Study Guide for Exam 3

Math 330: History of Mathematics

December 4, 2005.

1 Introduction

What follows is a list of topics that might be on the exam. Of course, the test will only contain a selection of these: there is simply too much to put on one exam. (Thus, some of the topics will appear on the final instead, and some topics will appear on this exam *and* the final).

History is full of names of people, place names, and dates. Don't worry: the only personal names, place names, or dates you need to memorize are given in this study guide. (Footnotes and parenthetical remarks in this study guide are given for information and clarification. You do not need to known them for the exam.)

2 Mathematicians

1. **Aristarchus (of Samos)**. Greek astronomer who lived several centuries before the more famous Ptolomy.[1] He proposed the heliocentric theory (but Ptolomy and the other Greek astronomers' geocentric theory won out, at least until Copernicus). Why did his contemporaries reject the heliocentric theory? Answer: lack of motion on earth and the apparent fixed positions of stars.

2. **Ptolomy**. Lived in Alexandria which was then part of the Roman Empire. He was an astronomer, mathematician, and geographer. His most famous book is commonly called the *Almagest* (original name: *Syntaxis Mathematica*). This book gives a mathematical treatment of planetary motion, and the motion of the sun and the moon. His geocentric theory did not use simple circular orbits. What did it use? answer: epicyles. His complicated system represents the observations quite well. It needed to be complicated to be somewhat accurate since it was based on a flawed premise: the geocentric model of the solar system.

   What was Ptolomy's theorem on cyclic quadrilaterals? how do you prove it? What type of trigonometric function did he use? answer: the chord function. How is it related to $\sin\theta$? answer: $\mathrm{Chord}(\theta) = 2\sin(\theta/2)$. Draw a picture illustrating this. (Today we use a unit circle. Ptolomy used a circle of radius of 60 for convenience). He used base sixty and the idea of a circle having 360 degrees. He approximated π using an inscribed 360-gon (value: $3\frac{17}{120}$). (He also cataloged the position of over a 1000 stars. Ptolomy also listed estimated latitudes and longtitudes of major known locations on the earth.)

[1] He lived before the base 60 system and the degree measure of angles were introduced to Alexandria from Mesopotamia.

3. **Heron**. Lived in Alexandria which was then part of the Roman Empire. He was mostly an applied mathematician and engineer. His works describe very interesting mechanisms using pneumatics, that is air pressure, steam, or water pressure, as well as other mechanisms. (He developed war engines, fire engines, fancy puppets, fancy toys, wind organs, coin-operated machines, sundials, fancy tricks with mirrors, etc.) What is Heron's formula for a triangle? How is it a limiting case of Brahmagupta's formula?

4. **Diophantus**. Lived in Alexandria which was then part of the Roman Empire. First person to use algebraic symbolism instead of just words (*syncopated* algebra instead of *rhetorical* algebra). His book, the *Arithmetica* was quite sophisticated, certainly the most sophisticated algebra in the Greek speaking world. Solving equations for him requires methods quite a bit different than the standard methods taught today in algebra. This is mainly because he didn't just want real number solutions, but wanted rational or integer solutions. His work later inspired Fermat to invent the study of *number theory.* There is a modern branch of number theory called *Diophantine equations*.

5. **Hypatia and the commentators** Hypatia was more than just a mathematician. She was also a neo-Platonist philosopher and was known as a charismatic teacher. She wrote commentary on earlier mathematicians. At this time, scholars were trying to keep alive the earlier scientific and mathematical culture and so would write commentaries to make the earlier works accessible, and to add their opinions to them. Hypatia was the last major intellectual figure of Alexandria and the Roman Empire. (Even after the western Roman Empire fell, Byzantine mathematicians wrote commentaries to keep the tradition alive.)

6. **Liu Hui** and **Nine Chapters on the Mathematical Art**. Liu Hui wrote (in 263 AD) commentary and additions to the Chinese classic: *Nine Chapters on the Mathematical Arts*. He lived after the Han dynasty.

