$ for all $i \geq k$. However, $b_i = b$, so, using the k mentioned above, if $i \geq k$ then

$$|a_i - b_i| = |a_i - b| < \epsilon.$$

Thus $(a_i) \sim (b_i)$ as desired. □

Theorem 21. *Let $b, c \in F$ where F is an ordered field. Suppose $b \neq c$. Then $(b)_{i\in\mathbb{N}}$ and $(c)_{i\in\mathbb{N}}$ are non-equivalent.*

Exercise 14. Prove the above theorem. Hint: start by showing the two sequences have distinct limits.

Corollary 22. *Let $b, c \in \mathbb{Q}$ be distinct. Then $[(b)] \neq [(c)]$ in $\mathbb{R}$.*

Proof. This follows from basic properties of equivalence classes: two non-equivalent elements give non-equal (and disjoint) equivalence classes. □

Definition 8 (Canonical embedding)**.** We define the *canonical embedding* $\mathbb{Q} \to \mathbb{R}$ by the rule $c \mapsto \big[(c)_{i\in\mathbb{N}}\big]$. In other words, the map sends c to the constant sequence with value c. By Theorem 15, the image is an equivalence class of Cauchy sequences, so the image is, indeed, in $\mathbb{R}$.

Theorem 23. *The canonical embedding $\mathbb{Q} \to \mathbb{R}$ is injective.*

Exercise 15. Prove the above theorem using Corollary 22.

Once we have a canonical embedding $\mathbb{Q} \to \mathbb{R}$ we can use this to identify elements of $\mathbb{Q}$ with certain elements of $\mathbb{R}$. Thus we can think of $\mathbb{Q}$ as being a subset of $\mathbb{R}$.

When we think of $\mathbb{Q}$ as a subset of $\mathbb{R}$, we need to check that the addition and multiplication of $\mathbb{R}$ extends the addition and multiplication of $\mathbb{Q}$ defined in Chapter 5. In other words, when we are working with addition and multiplication on $\mathbb{Q}$ we want to be assured that we get the same result whether we use the addition of $\mathbb{Q}$ (Chapter 6) or the addition of $\mathbb{R}$ (this chapter). This is demonstrated in the following lemma.

Lemma 24. *The definitions of addition and multiplication on $\mathbb{R}$ extend the definitions of addition and multiplication on $\mathbb{Q}$.*

Proof. We give the proof for addition; the proof for multiplication is similar. Let $a, b \in \mathbb{Q}$ be given and let $+_\mathbb{Q}$ be the addition defined in Chapter 5. Let $+_\mathbb{R}$ be the addition defined in the current chapter. We must show that $a +_\mathbb{Q} b$ is identified with the same real number as $a +_\mathbb{R} b$ (via the canonical embedding).

This is actually easy once we figure out what is involved. The canonical embedding identifies the rational number $a +_\mathbb{Q} b$ with the equivalence class of the constant sequence $(a +_\mathbb{Q} b)_{i\in\mathbb{N}}$. Since a is identified with the equivalence class of the constant sequence $(a)_{i\in\mathbb{N}}$ and b is identified with the equivalence class of $(b)_{i\in\mathbb{N}}$, the sum $a +_\mathbb{R} b$ is really equal to the sum $\big[(a)_{i\in\mathbb{N}}\big] +_\mathbb{R} \big[(b)_{i\in\mathbb{N}}\big]$. By the definition of $+_\mathbb{R}$

$$\big[(a)_{i\in\mathbb{N}}\big] +_\mathbb{R} \big[(b)_{i\in\mathbb{N}}\big] = \big[(a +_\mathbb{Q} b)_{i\in\mathbb{N}}\big].$$

The result follows. □

Remark 11. Since 0 in $\mathbb{Q}$ is identified with the equivalence class $[(0)]$ of the constant sequence (0), and since $[(0)]$ is the additive identity of $\mathbb{R}$, we usually write 0 for the additive identity of $\mathbb{R}$. This is consistent with the practice of writing 0 for the additive identity of any ring.

Similarly, we write 1 for the multiplicative identity of $\mathbb{R}$.

