# STUDY GUIDE FOR TEST 2

MATH 303. SPRING 2006. INSTRUCTOR: PROFESSOR AITKEN

The test will cover Chapters 4, 5, and 6.

## Chapter 4: The Mathematics of Voting

*Sample Exercises*: 1, 3, 5, 7, 8, 10, 14, 15, 17, 20, 23, 25, 28, 31, 32, 33, 36

**Concepts:** Understand the following.

- *Standard divisor.* This is the total population divided by the total number of seats $M$. (See page 140, and Exercises 1, 3 on pages 160–161.)
- *Standard quotas.* This is the state's population divided by the standard divisor. (See page 140, and Exercises 1, 3 on pages 160–161.)
- *Upper and lower quotas.* The lower quota and upper quota (of a state, etc.) are whole numbers related to the standard quota. Round the standard quota down and you get the lower quota. If you round the standard quota up then you get the upper quota. Usually the upper quota is one more than the lower quota. However, in the unusual case that the standard quota is a whole number then the lower quota equals the upper quota (and both equal the standard quota). (See page 141, and Exercises 1, 3 on pages 160–161.)
- *Modified Divisors and Modified Quotas.* These are used in many of the methods including Jefferson's, Webster's, Adam's, and Huntington-Hill. (They are not used in Hamilton's). (See page 149)

  The modified divisor is a number fairly close to the standard divisor, but must be found by trial and error.[1] There is usually a range of valid modified divisors. On the test I will give some hints to keep you from wasting time. The modified quota is the state's population divided by the modified divisor.
- *Balinsky and Young's impossibility theorem.* Know what it says, and that it arose around 1980. It states that any apportionment method that does not violate the quota rule must sometimes produce paradoxes, and any method that does not produce paradoxes must sometimes violate the quota rule. In other words, there is no perfect apportionment method.

**Apportionment Methods**: Be able to use the following.

- **Hamilton's Method**. Begins with lower quotas, but gives some states their upper quotas. Which states? (Page 142–143, and Exercises 7, 8, 10, 14, 15 on pages 161–163). Know that this method favors large states, and can produced several types of paradoxes. This method was the first ever proposed, and was commonly used in the 1800's.

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[1]There is an algebraic method that takes too much time to explain.

- **Jefferson's Method**. Uses modified divisors and modified quotas. In this method modified divisors are rounded down. (Page 148–150, Exercises 17 and 20 on page 163). Know that this method can violate the quota rule by giving large states more than their upper quota. This method was the first ever used, and was used several times before being discontinued.
- **Adam's Method**. Uses modified divisors and modified quotas. In this method modified divisors are rounded up. (Page 151–152, Exercises 25, 28, 31 on pages 163–164). Know that this method can violate the quota rule by giving large states less than their lower quota. Know that this method favors small states. It was proposed but never used by the House of Representatives.
- **Webster's Method**. Uses modified divisors and modified quotas. In this method modified divisors are rounded down if the decimal part is $< .5$ and rounded up if the decimal part is $\geq .5$. (Page 152–154, Exercises 33 and 36 on page 164). Know that this method can violate the quota rule, but does so rarely. This method was commonly used before the Huntington-Hill method.
- **Huntington-Hill Method**. You won't be asked to work any examples of the Huntington-Hill method, but know that it is the method used from 1941 to the present, that it is similar to Webster's method, and although it words well it it is still sometimes controversial. (Page 171 gives more than enough information)

  This method uses modified divisors and modified quotas. In this method modified divisors are sometimes rounded down and sometimes rounded up. (The dividing point between rounding up and rounding down is based on geometric means).

