A Chinese Classic: The Nine Chapters

Math 330: History of Mathematics

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1 Introduction

The *Jiuzhang suanshu*, or *Nine Chapters on the Mathematical Art* was written for the training of civil servants in the Han dynasty (206 BC - AD 220). However, some of the material might have dates from as far back as 1000 BC. It remained in use until 1600 AD. It was as important in Chinese mathematical education as Euclid was to Greek mathematical education.

In some ways this classic resembles the earlier Egyptian and Babylonian problem texts. For example, it consists of "word" problems, and their solutions, and very little in the way of proof. In all there were 246 problems relating to taxation, construction, land measurement, and other aspects of practical commerce.

The Chinese Mathematicians Liu Hui wrote a commentary in the 3rd century AD which justified many of the methods. His justifications can be considered to be proofs. He also added an addendum, called the *Sea Island Mathematical Manual* consisting of nine extra problems involving surveying.

The Nine Chapters contains accurate procedures for finding the volume of the pyramid and the truncated pyramid. For example, the truncated pyramid is given in exactly the same way as in the Egyptian papyrus (the Moscow Papyrus). It is equivalent to the formula $\frac{h}{3}(a^2 + ab + b^2)$. Unlike the Moscow Papyrus, Liu Hui's version of the Nine Chapters does contain commentary explaining why the solution is valid. Some areas and volumes were calculated correctly, others were only approximations. For example, for the area of the circle they often used $(c/2)(d/2)$ where c is the circumference and d is the diameter. This is correct (as we saw when we studied Archimedes). In other places they used $C^2/12$ or $3d^2/4$ which are correct only if you assume $\pi = 3$ (but don't pass judgement: later Chinese mathematicians came up with very good approximations for π).

The Chinese mathematicians were more sophisticated in working with fractions than their counterparts in Greece, Egypt, and Babylon. They had all the usual rules worked out for common fractions $\frac{a}{b}$. (The Egyptians and many Greeks used unit fractions. The Greeks were also very proficient with ratios $a : b$ which are similar to fractions, but the Greeks lacked some of the rules we use for manipulating, adding, and multiplying fractions. The Babylonians were proficient only with fractions involving denominators that were a power of 60.)

Below are problems adapted from this classic. I have changed the wording to make it easier to read, but I have kept the same premise and constants.

Since I do not have access to the primary source, I have made good use of high-quality secondary sources. My main source has been

Victor J. Katz, *A History of Mathematics*, Addison Wesley (second edition 1998).

Chapter 1 of the Nine Chapters

Contains several problems concerning areas and fractions.

Chapter 2 of the Nine Chapters

Contains several problems involving exchange rates.

Problem 3. Suppose that 50 units of millet has the same value as 24 units of white rice. How many units of white rice can you get for $4\frac{5}{10}$ units of millet? (The units are units of volume).

Chapter 3 of the Nine Chapters

Contains several problems related to proportional distribution.
Problem 1. There are five officials of different ranks. Suppose they obtained five deer to be divided among themselves. They decided that the proportions should be $5 : 4 : 3 : 2 : 1$. In other words, the highest ranking official gets five times as much as the lowest ranking official, the second highest gets four times as much as the lowest ranking official, etc. How much does each official get?
Problem 3. Three people are required to pay taxes. Suppose their taxes are proportional to their wealth. Suppose the first has 560 coins, the second has 350 coins, and the third has 180 coins. If the total taxes is 100 coins, how much did each pay?

Chapter 4 of the Nine Chapters

This chapter explains how to find square roots and cube roots. Also contains problems related to division by fractions, dimensions, areas and volumes of circles and spheres.

Chapter 5 of the Nine Chapters

Contains several volume problems.

Chapter 6

Contains more problems involving rates and proportions.

Problem 26. A reservoir has five channels bringing in water. If the first channel only is open it takes 1/3 of a day to fill it. If the second channel only is open it takes 1 day to fill the reservoir. If the third channel only is open it takes 21/2 days. If the fourth only is open it takes 3 days. If only the fifth is open it takes 5 days. How long does it take to fill the reservoir if all five channels are open at once?

Problem 28. A man carries a load of rice on a journey. He passes a custom station which takes 1/3 of his rice. Later he passes a second station which takes 1/5 of his current rice. At the third and final custom station, he has to give 1/7 of his current load of rice. He ends his trip with 5 units of rice. How much rice did he start with?