Remark 12. Subtraction in $\mathbb{R}$ extends the subtraction in $\mathbb{Q}$. This follows from the definition of $r - s$ (in any ring) as $r + (-s)$. As mentioned above, addition in $\mathbb{R}$ extends addition in $\mathbb{Q}$, and additive inverse in $\mathbb{R}$ extends additive inverse in $\mathbb{Q}$ since $-[(r)] = [(-r)]$.

8. Positive Real Numbers

An important part of showing that $\mathbb{R}$ is an ordered field is to define the set of positive real numbers P, and to show that this set has the required properties: closure and trichotomy. We will do this even before proving that $\mathbb{R}$ is a field.

Since a real number can be thought of as $[(a_i)]$ where (a_i) is Cauchy, we might be tempted to say that $x = [(a_i)]$ is positive if each a_i is positive. Warning: this does not work. For example, the sequence $(1/i^2)$ converges to 0, and so is equivalent to the constant sequence (0). Thus $[(1/i^2)]$ is zero even though all its terms are strictly positive.

Furthermore, if a sequence has a finite number of zero or negative terms, and the rest are positive, then the sequence could represent a positive number. Thus there are two ways in which the naive definition of positive is defective. The following definition corrects both deficiencies.

Definition 9 (Positive)**.** A *positive-type Cauchy sequence* in an ordered field F is a Cauchy sequence (a_i) with the following property: there is a positive $d \in F$ and a $k \in \mathbb{N}$ such that $a_i \geq d$ for all $i \geq k$.

A *positive* real number is a real number of the form $[(a_i)]$ where (a_i) is a positive-type Cauchy sequence with values in $\mathbb{Q}$.

Exercise 16. Suppose (a_i) and (b_i) are Cauchy sequences where $(a_i) \sim (b_i)$. Show that (a_i) is positive-type Cauchy if and only if (b_i) is.

Hint: suppose there is a $d > 0$ and a $k \in \mathbb{N}$ such that $a_i \geq d$ for all $i \geq k$. We must find d' and k' that work for (b_i). Let $\varepsilon = d/2$ and choose a k_0 so that $|a_i - b_i| < \varepsilon$ for all $i \geq k_0$. Why does such a k_0 exist? Choose k' as the maximum of k and k_0. What do you think d' should be? Prove that your choice of k' and d' work for (b_i).

Remark 13. Suppose $r \in \mathbb{Q}$. It is easy to check that r is a positive element of $\mathbb{Q}$ if and only if the real number $[(r)]$ is a positive element of $\mathbb{R}$. Thus the present definition of positive is compatible with the definition of Chapter 6

Theorem 25 (Closure)**.** *If $x, y \in \mathbb{R}$ are positive then so is $x + y$ and xy.*

Proof. Let $x = [(a_i)]$ and $y = [(b_i)]$ where (a_i) and (b_i) are positive-type Cauchy sequences of rational numbers. By definition there is a positive $d_1 \in \mathbb{Q}$ and a $k_1 \in \mathbb{N}$ such that $a_i \geq d_1$ for all $i \geq k_1$. Likewise, there is a positive $d_2 \in \mathbb{Q}$ and a $k_2 \in \mathbb{N}$ such that $b_i \geq d_2$ for all $i \geq k_2$. Let $d = d_1 + d_2$. We know that $d > 0$ (Chapter 6). Let k be the maximum of k_1 and k_2. If $i \geq k$, then

$$a_i + b_i \geq d_1 + b_i \geq d_1 + d_2 = d$$

by properties of rational numbers (Chapter 6). Thus $x + y = [(a_i + b_i)]$ is positive.

The proof for xy is similar. □

Exercise 17. Prove the above for the case of multiplication.

We now want to prove a trichotomy law: for all $x \in \mathbb{R}$ exactly one of the following occurs (i) x is positive, (ii) $x = 0$, or (iii) $-x$ is positive. In the third case we also say that x is *negative*.

