# ESSAY ON CAVALIERI'S PRINCIPLE

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This is an essay based on class notes for Math 330. It is still a relatively early draft. For this draft, many of the results I quote are drawn from memory, but I plan to add citations to more authoritative sources in future drafts.

## 1. Introduction

Cavalieri's principle is technically not just one principle, but several related principles. They were advocated by Bonaventura Cavalieri, an Italian Monk and student of Galileo, in the 17th century. Archimedes had a similar idea much earlier, but only used it to suggest results. Archimedes considered it as a method of discovery, not a method of proof. Archimedes used another method called the *method of exhaustion* in his final proofs.

In fact, in Cavalieri's day his method was controversial since it was not justified with a rigorous proof. In spite of this many mathematicians including Galileo thought it was a great idea. Today, we can rigorously justify his principle using real analysis.

## 2. Equal Volumes

One version of Cavalieri's principle is as follows: *given two solids both resting on the same plane, if the solids have equivalent cross sectional areas for every horizontal plane, then they have the same volume.* The solids do not have to be connected. In the theorem in the next section, the first solid is the union of a sphere and an inverted cone, and the second is a cylinder.

As an easy example, two cones with the same total height $H$ and base area $B$ can be shown to have the same cross sectional areas for every horizontal plane. The base of the cones do not have to be the same shape: they can be circular (a cone), rectangular, square (a classic pyramid), triangular (like a tetrahedron), a general polygon, etc. If you can prove that one cone or pyramid of a given height and base has volume $BH/3$, then you can use the principle to prove it for all cones and pyramids of the same $B$ and $H$.

The original justification for this principle is non-rigorous. You think of a solid as a "stack" of cross sections. If each layer of a stack has the same area, it is plausible that the total stack should have the same volume. The criticism of the principle is based on the fact that there are really an infinite number of layers, each of zero volume, and it is not obvious that

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something we take for granted for a finite stack would apply to infinite stacks as well. Today we know that volume is the integral of the cross sectional area function, so if two solids have the same cross section area function they must have the same volume. In other words, modern analysis vindicates Cavalieri.

## 3. Volume of the Sphere

Earlier in class we learned about Archimedes theorem on the sphere and the cylinder, but we did not study the proof due to its complexity. However, if you accept Cavalieri's principle and coordinate geometry, there is a very short proof of this result. (The coordinate geometry is not strictly necessary, but it helps make the proof accessible).

Imagine that we have three solids: a hemisphere $A$, an inverted cone $B$, and a cylinder $C$. Assume that the hemisphere has radius $R$, and that the cone and cylinder both have base radius $R$ and height $R$. So all three solids have the same height.

We begin by looking at the cross-sectional areas.

**Lemma 1.** *The cross section of the hemisphere $A$ with the plane $z = z_0$ is a circle with area*

$$\pi(R^2 - z_0^2).$$

*Here $0 \le z_0 < R$.*

*Proof.* The hemisphere is defined by $x^2 + y^2 + z^2 = R^2$ where $0 \le z \le R$. (We can ignore $z = R$ since it just gives a point). If we intersect this with the plane $z = z_0$ then we get the region in the $z = z_0$ plane defined by the equation

$$x^2 + y^2 = R^2 - z_0^2$$

Recall that the equation $x^2 + y^2 = c$ yields a circle of radius $\sqrt{c}$. Thus the cross section is a circle of radius

$$r = \sqrt{R^2 - z_0^2}.$$

This circle has area

$$\pi r^2 = \pi \left(\sqrt{R^2 - z_0^2}\right)^2 = \pi(R^2 - z_0^2)$$

□

**Lemma 2.** *The cross section of the inverted cone $B$ at height $z = z_0$ is a circle of radius $z_0$ and so has area $\pi z_0^2$. (Here $0 < z_0 \le R$).*

*Proof.* An inverted cone has equation $x^2 + y^2 = kz^2$. In order for the radius and height to be equal, we must have $k = 1$.

The intersection with the plane $z = z_0$ yields the equation $x^2 + y^2 = z_0^2$, which is a circle of radius $z_0$. □

*Remark* 1. Our proofs of the above two lemmas use coordinate geometry, which was not available in Cavalieri's time. However, you do not really need coordinate geometry to prove these lemmas. The lemma for the hemisphere can be proved with the Pythagorean theorem instead, and the lemma for the cone can be proved with similar triangles instead.

**Lemma 3.** *The cross section of the cylinder $C$ at height $z = z_0$ is a circle of radius $R$ and area $\pi R^2$*

*Proof.* Every cross section of a cylinder is a circle of the same radius and area. Since the base has radius $R$ and area $\pi R^2$, the same is true of all the cross sections of the cylinder. □

Cavalieri's principle can now be used:

**Theorem 4.** *Let $V_A, V_B, V_C$ be the volumes of the solids $A, B$, and $C$ described above. Then*

$$V_A + V_B = V_C.$$

*Proof.* We take Cavalieri's principle as given. By the above lemma the combined cross section of $A$ and $B$ with the horizontal plane $z = z_0$ has area

$$\pi(R^2 - z_0^2) + \pi z_0^2 = \pi(R^2 - z_0^2 + z_0^2) = \pi R^2.$$

The right hand side of the equation is the cross sectional area of $C$. This equality holds for all horizontal planes, so, by Cavalieri's principle, the solids $A$ and $B$ together have the same volume as $C$. □

Suppose that the volume formulas for cones and cylinders are known, but the formula for a sphere is not known. This is historically accurate: the volume of the sphere was discovered last. Then we can use the above theorem to derive the volume formula for the sphere.

**Corollary 5.** *The volume of the sphere of radius $R$ is $\frac{4}{3}\pi R^3$.*

*Proof.* We know that $V_A = V_C - V_B$ by the above theorem. Both $B$ and $C$ have base area $\pi R^2$ and height $R$. Thus

$$V_A = V_C - V_B = (\pi R^2)R - \frac{1}{3}(\pi R^2)R = \frac{2}{3}\pi R^3$$

This the the volume of the hemisphere. Double it and we get the volume of the sphere. □

*Remark* 2. There are slight variations to Theorem 4 that involve only two solids, and so may be appealing. You can define one solid to be the hemisphere, and the second to be a cylinder with an inverted cone carved out of it. Or you can define one solid to be the cylinder with a hemisphere carved out, and the other to be the inverted cone.

## 4. Proportional Versions of Cavalieri's principle

There is an area version of Cavalieri's principle:

*Suppose two two-dimensional regions $A$ and $B$ have equal cross sectional lengths for all horizontal lines. Then $A$ and $B$ have the same area.*

One can use this to show, for instance, that all triangles with the same base length and height have the same area. Of course, this basic result can be proven without Cavalieri's principle.

A more sophisticated and powerful version is as follows: *Suppose $A$ and $B$ are two two-dimensional regions and $k$ is a constant such that, for every horizontal line, the length of the cross section of $B$ is $k$ times the length of the cross section of $A$. Then the area of $B$ is $k$ times the area of $A$.*

This can be used to prove the following theorem (by comparing the ellipse with the circle having equation $x^2 + y^2 = b^2$).

**Theorem 6.** *The ellipse defined by*

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

*has area $\pi ab$.*

*Remark* 3. For students in my Fall 2007 class: the proof of Theorem 6 will not be on the final exam. However, the proof of Theorem 4 will be on the final.