   The *Nine Chapters on the Mathematical Arts* (*Jiuzhang suanshu*) was a practical textbook of mathematics dating from the Han dynasty or earlier used to train civil servants who had to take tests to get their positions. The *Nine Chapters* is the most important early work of Chinese mathematics. Its influence was so strong that it might be called the "Euclid of China", but without proofs. It consists of word problems (246 problems, surveying, trade, taxation, distribution problems, construction of canals, and other problems you might see in as word problems in a textbook today.) Topics: Arithmetic including rules for common fractions. Areas. Proportion, percentages and exchange rates. Square and cube roots. Volumes (but not the volume of a sphere). Linear and quadratic equations. Pythagorean theorem (which the Chinese called the *Gougu rule*), and examples of Pythagorean triples. Similar triangles.

   Some interesting features; use of matrices to solve systems of linear equations (up to six equations and six unknowns). The use of negative numbers, especially for matrix manipulations. Correct rules for adding and subtracting negative numbers.

   Liu Hui supplied explanations, some of which can be regarded as proofs, for much of the *Nine Chapters*. He also pointed out that 3, the value of π used in the book, was not a good approximation and gave an excellent approximation (3.14159) using n-gons (in fact a 3072-gon) in a manner similar to Archimedes.

7. **Zu Chongzhi** (439-501). From a family of astronomers. He was a mathematician, astronomer, engineer, and even wrote ten novels. He worked for the Chinese emperors. He is famous for his work on the Chinese calendar. He gave excellent approximations for π ($\frac{355}{113}$ and $3.1415926 < \pi < 3.1415927$. He probably used a 24576-gon).

8. **Aryabhata** (476 - about 550). Indian mathematician and astronomer. Like most Indian mathematicians, his mathematics and trigonometry was written as part of an astronomy book. He wrote a poem in Sanskrit on astronomy containing a section on mathematics (118 verses total, 33 devoted to mathematics. These verses contain 66 rules of mathematics.) He used the idea of sine instead of the idea of chord. The Indian word for sine means "half-chord". He worked on number theory problems that were motivated by cycles occurring in astronomy. For example, he used a form of the Euclidean Algorithm. He developed a sine table (his circle had radius 3438 and circumference 21600 which is the same as the earlier Greek astronomer Hipparchus). He also worked on quadratic and indeterminate equations.

9. **Brahmagupta** (died 670). Indian mathematician and astronomer (head of an astronomical observatory). What is Brahmagupta's formula for cyclic quadrilaterals? He was an early user of negative numbers and zero, and gave rules for them. In modern notation: $a + 0 = a$, $a - 0 = a$, $a0 = 0$, $0 - 0 = 0$, product or quotient of two negatives is a positive, product or quotient of a positive with a negative is a negative, $0/a = 0$. He was the first that I know of to give rules for multiplying and dividing negative numbers. He even has rules that we don't quite agree with today: $0/0 = 0$.

   Interestingly enough, later mathematicians did not always accept negative numbers. For example, Arabic mathematicians avoided negative numbers, and treated zero as just a place holder.

   (His works discuss geometry, algebra, Pythagorean triples, indeterminate equations including $8x^2 + 1 = y^2$ and $61x^2 + 1 = y^2$ with solution $x = 226153980$ and $y = 1766319049$, sine functions. Even has the formula for the first n squares. He used $\sqrt{10}$ for π.)

10. **Al-Khwarizmi** (c. 780 - c. 850). One of the earliest and most important Arabic mathematicians. Lived in Baghdad which was the capital of the Islamic world, and he worked at the famous House of Wisdom.

    The word "algebra" comes from the title of his most famous work *Hisab al-jabr w'al-muqabala* concerning calculation using the operations of *al-jabr* and *al-muqabala*. (The term *al-jabr* means "completion" or "restoration". It refers to adding to both sides of an equation to compensate for a subtraction. For example, going from $x^2 - 5 = x$ to $x^2 = x + 5$. The term *al-muqabala* means "balancing". It refers to the operation of canceling from both sides of the equation. For example, going from $x^2 + 3x = 5x + 1$ to $x^2 = 2x + 1$. With these two operations every quadratic equation can be reduced to one of al-Khwarizmi's standard forms.) This book discusses how to solve quadratic equations, but does not use symbolism: everything is written out in words. It justifies the procedures using geometric arguments: when al-Khwarizmi completes the square, he really completes the square. Of course, quadratic problems had been solved long before al-Khwarizmi, but he lays things out in a very organized, practical, and accessible way. Because of this his book was very influential. This book was very influential in Europe as well.[2] (This book also discusses linear equations, areas and

[2] It was translated by Robert of Chester into Latin in 1145.

volumes, and algebraic problems arising from dividing inheritances.)