We divide the proof of this law into lemmas:

Lemma 26. *Let $x \in \mathbb{R}$. If x is positive, then $x \neq 0$.*

Proof. Suppose otherwise that x is positive and $x = 0$ at the same time. Since 0 is the equivalence class $[(0)]$ we have $x = [(0)]$. Since x is positive, there is a positive-type Cauchy sequence (a_i) such that $x = [(a_i)]$. By properties of equivalence classes, $(0) \sim (a_i)$. By Exercise 16, (0) is a positive-type Cauchy sequence. This is clearly a contradiction. □

Lemma 27. *Let $x \in \mathbb{R}$. It is not possible for both x and $-x$ to be positive.*

Exercise 18. Prove the above. Hint: suppose not. Write $x = [(a_i)]$. Then $-x = [(-a_i)]$. Define a d_1 and k_1 for (a_i) and d_2 and k_2 for $(-a_i)$. Let i be the maximum of k_1 and k_2, and show that a_i is both positive and negative in $\mathbb{Q}$. This contradicts a certain law in Chapter 6.

Remark 14. Notice how we use a trichotomy law for $\mathbb{Q}$ (Chapter 6) to help prove a trichotomy law for $\mathbb{R}$.

Now we come to a lemma that is easy to state, but a bit tricky to prove. Part of its trickiness comes from the need to manipulate quantifiers.

Lemma 28. *If $x \neq 0$ is a real number, then either x or $-x$ is positive.*

Proof. Write $x = \big[(a_i)\big]$ where (a_i) is a Cauchy sequence in $\mathbb{Q}$. This implies that $-x = \big[(-a_i)\big]$.

By assumption $x \neq 0$. In other words, $\big[(a_i)\big] \neq \big[(0)\big]$. Thus we have that (a_i) and the constant sequence (0) are non-equivalent. When we negate the definition of equivalence (see Exercise 2 for a similar negation) we find that there exists a positive $\varepsilon \in \mathbb{Q}$ such that for all $k \in \mathbb{N}$ there is an integer $i \geq k$ with $|a_i - 0| \geq \varepsilon$. Fix such an ε for what follows.

Let $\varepsilon' = \varepsilon/2$. Since (a_i) is Cauchy, there is a $k_0 \in \mathbb{N}$ such that

$$i, j \geq k_0 \Rightarrow |a_i - a_j| < \varepsilon'.$$

Above we determined that for all $k \in \mathbb{N}$ there is an integer $i \geq k$ with $|a_i - 0| \geq \varepsilon$. In particular, there is an integer $i_0 \geq k_0$ with $|a_{i_0}| \geq \varepsilon$. By a result of Chapter 6, this implies that $a_{i_0} \geq \varepsilon$ or $a_{i_0} \leq -\varepsilon$.

We first consider the case where $a_{i_0} \geq \varepsilon$. Suppose $j \geq k_0$. Since $i_0, j \geq k_0$ we have $|a_j - a_{i_0}| < \varepsilon'$ by the Cauchy condition. By a result of Chapter 6, $-\varepsilon' < a_j - a_{i_0}$. This gives us $-\varepsilon' + a_{i_0} < a_j$. Therefore,

$$a_j > a_{i_0} - \varepsilon' \geq \varepsilon - \varepsilon' = \varepsilon - \varepsilon/2 = \varepsilon/2 = \varepsilon'.$$

In summary, we have $\varepsilon' > 0$ and k_0 such that $j \geq k_0$ implies $a_j \geq \varepsilon'$. By definition of positive, we conclude that $x = \big[(a_i)\big]$ is positive.

Next we consider the case where $a_{i_0} \leq -\varepsilon$. Suppose $j \geq k_0$. Since $i_0, j \geq k_0$ we have $|a_j - a_{i_0}| < \varepsilon'$ by the Cauchy condition. By a result of Chapter 6, $a_j - a_{i_0} < \varepsilon'$. This gives us $a_j < \varepsilon' + a_{i_0}$. Thus

$$a_j < \varepsilon' + a_{i_0} \leq \varepsilon' - \varepsilon = \varepsilon/2 - \varepsilon = -\varepsilon/2 = -\varepsilon'.$$

Hence, $-a_j \geq \varepsilon'$. In summary, we have $\varepsilon' > 0$ and k_0 such that $j \geq k_0$ implies $-a_j \geq \varepsilon'$. By definition of positive, we conclude that $-x = \big[(-a_i)\big]$ is positive.