Chapter 7

This chapter contains 20 problems: each of which involves two linear equations and two unknowns. They are solved using a method called "surplus and deficiency".

Problem 1. Suppose a group of people want to join together to buy an item. If each contributes 8 coins then there is 3 more coins than needed. If each contributes 7 coins, then there are 4 coins too few. How much does the item cost, and how many people are in the group?

Problem 17. Suppose one unit of good land costs 300 gold pieces, but 7 units of bad land can be purchased for 500 gold pieces. Someone buys 100 units of land for 10,000 gold pieces. How much of each type of land was purchased?

Chapter 8

This chapter contains 20 problems: each of which involves n linear equations and n unknowns with $3 \leq n \leq 5$. They are solved by writing something like a matrix and using something like Gaussian elimination. Negative numbers were allowed: the Chinese had correct rules for dealing with addition and subtraction of negative numbers.

Problem 1. There are three types of grain. Three bundles of the best, plus two bundles of the second best, plus one bundle of the worst give a total of 39 units of flour. But two bundles of the best, plus three bundles of the second best, plus one bundle of the worst give a total of 34 units of flour. Finally, one bundle of the best, plus two bundles of the second best, plus three bundles of the worst give a total of 26 units of flour. How many units of flour can be made from one bundle (for each type of grain)? The Chinese mathematicians wrote each equation as a column:

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 1 & 1 \\ 26 & 34 & 39 \end{matrix}$$

Of course we would write each equation as a row:

$$\begin{bmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{bmatrix}$$

Problem 3. Using 2 bundles of the best grain does not quite make up one unit of flour. Likewise, 3 bundles of ordinary grain, or 4 bundles of inferior grain are not enough for one unit of flour. To make one unit we need to add one bundle of the good to the 3 bundles of the ordinary. Or we could add one bundle of the ordinary to the 4 bundles of the inferior. Finally, we could add one bundle of the inferior to the 2 bundles of the best to get the one unit of flour. How much flour does one bundle of each type of grain yield. (The solution involves negative numbers).

Problem 13. Has five equations and five unknowns. In modern notation: $2x + y = 721$, $3y + z = 721$, $4z + u = 721$, $5u + v = 721$, $x + 6v = 721$.

Chapter 9

This chapter contains problems concerning right triangles and quadratic equations. It assumes as known the Pythagorean theorem (another Chinese text gives an argument for the Pythagorean theorem). All the answers to the questions are rational since the authors were careful to base the problems on Pythagorean triples. This shows that the Chinese knew how to generate Pythagorean triples. Some examples in Chapter 9 are $(3, 4, 5)$, $(5, 12, 13)$, $(48, 55, 73)$, and $(60, 91, 109)$.

Quadratic equations were probably solved with procedures based on dissections of squares and rectangles (similar, but different than the dissections probably used by the Babylonians. Book 2 of Euclid's elements also gives such dissections).

Problem 6. Suppose a pond is shaped like a square with sides of length 10 units. Suppose in the exact center of the pond, a reed with roots in the bottom of the pond grows up 1 unit above the water level. Suppose you grab the top of the reed and drag it to the side of the pool and find that the top of the reed is exactly at water level. What is the depth of the water and the length of the reed?

Problem 8. A pole leans against a wall so the top of the pole is even with the top of the wall. The wall is 10 units high. You only need to pull the base of the pole 1 unit further away from the wall in order to force the top of the pole to the ground. What is the length of the wall?

Problem 9. I do not know what this problem is, but solving this problem results in the equation $x^2 + 34x = 71,000$ which has solution 250.

Problem 11. Suppose the corners of a certain door are 10 units apart. Suppose also that the height is $6\frac{8}{10}$ units larger than the width. Find the dimensions of the door.

Problem 14. Suppose two people, X and Y, start at the same point. X goes east with a speed of 3. But Y goes south for 10 units, then heads in a northeasterly direction and meets X at a certain point. If Y has a speed of 7, how far did X and Y go?

Problem 30. A square shaped city is surrounded by walls. Each side has a gate each located in the center of a side of the square. There is a tree 20 units north from the north gate. If you walk 14 units southward from the south gate, then you have to walk 1775 units west before you can see the tree. What are the dimensions of the city?