**Proplems and Paradoxes**: Know the following, and in what methods they can arise. (See Table 4-22 on page 155).

- **Quota rule violation**. Ideally every state should get either their lower quota or their upper quota. However, many methods (in fact all the methods we studied except Hamilton's) can lead to violations of this rule. (Pages 150–151, Exercises 23 and 31 on pages 163–164).
- **Alabama Paradox**. Normally if we increase $M$ (the number of seats) by one, one state goes up in seats but the rest stay the same. The Alabama Paradox is the unusual situation where $M$ goes up by one, but two (or more) states increase seats while one (or more) decrease seats. (Pages 144–145. Exercise 14 on page 162. Exercise 32 on page 164. The answer to Exercise 32 is *no* since Adam's method never gives the Alabama paradox.)
- **Population Paradox**. Normally if some populations go up and others stay the same (or go down) the growing states should not have to give up seats to the non-growing states (when keeping $M$ the same). The Population Paradox is where a growing state loses a seat to a state that is not growing. (Pages 145–147.)
- **New State Paradox**. Normally if we add a new state, and increase $M$ to give the new state the correct number of seats, then the other states should stay the same. The New-State Paradox is where one of the old states loses a seat and another old state gains a seat. (Pages 147–148. Exercise 15 on page 162.)
- **Bias to large or small states**. Hamilton and Jefferson's methods favor large states, while Adam's method favors small states (see page 155.)

**History**: Study the historical section on pages 156–158. Be able to answer the following questions.

- Does the constitution stipulate an apportionment method? (Answer: no, it just states that the number of seats in the House of Representatives should be based on population census carried out every 10 years.)
- Who proposed the first apportionment bill (in 1792)? (Hamilton)
- Who vetoed that bill?
- What was the first ever presidential veto in U.S. History?
- What was the first method proposed in U.S. history? (Hamilton) What was the first method actually used in U.S. history? (Jefferson)
- When was Jefferson's method discontinued? (In the early 1800's)
- Why was Jefferson's method discontinued? (New York got more than its upper quota, violating the quota rule)
- What methods were used after Jefferson's method was discontinued? (Webster's and Hamilton's)
- What method is used today, since 1941? (Huntington-Hill)
- Does the constitution mandate the number of seats $M$ ? (No)
- Has the number of seats $M$ been the same throughout U.S. History?
- Since 1941, how many seats does congress have? ($M = 435$)
- Which state recently challenged the use of Huntington-Hill? (Montana in 1991 after dropping from 2 seats to 1 seat.)

**Other**: Be able to use these methods for situations other than apportionment of congress. (For example, see Exercises 3, 4, 10, 14, 20, 28, 36 on pages 160–164).

## Chapter 5: Euler Circuits

*Sample Exercises:* 5, 7, 8, 15, 19, 20, 21, 25, 26, 29, 30, 33, 35, 36, 37, 38, 41, 43, 51, 52, 53, 60, 63, 66.

**Concepts:** Understand the following..

- *Graph.* (See Definition and examples on pages 183–186. Exercises 5, 7, 8, page 204)
- *Vertices.* (See Definition and examples, pages 183–186. Exercises 5, 7, 8, page 204)
- *Edges.* (See Definition and examples on pages 183–186.)
- *Degree.* (Page 186. Exercises 7 and 8 on page 204.)
- *Path.* (Pages 186–187)
- *Circuit.* (Page 187)
- *Connect Graphs.* (Page 187)
- *Bridge Edges.* (Bottom of page 187, top of page 188)
- *Euler Path.* (Page 188. Exercises 25, 26, 33, 35, 36 on pages 208–210)
- *Euler Circuit.* (Page 188. Exercises 25, 26, 29, 30 on pages 208–209)
- *Euler's Theorem.* Know these, and be able to answer questions based on them. (Pages 190–192. Exercises 25 and 26 on page 208. Exercises 37, 39 on page 210.)
- *Routing Problem.* This is the problem of finding efficient delivery routes. Our solutions involve making a graph of the situation and Eulerizing if necessary. (Page 180, Examples 5.1 and 5.2 on pages 180–181. See also Examples 5.23 and 5.24 on pages 199–200. Exercise 51, 52, 53 on page 213.)