So either x or $-x$ is positive, assuming $x \neq 0$. □

Theorem 29 (Trichotomy: version 1)**.** *For every $x \in \mathbb{R}$ exactly one of the following occurs: (i) x is positive, (ii) $x = 0$, or (iii) $-x$ is positive.*

Exercise 19. Prove the above. Hint: if you make generous use of the previous lemmas, the proof should be fairly short.

9. The Real Numbers $\mathbb{R}$ as an Ordered Field

We will now see that every non-zero element of $\mathbb{R}$ has a multiplicative inverse. We first focus on positive elements. Recall that a sequence (a_i) is *positive-type Cauchy* if there is a k and and $d > 0$ such that $i \geq k$ implies $a_i \geq d$. In particular, such sequences are allowed to have a finite number of zero (or even negative) terms. This is why we have to change from n_0 to k_0 in the following lemma.

Lemma 30. *Suppose $(a_i)_{i \geq n_0}$ is a positive-type Cauchy sequence in an ordered field F. Then there is an integer k_0 such that $\left(a_i^{-1}\right)_{i \geq k_0}$ is a Cauchy sequence.*

Proof. By definition of positive-type Cauchy sequence, there is a $k_0 \in \mathbb{N}$ and positive $d \in F$ such that $a_i \geq d$ for all $i \geq k_0$. If $i \geq k_0$ then $a_i \neq 0$, so a_i has a multiplicative inverse in F. By properties of ordered fields (Chapter 6) $a_i^{-1} \leq d^{-1}$, and a_i^{-1} is positive.

We wish to show that $\left(a_i^{-1}\right)_{i \geq k_0}$ is Cauchy. In other words, let $\varepsilon \in F$ be positive. We want a k such that if $i, j \geq k$ then $\left|a_i^{-1} - a_j^{-1}\right| < \varepsilon$.

Let $\varepsilon' = \varepsilon d^{-2}$. Since d and ε are positive, so is ε'. Since (a_i) is Cauchy, there is a k' such that $|a_i - a_j| < \varepsilon'$ for all $i, j \geq k'$. Let k be the maximum of k' and k_0. If $i, j \geq k$ then

$$\begin{aligned} |a_i^{-1} - a_j^{-1}| &= \left|(a_j - a_i) a_i^{-1} a_j^{-1}\right| \\ &= |a_j - a_i| \left|a_i^{-1}\right| \left|a_j^{-1}\right| \\ &= |a_j - a_i| \, a_i^{-1} a_j^{-1} \\ &\leq |a_j - a_i| \, d^{-1} d^{-1} \\ &< \varepsilon' d^{-2} = \varepsilon \end{aligned}$$

So $|a_i^{-1} - a_j^{-1}| < \varepsilon$ as desired. So $\left(a_i^{-1}\right)_{i \geq k_0}$ is Cauchy. □

Lemma 31. *Let $x \in \mathbb{R}$. If x is positive, then x has a multiplicative inverse.*

Proof. Write $x = [(a_i)]$ where (a_i) is a positive-type Cauchy sequence of rational numbers. By the previous lemma, there is a k_0 such that $\left(a_i^{-1}\right)_{i \geq k_0}$ is Cauchy. Thus

$$y = \left[(a_i^{-1})_{i \geq k_0}\right]$$

is a real number. By definition of multiplication in $\mathbb{R}$,

$$xy = [(a_i)] \left[(a_i^{-1})\right] = \left[(a_i a_i^{-1})\right] = [(1)].$$

Thus $xy = 1$. By the commutative law for multiplication, $yx = xy = 1$. We have shown that x has a multiplicative inverse. □

Theorem 32. *Let $x \in \mathbb{R}$. If $x \neq 0$, then x has a multiplicative inverse.*

Exercise 20. Prove the above theorem. Hint: if x is positive, use the lemma. If $-x$ is positive, try $-y$ where y is the multiplicative inverse of $-x$. Use the fact that $x(-y) = (-x)y$ (true in any ring).

We now come to one of the main theorems of this Chapter.

Theorem 33. *The real numbers $\mathbb{R}$ are an ordered field.*

Proof. We know that $\mathbb{R}$ is a commutative ring by Theorem 19. We know that $0 \neq 1$ by Corollary 22. Multiplicative inverses exist by Theorem 32. We conclude that $\mathbb{R}$ is a field.