Another important book, which we only know about through its Latin translation, is *Algoritmi de numero Indorum*. The word *Algoritmi* is a Latinized version of Al-Khwarizmi's name. This translated book was influential in Europe in spreading Hindu-Arabic numbers. It discusses how to calculate with these numbers, which were new at the time. It is from this book that we derive our word *algorithm*.

(Al-Khwarizmi also wrote on astronomy and trigonometry, based on Indian sources, geography, the sundial, the astrolabe, the Jewish calendar, and history. His geography improves on Ptolomy's).

11. **Omar Khayyam** or **'Umar al-Khayyami** (1048-1131). Famous Persian poet, mathematician, scientist, and philosopher. Wrote mathematics in Arabic but poetry in Persian. He also wrote a book on *al-jabr* and *al-muqabala* the first to take on general cubic equations. His solutions were geometric: involving the intersection of curves such as conics and circles. (He wrote this book in Samarkand, a famous ancient city now in Uzbekistan).

    He wrote a critique of Euclid (discussing the fifth postulate and the theory of proportion). He also reformed the calendar based on his extremely accurate estimates for the length of a solar year. (His idea was to have 8 leap years every 33 years. He is famous worldwide outside of mathematics for writing the poem called the *Rubaiyat*).

12. **Gerbert and the Medieval Translators**. Gerbert, who lived about the year 1000, was one of the earliest Medieval era Europeans to show an interest in mathematics. He studied in Spain, and knew about the Arabic contributions to mathematics. His math books were elementary by our standards, but they stimulated interest in mathematics. Gerbert later became the Pope. Because of his influence, there were several famous scholars who went to Spain during the 12th century to translate mathematical works from Arabic to Latin. Later translators translated Greek works into Latin. (Some of the Arabic writings were themselves translations of Greek works, so the Latin translations were two steps removed from the original source). Latin was the universal language of scholars of Western Europe, just as Greek was the universal scholarly language of Eastern Europe, and Arabic was the universal scholarly language of the Islamic world. The scholarly language of India was Sanskrit.

13. **Fibonacci**. His real name was **Leonardo of Pisa**. An Italian mathematician, the most famous and important medieval European mathematician. From a merchant family, he travelled widely in the European and Arabic worlds, and learned some Arabic mathematics. He advocated for the Hindu-Arabic numerals instead of the then current Roman numerals. Despite the efforts of Fibonacci and others, Roman numerals remained popular for several more centuries. He is most famous for his Fibonacci sequence motivated by a problem involving the population of rabbits. What are the Fibonacci numbers? How are they formed? His performance at a mathematics contest made him famous. (For example, he showed that the cubic $x^3 + 2x^2 + 10x = 20$ cannot be solved with square roots, but he gave a very accurate approximate solution 1.3688081075, good to 9 places. He also found a rational square such that if you add or subtract five, you still have a rational square: $(41/12)^2$. He realized, as did others, that if p and q are the sum of two squares, then so is pq.)

14. **Oresme** A French mathematician, and later a bishop. He did three important things, all

of which were ahead of his time: (i) first use of fractional exponents such as $8^{2/3}$ (not with this notation). (ii) developing the idea of a graph of a function displaying the independent variable (the input) as the first coordinate and the dependent variable (the output) as the second coordinate (which may have influenced Descartes). (iii) argued that the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

diverges. What was his proof of this divergence?

15. **Regiomontanus**. German mathematician of the early Renaissance. (His real name was Johann Müller). The best mathematician and astronomer of his generation. The first European to treat trigonometry as its own subject, and not just as a part of astronomy. (The Arab mathematician Nasir Eddin did this earlier). He learned ancient Greek in order to translate the Greek mathematical and astronomical classic. He even set up a printing press.

