Artin's First Article on the Artin L-Series (1924): *Paraphrasis* and Commentary

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This document started as a kind of mathematically oriented freestyle translation of E. Artin's *Über eine neue Art von L-Reihen* ("A New Kind of L-Series") [3] with added commentary.[1] However, I made enough changes in notation, terminology and even in the details of proofs that the term *translation* is perhaps not entirely appropriate. Instead, I decided to borrow the Latin term *paraphrasis* which was in turn borrowed from Greek. This term connotes a very close connection to the original article, while giving me license to make adaptations here and there for the benefit of a modern reader. It also allows me to focus on the mathematical description and development without the responsibility capturing the subtle nuances of German mathematical writing style from almost one hundred years ago.[2] This license allows me to sneak in clarifying devices, such as commutative diagrams, which were not current in the 1920s.

My goal is to give a modern reader access to the mathematics of [3], but not necessarily to follow the stylistic and mathematical conventions of the 1920's. For example, in his article Artin does not use the usual notation $\mathcal{O}_k$ for the ring of integers of a number field k, but instead just speaks of integers in k. He speaks of prime ideals in k instead of in the the ring of integers $\mathcal{O}_k$. This is just one of several conventions that I have chosen not to follow in this *paraphrasis*. In one case I have even changed the statement of a result, and the proof, to be a bit more general since it was easy to do so using Artin's methods. I have even added section titles to supplement Artin's simple section numbering, and I have added a bibliography. So the reader should expect these sorts of changes from Artin's original. However, I hope these changes are in a mathematical sense all minor, and that I have captured the spirit of things well. My intent is to open up this great landmark of mathematics to modern number theorists and transport the reader to the genesis of the all important Artin L-Series. I have been faithful the order

 Thanks to Jason Martin for his comments and suggestions on earlier drafts.

[1] Artin's article is 20 pages long. As the reader will notice, this version has gained some length since it departs from Artin's elegant but succinct style, and includes a generous amount of commentary.

[2] A good thing too; my knowledge of German and the mathematical conventions of the period is not good enough for a faithful literal translation. Readers with a strong historical interest are encouraged to read this paraphrasis alongside the German original to get a fuller picture.

of presentation; the sectioning and section numbers, numbered equations, and the numbering of main results (*Sätze*) are all faithful to [3].

In this paper Artin introduces what are known as *Artin L-series*, but the paper has much more. It has the first statement, and proofs in many cases, of *Artin's Reciprocity Law*, arguably the most important result in class field theory. It has the analytic continuation and functional equations for these new L-series: actually he gives meromorphic continuations but only for powers of the L-series. So the analytic continuations he gives here are viewed as possibly "multivalued". He conjectures that these are single-valued, i.e., that the L-series are indeed meromorphic on $\mathbb{C}$, and proves this in the significant case where the Galois group is A_5 (the Icosahedral case). Note that the Abelian case of the single-valuedness claim follows from Hecke's earlier work together with Artin's reciprocity law; the general case was handled much later by Brauer in 1947. Artin goes further and conjectures that primitive L-series are entire (holomorphic) when they are not equal to Dedekind zeta functions, and gives some evidence in the the A_5 case. This conjecture remains open, even for the specific case of A_5 extensions. Finally, Artin gives a proof (assuming the Reciprocity Law) of the Chebotaryov density theory. Unknown to Artin, at about the same time as Artin was writing [3], Nikolai Chebotaryov proved this same result, a conjecture of Frobenius, but with a different method that, in an interesting twist, would be the inspiration for Artin's definitive proof of the Reciprocity law of 1927. So all in all, this is an amazingly rich and interesting paper.

The paper, published in 1924, reflects a seminar in Hamburg in July 1923. This was the initial presentation of the theory, but is not the final word on the birth of the theory. It suffered from gaps that were soon fixed by Artin himself during his Hamburg years:

1. It depended on a general reciprocity law that Artin did not prove until 1927. This is the partially proved Satz 2 in the current paper.

2. The ramified primes were not suitably handled yet. This was fixed in 1930, with a factor for the "infinite prime" and a theory of conductors. (See [5])

The work on conductors made an impact on the number theorist Hasse who was essentially the same age as Artin, and inspired Noether's work on her theorem on normal integral bases. So the work on reciprocity and conductors makes this work of interest beyond concern for Artin L-series per se.

At the time of this paper, Artin was just starting his mathematical career. He received his PhD under Herglotz in 1921, in Leipzig. After a year at Göttingen he accepted a permanent position in Hamburg in 1922 and stayed there for 15 years. It was a very rich and productive time in Artin's career and which came to an end when Artin moved to the United States to escape the Third Reich. (See [14] for additional historical perspective.)

"Concerning a new kind of L-Series": Introduction

By E. Artin in Hamburg.