To show that $\mathbb{R}$ is an ordered field we need to check that (i) the positive elements are closed under addition and multiplication, and (ii) the positive elements satisfy the trichotomy law. Both these were done in the previous section (Theorem 25 and Theorem 29). □

Remark 15. Now that we know that $\mathbb{R}$ is an ordered field, we can use all the definitions and results about ordered fields F from the end of Chapter 6 including facts about $<$ and absolute values. You should review these definitions and results from Chapter 6.

Remark 16. As mentioned above, positivity defined for $\mathbb{R}$ is compatible with the earlier concept of positivity defined for $\mathbb{Q}$. Since $x < y$ means $y - x$ is positive, if follows that inequality in $\mathbb{R}$ is compatible with inequality in $\mathbb{Q}$.

10. Relationship between $\mathbb{R}$ and $\mathbb{Q}$

In this section we will consider a few useful results relating $\mathbb{R}$ and $\mathbb{Q}$. For example, we will see that Cauchy sequences of rational numbers always converge to real numbers, and that all real numbers are limits of rational sequences. We will also see that $\mathbb{Q}$ is dense in $\mathbb{R}$; in other words, between any two distinct real numbers we can always find a rational number.

We begin with a lemma that can be used to establish $x \leq y$ where $x, y \in \mathbb{R}$.

Lemma 34. *Suppose (a_i) and (b_i) are two Cauchy sequences of rational numbers, and let $x = [(a_i)]$ and $y = [(b_i)]$ be the corresponding real numbers. If there is a $k \in \mathbb{N}$ such that $a_i \leq b_i$ for all $i \geq k$, then $x \leq y$.*

Remark 17. If we have $a_i < b_i$ instead, we cannot necessarily conclude that $x < y$. Without extra information, we can only conclude that $x \leq y$.

Proof. Suppose otherwise that $x > y$. Then $x - y$ is positive. Now $x - y$ is given by $[(a_i - b_i)]$. So there is a $k' \in \mathbb{N}$ and a positive $d \in \mathbb{Q}$ such that $a_i - b_i \geq d$ for all $i \geq k'$. Since $d > 0$ this means that $a_i > b_i$ for all $i \geq k'$.

Let i be the maximum of k and k', where k is as in the statement of the lemma. Then $a_i \leq b_i$ by hypothesis and $a_i > b_i$ by the above argument. This contradicts trichotomy of Chapter 6. □

Corollary 35. *Suppose $x \in \mathbb{R}$ is given by $x = [(a_i)]$. Suppose that b is a rational number. If there is a $k \in \mathbb{N}$ such that $a_i \leq b$ for all $i \geq k$, then*

$x \leq b$ (where here we are thinking of b as a real number). If, instead, there is a $k \in \mathbb{N}$ such that $a_i \geq b$ for all $i \geq k$, then $x \geq b$.

Proof. For the first statement, apply Lemma 34 to the sequences (a_i) and (b). For the second statement, switch the order and apply Lemma 34 again. $\square$

The following lemma gives a special case of the density of $\mathbb{Q}$ in $\mathbb{R}$. The general case will come later.

Lemma 36. *If ε is a positive real number, then we can find a rational number $r \in \mathbb{Q}$ such that $0 < r < \varepsilon$.*

Proof. Write ε as $[(a_i)]$ where (a_i) is a positive-type Cauchy sequence of rational numbers. By definition, there is a $k_0 \in \mathbb{N}$ and a positive $d \in \mathbb{Q}$ such that $a_i \geq d$ for all $i \geq k_0$. Identify d with the constant sequence (d_i) where $d_i = d$ for all $i \in \mathbb{N}$. Observe that $d_i \leq a_i$ for all $i \geq k_0$ since $d_i = d$. By Lemma 34, we have that $d \leq \varepsilon$.

From Chapter 6 we know that between the rational number 0 and d there is a rational number r. Thus $0 < r$ and $r < d \leq \varepsilon$. $\square$

The following says that if a Cauchy sequence of rational numbers represents a certain real number, then the Cauchy sequence converges to the real number.