16. **Albrecht Dürer and the Renaissance artists**. Not only mathematicians, but artists became interested in mathematics during the Renaissance. For example, the artist Leonardo da Vinci became interested in the golden ratio, and the artist Dürer, from Germany, used a famous magic square in his art. These two artists, and many others, became interested in geometry, especially the theory of perspective. Two important perspective facts: (i) parallel lines appear to intersect in a point on the horizon, (ii) circles appear to be ellipses or other conic sections. In fact, parabolas can appear, in perspective, to be ellipses!

17. **Pacioli**. Renaissance Italian mathematician. A roommate of the artist, Leonardo da Vinci. He taught Leonardo mathematics. Leonardo drew illustrations for Pacioli's book *De divina proportione*. This book is responsible for modern interest in idea of the golden ratio. Wrote the first *published* algebra book called the *Summa* (1494). It has algebraic symbolism that became popular for a while, but different than our modern notation. It also had arithmetic, some elementary geometry, and bookkeeping!

18. **Robert Recorde and the Cossic Artists**. The term *coss* is German for "unknown" (the Italian term was *cosa* meaning "thing"). For much of the Renaissance, algebra was called the *cossic art*. Most cossic artists were German. The cossic artists were responsible for much of our modern symbolism including $+$ and $-$ and $\sqrt{\ }$. Robert Recorde was an English "cossic artist" who was responsible for the symbol $=$. He chose this since "no two things can be more equal" than parallel lines of the same length. Recorde was also a physician to royalty (including King Edward VI, and Queen Mary).

19. **Thomas Harriot** (1560-1621). English mathematician of the late Renaissance. Science advisor of Sir Walter Raleigh (where his duties included ship design, developing and teaching navigational techniques, surveying, keeping the financial accounts, and solving various mathematical problems put to him.) He was one of the first Englishmen to travel to North America: he was on Raleigh's expedition to Virginia (1585-86, perhaps also to Roanok Island 1584). He died of cancer of the nose caused by his habit of inhaling tobacco smoke, a practice he learned in Virginia.

    In algebra he introduced the symbols $>$ and $<$. Harriot advocated moving all terms of an algebraic equation to one side of the equation and setting this equal to zero (he was very comfortable with negative numbers). He made several other advances in algebra as well.

He came up with an interesting formula for the area of spherical triangle. Let H be the area of a hemisphere, and T the area of a triangle on a sphere.Then

$$\frac{T}{H} = \frac{E}{360}$$

where E is the excess angle sum: the sum of the three angles minus 180 degrees.

(He made observations with telescopes about the same time as Galileo: mountains on the moon, moons of Jupiter, and sunspots — he even found the period of rotation of the sun based on the sunspots. His interests included algebra, astronomy, optics, chemistry, trajectories of projectiles, the rainbow.)

20. **Cardano** (1501-1576). Famous Renaissance Italian physician, mathematician, notorious gambler, and astrologer (at one time he was an astrologer to the Pope). He convinced another mathematician (the famous Tartaglia) into telling him his method for cubic equations (1539). Cardano swore he would not tell anyone. Yet in his influential algebra book *Ars magna* "the great art" (1545) he explained how to solve cubics. (Cardano's student, Ferrari, claimed that Cardano also learned it from another source, and then extended it in original ways so he could legitimately write about it.) The *Ars magna* also has a solution to the fourth degree polynomial (discovered by his student Ferrari). It was only after 1800 that mathematicians (including Abel) were able to show that general polynomials of degree five or more cannot be solved with these types of algebraic methods. (He also was a forerunner to probability, due partly to his gambling.)

21. **Bombelli** (1526-1572). Renaissance Italian mathematician. Wrote a famous algebra book that explains the role of imaginary and complex numbers in solving cubic (and other types of) equations. (The book is called *Algebra* and it was published in 1572.)

22. **Copernicus** From Poland. The most famous Renaissance astronomer. He contributed to trigonometry as well. He revolutionized astronomy with the heliocentric theory. (It had not been proposed seriously since Aristarchus).This theory, now the basis for our idea of the Solar System, was very controversial at the time.

23. **Kepler**. Late Renaissance. He was a big fan of Copernicus' theory. He tried to explain the paths of the planets in several flawed ways (including comparing inscribed and circumscribed circles). Finally he hit on his three laws. What are the three laws of planetary motion that he discovered? Are they based on observational data, or only speculation? Answer: data. Did he explain these laws in terms of inverse square laws (or did that come later)? Answer: that came later with Newton.