In what follows the black paragraphs give my very free translation of Artin's original paper. The blue paragraph gives my notes and comments. A similar convention will hold for footnotes.

We start with the with the introduction, which consists of a short paragraph:

For investigating non-Abelian algebraic number fields one needs a new kind of L-series that generalizes the usual L-series for Abelian algebraic number fields. These analytic functions are formed with Frobenius style group characters. This article is dedicated to the investigation of such functions.

1 Frobenius Style Group Characters: Review

For the convenience of the reader, I will begin by briefly giving the formulas and notation that we will need from the theory of group characters.[3]

Let G be a finite group of order n. Decompose G into x conjugacy classes $C_1, \ldots, C_x$, and let h_i be the number of elements of C_i.

Let Γ be a representation of the group G as nonsingular matrices. Given Γ we get a *character* χ which is a function $G \to \mathbb{C}$ that assigns to $\sigma \in G$ the trace of the associated matrix. There are x irreducible representations $\Gamma_1, \ldots, \Gamma_x$, and let $\chi^1, \ldots, \chi^x$ be their associated characters. These characters are called *simple characters*. Every character χ is in fact the linear combination of simple characters:

$$\chi(\sigma) = \sum_{i=1}^{x} r_i \chi^i(\sigma) \tag{1}$$

where r_i are nonnegative integers associated with the decomposition of Γ into irreducible representations.

The simple characters satisfy the following formulas

$$\sum_{\sigma} \chi^i(\sigma)\chi^k(\sigma^{-1}) = n\delta_{ik} \tag{2}$$

and

$$\sum_{i=1}^{x} \chi^i(\sigma)\chi^i(\tau^{-1}) = \begin{cases} 0 \text{ if } \sigma \text{ and } \tau \text{ are in different classes,} \\ \frac{n}{h_r} \text{ if } s \text{ and } \tau \text{ are both in the class } C_r. \end{cases} \tag{3}$$

Furthermore, suppose H is a subgroup of G and that

$$G = \sum_{i=1}^{s} HS_i \tag{4}$$

is the decomposition into cosets (here $S_i \in G$).

Let Δ be a representation of the subgroup H of degree δ, and let A_σ be the matrix associated to $\sigma \in H$. If $\sigma \in G$ is not in H we take A_σ to be the zero matrix. We build the matrix B_σ out of blocks in the following way:

$$B_\sigma = \left(A_{S_i \sigma S_k^{-1}}\right). \tag{5}$$

[3] See J. Schur 1905, *Neue Begründung der Theorie der Gruppencharaktere* (New foundation for the theory of group characters), Sitzungsberichte (conference reports), Berlin, and Speiser [16] Chapters 10-12.

As stated this is an s by s square matrix with entries equal to δ by δ square matrices, where s is the index of H in G. But we regard B_σ as defining a square $s\delta$ by $s\delta$ matrix, and it turns out that this gives a representation of G called the representative of G induced by the representation of H.[4]

Remark. In the above $A_{S_i\sigma S_k^{-1}}$ designates the (i,k) block (using the ith row partition and kth column partition). There are s^2 such blocks total, and each block is a δ by δ matrix. So the induced representation is given concretely in terms of $s\delta$ by $s\delta$ matrices associated to each $\sigma \in G$.

Artin is viewing the group acting on the right of vectors. If we act on the left, which is common today, we end up with the (i,k) block looking like $A_{S_i^{-1}\sigma S_k}$.

If ψ is the character of the representation Δ then the character χ_ψ associated to the representation (5) is called the *character of G induced by the character ψ of H*.

Let $\psi_1, \ldots, \psi_\lambda$ be the simple characters of the subgroup H. Then we can express the restriction of χ^i to H as a nonnegative integral linear combination of $\psi_1, \ldots, \psi_\lambda$ with nonnegative integer coefficients $r_{1i}, \ldots, r_{\lambda i}$:

$$\chi^i(\tau) = \sum_{\nu=1}^{\lambda} r_{\nu i}\psi_\nu(\tau) \qquad (i = 1, \ldots, x) \tag{6}$$

for all $\tau \in H$. Similarly, we can express the induced character χ_{ψ_i} as a nonnegative integral linear combination of the simple characters $\chi_1, \ldots, \chi_x$ of G, and in fact the nonnegative coefficients are just the coefficients that arise in (6):

$$\chi_{\psi_i}(\tau) = \sum_{\nu=1}^{x} r_{i\nu}\chi^\nu(\tau) \qquad (i = 1, \ldots, \lambda). \tag{7}$$

for all $\tau \in G$.

Remark. The above is an expression of the Frobenius reciprocity law. The version of Serre [15] Section 7.1 can be written

$$\langle \psi, \operatorname{Res}\chi \rangle_H = \langle \operatorname{Ind}\psi, \chi \rangle_G .$$

The above statement can be derived from this.