Theorem 37. *Let (a_i) be a Cauchy sequence of rational numbers. Then (a_i), considered as a sequence of real numbers, converges to the real number x where $x = [(a_i)]$.*

Proof. Let ε be an arbitrary positive real number. We must find a $k \in \mathbb{N}$ such that $|a_i - x| < \varepsilon$ for all $i \geq k$. It seems like we should be able to use the definition of Cauchy sequence to find such a k. There is a slight problem: (a_i) is a Cauchy sequence in $\mathbb{Q}$, but $\varepsilon \in \mathbb{R}$. We first need to replace ε with a rational number. Lemma 36 gives us a positive $\varepsilon' \in \mathbb{Q}$ such that $\varepsilon' < \varepsilon$. By definiton of a Cauchy sequence in $\mathbb{Q}$, we can use this ε' and find a $k \in \mathbb{N}$ such that $|a_i - a_j| < \varepsilon'$ for all $i, j \geq k$. We must show that this k has the desired properties.

So let k be as above, and let $i \geq k$. We must show that $|a_i - x| < \varepsilon$. We will actually show that $|a_i - x| < \varepsilon'$. Since we are fixing i, write a_i as c. If $j \geq k$ then (in $\mathbb{Q}$)

$$|c - a_j| = |a_i - a_j| \leq \varepsilon'$$

since (a_j) is Cauchy. Thus $-\varepsilon' \leq c - a_j \leq \varepsilon'$ for all $j \geq k$ (Chapter 6: properties of absolute values). By Corollary 35 (in $\mathbb{R}$)

$$-\varepsilon' \leq [(c - a_j)] \leq \varepsilon'.$$

Now $[(c - a_j)] = [(c)] - [(a_j)]$, so $[(c - a_j)]$ is the real number $c - x$. Recall $c = a_i$. Thus

$$-\varepsilon' \leq a_i - x \leq \varepsilon'.$$

This implies that $|a_i - x| \leq \varepsilon'$ (Ch. 6). Since $\varepsilon' < \varepsilon$ we get $|a_i - x| < \varepsilon$. $\square$

Corollary 38. *Every Cauchy sequence of rational numbers converges to some real number.*

Proof. Let (a_i) be a Cauchy sequence of rational numbers. Let $x = [(a_i)]$. By Theorem 37, (a_i) has limit x. $\square$

Remark 18. Let (a_i) be a sequence of rational number. There is some ambiguity of what *Cauchy* means for (a_i) when we embed $\mathbb{Q}$ into $\mathbb{R}$. We can mean the Cauchy condition holds for all positive ε in $F = \mathbb{Q}$, or we can mean that the Cauchy condition holds for all positive ε in $F = \mathbb{R}$. In the above theorem and corollary we are thinking of the former. However, since such sequences converge to a real number (in the sense of $F = \mathbb{R}$), they are also Cauchy for $F = \mathbb{R}$ (Theorem 4).

Conversely, if (a_i) is Cauchy in the sense of $F = \mathbb{R}$, then it is automatically Cauchy in the sense of $F = \mathbb{Q}$ since any rational ε is automatically real.

We conclude that if (a_i) is a sequence of rational numbers, there is no difference between being Cauchy for $F = \mathbb{Q}$ or for $F = \mathbb{R}$.

Corollary 39. *Every real number is the limit of a sequence of rational numbers*

Proof. Let $x = [(a_i)]$ be a real number. By Theorem 37, (a_i) has limit x. $\square$

Corollary 40. *If $x \in \mathbb{R}$ and if $\varepsilon \in \mathbb{R}$ is positive, then there is a rational number $r \in \mathbb{Q}$ with $|x - r| < \varepsilon$.*

Proof. Since x is the limit of a sequence (a_i) of rational numbers, there is a $k \in \mathbb{N}$ such that $|a_i - x| < \varepsilon$ for all $i \geq k$. Let $r = a_k$. $\square$

Theorem 41. *The field $\mathbb{Q}$ is dense in $\mathbb{R}$. In other words, if $x, y \in \mathbb{R}$ are such that $x < y$, there is a $r \in \mathbb{Q}$ with $x < r < y$.*

Proof. Let $\varepsilon = (y - x)/2$. By Corollary 40 there are rational numbers $a, b \in \mathbb{Q}$ such that $|x - a| < \varepsilon$ and $|y - b| < \varepsilon$. Let $r = (a + b)/2$. Observe that $r \in \mathbb{Q}$ since $a, b, 2 \in \mathbb{Q}$.