**Methods**: Be able to use the following methods.

- **Fleury's Algorithm**. A method for finding Euler circuits and paths (when they exist). (See pages 193–196. Exercises 29, 30, 33, 35, 36 on page 209–210)
- **Eulerize**. Adding extra, redundant edges so as to make each degree even. (Pages 196–200. Exercise 41 on page 211. Exercises 52, 53 page 213.)
- **Semi-Eulerize**. Adding extra, redundant edges so as to make all but two degrees even. (Page 198–199. Exercise 43 on page 211.)

**Other**:

**1**. Know how to find degrees if your graph has loops connecting to one vertex (page 186).

**2**. Be able to tell if a bridge problem (like the Köningsberg bridge problem) has a solution. (Exercise 60 and 66, page 214–216).

**3**. Know about the history of the Königsberg bridge problem, and Euler's role in its solution.

**4**. Be able to tell if a graph has an Euler circuit or an Euler path.

**5**. Be able to tell where the starting and ending vertices of an Euler path will be.

**6**. Be able to tell if a unicursal tracing is possible. (Page 183, 191. Exercises 37, 39 on page 210).

**7**. Find graph models to represent information (see Exercises 15–22).

**8**. Find Euler paths and circuits involving house plans (see Exercise 63 on page 215).

**9**. Given a bridge problem or a house plan problem (see Exercise 63 on page 215) be able to tell me how to modify it so that Euler circuits or paths are possible. For example, what bridges can you blow-up, or what doors can you add.

## Chapter 6: TSP Problems

*Sample Exercises:* 1, 3, 5, 11, 15ab, 19, 23, 24a, 25, 29, 31, 32, 37, 39, 41, 43.

**Concepts:** Understand the following.

- *Hamilton Circuits and Paths.* (Pages 223–224. Exercises 1, 3, 5, 11 on pages 247–250.)
- *Complete Graph.* In complete graphs, there are lots of Hamilton circuits. In fact there are $(N-1)!$ of them where $N$ is the number of vertices. The fact that there are so many is one reason that there are no algorithms that are efficient and optimal. (Pages 225–227. Exercises 15ab, 19 on page 251.)
- *Weighted Graph.* (Pages 229–230. Problem 11 on page 250. Problem 43 on page 259.)
- *The TSP problem.* (Pages 227-229, and page 230).
- *Efficient Algorithm.* (Page 236).
- *Inefficient Algorithm.* (Page 235).
- *Optimal Algorithm.* (Page 236).
- *Approximate Algorithm.* (Pages 236–237).

**Methods**: Be able to use the following methods.

- **Brute-force Algorithm**. Also know the short-cut that cuts this in half. Know that this method is impossible when $N$ gets pretty large. This is an inefficient, optimal algorithm. (Page 234. Exercise 23a and 24a on page 252.).
- **Nearest-Neighbor algorithm**. This is an efficient, approximate algorithm. (Page 234. Exercise 23, 25, 29 on pages 252–254.)
- **Repetitive Nearest-Neighbor algorithm**. This is an efficient, approximate algorithm. (Page 237–239. Problems 31, 32 on page 255.).
- **Cheapest-Link algorithm**. This is an efficient, approximate algorithm. (Pages 239–241. Problems 37, 39, 41 on pages 257–258.)

**Other**:

**1**. Be able to give several examples of TSP problems that do not involve a salesperson. (Pages 228–230.)

**2**. Give an example of a graphs with an Euler circuit, but no Hamilton circuit. Give an example of a graphs with a Hamilton circuit, but no Euler circuit. Example with neither. Example with both. (Page 224).

**3**. Know how many Hamilton circuits can be produced for a complete graph with $N$ vertices. (Page 237. Problem 15ab, 19b on page 251).

**4**. Be able to apply the algorithms to mileage charts. (problem 29 on page 254, and problem 41 on page 258).

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