3 Historical Dates, Events, and People

This test covers three periods: Ancient, Medieval, and Renaissance. The Ancient period lasts until the fall of the Roman Empire. In the Roman Empire, most mathematics was done by Greek writers, often in Alexandria. The Medieval Period includes Chinese, Indian, Arabic, Mayan, and European accomplishments. At the beginning of this period, the (western) Roman Empire had fallen, and western Europe was in the dark ages. The eastern Greek speaking half of the Roman Empire remained and evolved into the *Byzantine Empire*. Their mathematics was not of the highest rank, but at least they kept the tradition alive. The term *Renaissance* is used mainly to denote the flowering of culture that occurred in Italy and other parts of western and central Europe.

1. **The Roman empire**. The major cities included Rome, of course, but also Constantinople (or Byzantine, today Istanbul) in Asia Minor (today Turkey). However, Alexandria remained a center of mathematics and culture. In the western half of the Roman Empire, people spoke Latin. (In fact, Spanish, French, Italian, Portuguese and even Romanian all descend directly from Latin. These languages are called *Romance Languages*.) In the eastern half of the Roman Empire, including Alexandria, people continued to write in Greek.

2. **476: The fall of the (western) Roman Empire**. The eastern Roman Empire lasted for almost 1000 years after this, and evolved into the Byzantine Empire. Rome was conquered by Germanic tribes.[3]

3. **The growth of Islam**. The start of the Muslim calendar was our year 622. During the dark ages in Europe, Muslim culture prospered. The center of culture was Baghdad (although the religious center was Mecca). The House of Wisdom was in Baghdad.

4. **The development of the Hindu-Arabic number system**. From about 600-800. Until Al-Khwarizmi, our information is very sketchy.

5. **Renaissance** in Europe. Know the date 1450: the approximate date of the fall of Constantinople (eastern Roman empire) and the beginning of printing in Europe. The height of the Renaissance (in Italy and Central Europe. It would wait to come to England until the late 1500's).

4 Mathematical Topics

1. **Number Systems**. Be able to use Mayan, Roman, and Chinese rod numbers. (Roman up to 4000). See Sketch 1 in *Mathematics through the ages*. For the Chinese rod numbers see the Chinese numerals entry at Mac-Tutor.

   `www-groups.dcs.st-and.ac.uk/~history/HistTopics/Chinese_numerals.html`

2. **Zero**. The Babylonians used a space to indicate a zero in a given sexagesimal position. Later Babylonians used a dot to indicate such a space. Greek astronomers including Ptolomy used a symbol for zero. Indian mathematicians treated zero as a number by itself, and not just as a placeholder. Brahmagupta and others gave rules for using zero. Of course, zero became an essential ingredient of the Hindu-Arabic number system (base ten positional). However, most civilizations outside of India treated zero as just a placeholder until after the Renaissance. Harriot (and Descartes) changed all that by advocating solving polynomial equations by moving all the terms to one side and setting them equal to zero. This way of finding roots shows how useful zero is in algebra. The Mayan's independently invented zero as a placeholder in their base 20 system

[3] England was also conquered by Germanic tribes, the Angles and the Saxons. The English language is a Germanic language, but it has a huge amount of French and Latin influences as well. In fact, English draws from many languages for its vocabulary. We have seen the importance of Greek words to our English vocabulary.

3. **Negative numbers**. Chinese and Indian mathematicians were comfortable, more or less with negative numbers. However, later Arabic and European mathematicians were not (they were comfortable with subtraction, but only if the result is positive). It was only after books such as Cardano's *Ars Magna* were published that European mathematicians became comfortable with negative numbers.

   Negative number, and even complex numbers, help tremendously in the solution of polynomial equations.

4. **Positional Decimal System** (and the **Hindu-Arabic Numerals**). Developed in India from 600 to 800. This system consists of three ideas: (i) base 10, (ii) positional notation, it only needs symbols for $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ since the power of 10 is indicated by the position, and (iii) the use of 0 as a placeholder. The Chinese almost had a positional decimal system. The Babylonians had (ii) and laer (iii), but without (i). Many civilizations had (i) but not (ii) or (iii).