Observe that $-\varepsilon < x - a < \varepsilon$ (Chapter 6). Since $x - a < \varepsilon$, we get $x - \varepsilon < a$. Likewise, $y - \varepsilon < b$. Adding inequalities gives $x + y - 2\varepsilon < a + b$. However, $2\varepsilon = y - x$. Thus $2x = x + y - (y - x) < a + b$. So $x < (a + b)/2$. This shows $x < r$.

A similar argument shows that $r < y$. $\square$

Exercise 21. Complete the above theorem: show that $r < y$.

11. Complete Ordered Fields

As we have discussed before, $\mathbb{Q}$ has "holes". For example, $\mathbb{Q}$ is missing a square root for 2. In an ordered field without holes, we expect every Cauchy sequence to converge. In contrast, $\mathbb{Q}$ has Cauchy sequences that do not converge. For example, consider $(n_i/10^i)$ where n_i is the largest integer such that $(n_i/10^i)^2 \leq 2$. This sequence is related to the decimal expansion of $\sqrt{2}$.

In the next chapter, we will see that this sequence is Cauchy. This Cauchy sequence does not converge in $\mathbb{Q}$, but does converge in $\mathbb{R}$ with limit $\sqrt{2}$.

We will now show that the real numbers $\mathbb{R}$ do not have such Cauchy sequences that fail to converge. This means that $\mathbb{R}$ is "complete": it does not have "holes". In order to make this discussion formal, we need the following:

Definition 10. An ordered field F is said to be *complete* if every Cauchy sequence in F converges to a limit in F.

We already know, from the previous section, that every Cauchy sequence of rational numbers converges in $\mathbb{R}$. However, we need to show that every Cauchy sequence of real numbers converges in $\mathbb{R}$ in order to be able to assert that $\mathbb{R}$ is complete. We begin with a lemma.

Lemma 42. *If (a_i) is a sequence of real numbers, then there is a sequence (b_i) of rational numbers such that $(a_i) \sim (b_i)$. (Equivalence is taken with $F = \mathbb{R}$).*

Proof. For each a_i, we know by Corollary 40 that there is a rational number b_i such that $|a_i - b_i| < 1/i$. Consider the sequence (b_i) formed from such rational numbers.[4] We must show that $(a_i) \sim (b_i)$.

Let $\varepsilon \in \mathbb{R}$ be an arbitrary positive real number. We must find a $k \in \mathbb{N}$ such that $|a_i - b_i| < \varepsilon$ for all $i \geq k$. By Lemma 36 we can find a positive rational number $m/n < \varepsilon$. Here m, n are chosen to be positive integers. Then $1/n \leq m/n$. So $1/n < \varepsilon$. Let $k = n$. If $i \geq k$ then

$$\begin{array}{rcll} |a_i - b_i| & < & 1/i & \text{(choice of } b_i) \\ & \leq & 1/n & (i \geq k \text{ and } k = n) \\ & < & \varepsilon & \text{(choice of } n) \end{array}$$

Thus $(a_i) \sim (b_i)$ as desired. □

Now for the main theorem.

Theorem 43. *The field $\mathbb{R}$ is a complete ordered field.*

Proof. We must show that if (a_i) is a Cauchy sequence of real numbers, then (a_i) converges. By Lemma 42 there is a sequence (b_i) of rational numbers such that $(b_i) \sim (a_i)$. By Remark 6, the sequence (b_i) must also be Cauchy. By Corollary 38 (b_i) has a limit b. By Exercise 8 and the fact that $(a_i) \sim (b_i)$ we conclude that (a_i) has limit b. Thus (a_i) converges. □

[4] In order to avoid using the axiom of choice, we can select b_i to have the smallest possible denominator, and among fractions with the smallest possible denominator we choose the smallest possible numerator.