SQUARE ROOTS ARE USUALLY IRRATIONAL

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In class we proved the following handy fact:

Lemma. *A positive integer n is a perfect square if and only if every prime in its prime factorization occurs an even number of times.*

Example. Observe that $144 = 2^4 3^2$ is a perfect square, but $72 = 2^3 3^2$ is not.

Most of what we have done in this class can be done purely in $\mathbb{Z}$: fractions can be avoided by using the cancellation laws and the Quotient-Remainder Theorem instead. The following theorem (from Sept. 12) is an exception to our policy of staying in $\mathbb{Z}$: we must start with real and rational numbers and their properties. But we will immediately square and clear denominators so we can go back to the world of $\mathbb{Z}$. (Note: this result is an application of number theory to real numbers. We will not need it to prove any results later in this course.)

Theorem. *Suppose $n \geq 2$ is an integer which is not a perfect square. Then $\sqrt{n}$ is irrational.*

Proof. (By contradiction) Suppose that $\sqrt{n}$ is rational. We know that $\sqrt{n}$ is positive, so $\sqrt{n} = a/b$ where a and b are positive integers. Squaring both sides gives us $n = a^2/b^2$ and multiplying both sides by b^2 gives us $nb^2 = a^2$.

By the above lemma, since n is not a perfect square, there must be a prime q that occurs in the prime factorization of n an odd number of times. By the same lemma, the number of times q occurs in the factorization of b^2 is even (it could occur 0 times, but 0 counts as even since $2 \mid 0$). Thus the number of times q occurs in the factorization of nb^2 is odd because an odd number plus an even number is an odd number.

Similarly, the number of times q occurs in the factorization of a^2 is even (it could occur 0 times, but 0 is even). Let $N = nb^2 = a^2$. There is one factorization where q occurs an odd number of times (using nb^2) and another factorization where q occurs an even number of times (using a^2). This contradicts the uniqueness of the prime factorization (The Fundamental Theorem of Arithmetic). □

Here is a concise version of the proof:

Proof. Suppose $\sqrt{n}$ is rational: $\sqrt{n} = a/b$ for positive integers a, b. Then $nb^2 = a^2$. Let q be a prime the occurs to odd power in the prime factorization of n. Then q occurs to odd power in nb^2 as well: contradicting the equation $nb^2 = a^2$. □

(One strategy for learning proofs is to learn concise versions, and test yourself to see if you can justify each step.)

As an exercise, try to generalize this: show that $n^{1/m}$ is irrational if n is not a perfect mth power.

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