   Although the symbols for $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ evolved over time, the numerals built out of them are called the *Hindu-Arabic numerals* since they were invented by Hindu mathematicians in India and popularized by the Arabic mathematicians including al-Khwarizmi in his book *Algoritmi de numero Indorum.* In Medieval Europe, Fibonacci and others advocated for its use insead of the clunkier Roman numerals. However, it took centuries before the Roman numerals were completely replaced. (In fact, we still use Roman numerals for special purposes. I have used Roman numerals several times in this study guide!).

5. **Fractions**. Four types; (i) Unit fractions used in Egypt, and even by the Greeks and some Europeans before the Renaissance. (ii) Common fractions $\frac{a}{b}$ used in China (in the *Nine Chapters*) and India. Common fractions then were used in the Arab world and in Europe. (iii) Sexagesimal fractions from Babylon, but popular with Greek astronomers, and used even today in astronomy, geography and in telling time — factions of an hour are given in sexagesimal by using minutes and seconds, (iv) Decimal fractions. Chinese mathematicians liked decimal fractions. The Arabic mathematician Al-Khashi popularized decimal fractions. In the late Renaissance, they started catching on in Europe. In fact, the only type of fraction that has fallen out of use is the unit fractions (but mathematicians are still interested in some of their mathematical properties, but mainly as a curiosity).

6. **Trigonometry**. Astronomers were the pioneers of trigonometry. The Greek geometers (including Ptolomy) used chord tables (with sexagesimal fractions). The Indian mathematicians, who were very interested in astronomy and knew Greek astronomy, developed the idea of using half-chords, which give us the sine function. Sines are more convenient than chords. Arabic mathematicians learned the idea of sines from India. During Medieval times, the European mathematicians learned about sines from the Arabic mathematicians. Can you draw a figure showing that $\text{chord}(\theta) = 2\sin(\theta/2)$? (Can you draw a figure showing a tangent line segment of length $\tan(\theta)$?)

7. **Systems of Linear Equations**. All cultures could handle one equation and one unknown, or two equations and two unknowns. The Chinese classic *The Nine Chapters* handled up to six equations and six unknowns. It used matrix manipulations to solve such systems.

8. **Cubic Equations**. There are three old methods for solving cubic: the geometric method of Omar Khayyam, the algebraic method of Cardano (et al.), and the numerical method (for example, Fibonacci). Be able to solve the equation $x^3 + 2x - 5 = 0$ using Cardano's method. Realize that sometimes finding even simple real roots involves using the complex numbers and that Bombelli explained how to use complex numbers.

9. **Cyclic Quadrilaterals**. Can you state Ptolomy's theorem? Can you prove it? Can you state Brahmagupta's formula? Can you relate it to Heron's formula?

10. **Harmonic Series**. What is it? What was Oresme's argument for its divergence.?He was ahead of his time: diverging series were studied much later.

11. **Approximations of** π. Remember that Archimedes gave an approximation for π by using inscribed and circumscribed 96-gons. This gives the famous 22/7 approximation for π. (His actual estimate was $3\frac{10}{71} < \pi < 3\frac{1}{7}$). Later mathematicians improved on this bound. However, until modern times, the method used was essentially Archimedes' but with more and more sides on the polygon. For example, Ptolomy got a slight improvement using 360-gons. Chinese mathematicians were very good at approximating π, and later Arabic mathematicians were very good as well. (Zu Chongzhi approximated to seven decimal places with 355/11 in the year 480, a record that was not surpassed for almost 1000 years. The arab mathematician al-Kashi calculated π accurately to 14 digits in 1430, a record that was not surpassed for over a century and a half.[4].)

12. **Spherical triangles.** Angles do not add up to 180 degrees. In fact, you can measure area by finding the sum of the angles; be able to do this. This is a simple example of non-Euclidean geometry. Spherical trigonometry was an important topic in astronomy and trigonometry in the history of mathematics.

[4]In fact, there is no practical reason to get more accuracy than Al-Khashi. However, determined people kept going. Notable steps after Al-Khashi include 35 places in 1596, 100 places in 1706, and 200 places in 1824. Today, computers can calculate billions of digits of π. In fact, calculating π used to be a method of testing computers. Brahmagupta had an interesting, but not very accurate, irrational approximation: $\sqrt{10}$.