2 Construction of the L-Series

From now on let k be an algebraic number field, let K be a Galois extension of k, and let G be the Galois group of K/k.

Let $\mathfrak{p}$ be a prime ideal in the ring of integers of k not dividing the relative discriminant of K/k. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_K$ dividing $\mathfrak{p}\mathcal{O}_K$.

[4] See Speiser [16] §52 from which formula (44) can easily be easily derived. See also an 1898 report by Frobenius called *Über Relationen zwischen den Charakteren einer Gruppe und denen ihrer Untergruppen* (Concerning the connection between the characters of a group and those of its subgroups).

We chose an element $\sigma \in G$ such that for all algebraic integers A in K we have

$$\sigma A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{P}} \tag{8}$$

where $N\mathfrak{p}$ is the norm of $\mathfrak{p}$ in k. For the existence of such a σ see Weber's Algebra [20], §178 (volume 2).

This congruence determines σ uniquely given a choice of $\mathfrak{P}$, since if σ_1 satisfies the same congruence then, for all algebraic integers A in K,

$$\sigma^{-1}\sigma_1 A \equiv A \pmod{\mathfrak{P}},$$

and so $\sigma^{-1}\sigma_1$ belongs to the inertia group (Trägheitsgruppe) of $\mathfrak{P}$. By our assumption ($\mathfrak{p}$ not dividing the relative discriminant) the inertia group is trivial.

Next suppose one chooses $\mathfrak{P}'$ instead of $\mathfrak{P}$ as a designated prime divisor of $\mathfrak{p}\mathcal{O}_K$. Since G acts transitively on primes above $\mathfrak{p}$, we have $\tau\mathfrak{P} = \mathfrak{P}'$ for some $\tau \in G$. It is easy to check that one gets $\tau\sigma\tau^{-1}$ as the corresponding element of G (where σ is the corresponding element for $\mathfrak{P}$).

So we have a way to associate to $\mathfrak{p}$ a well-defined conjugacy class C of G. It is well-known that each element of C generates the decomposition group for some $\mathfrak{P}$ above $\mathfrak{p}$ but this property does not in general completely determine the class C (in fact certain powers of this class C with have this property).[5] We will say that the prime ideal $\mathfrak{p}$ *belongs to the class* C and we will write this class as $C_\mathfrak{p}$.

Remark. We call each element of $C_\mathfrak{p}$ a *Frobenius element*, and the class as a whole the *Frobenius class*, in honor of Frobenius who, as Artin points out in the footnote, developed this idea earlier. Artin does not really use these terms in the German original of this paper, but I will use them in the translation below for the convenience of the modern reader.

From now on let Γ be a linear representation of G. For $\mathfrak{p}$ as above let $A_\mathfrak{p}$ be a matrix associated to an element of $C_\mathfrak{p}$ via Γ. Since the elements of $C_\mathfrak{p}$ are conjugate, the characteristic polynomial

$$|E - tA_\mathfrak{p}|$$

of $A_\mathfrak{p}$ does not depend on the choice of $A_\mathfrak{p}$. Here E is the identity matrix and, as usual, the absolute values indicates determinant. Note that A_p will change by a conjugate if Γ is replaced by an equivalent representation, so the characteristic polynomial only depends on the representation Γ up to equivalence.

We define the associated L-series by the formula

$$L(s, \chi; k) = \prod_\mathfrak{p} \frac{1}{|E - (N\mathfrak{p})^{-s} A_\mathfrak{p}|} \tag{9}$$

where s is a complex variable and χ denotes the character associated with the representation Γ. Here, the product varies only for the set of prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ that do no divide the relative discriminant of K/k.

[5] This assignment of conjugacy classes to prime ideals was already carried out by Frobenius. See the 1896 Berlin report called *Über Beziehungen zwischen Primidealen eines algebraischen Körpers und den Substitutionen seiner Gruppe* (concerning the relationships between prime ideals of an algebraic field and the elements of its Galois group).

Remark. In a later paper, Artin gives an explicit formula for terms associated to primes that do divide the relative discriminant of K/k. Note that the above L series is expressed using χ instead of Γ, since χ determines Γ up to equivalence and so determines the expression on the right-hand side of (9).

The function $L(s, \chi; k)$ converges absolutely and uniformly on any closed and bounded region in the half plane $\Re(s) > 1$. To see this observe that every root of the characteristic polynomial $|E - tA_{\mathfrak{p}}|$ is a root of unity. Thus $L(s, \chi; k)$ is a product of terms of the form

$$\frac{1}{1 - (N\mathfrak{p})^{-s}\,\varepsilon}$$

where ε is a root of unity.

Remark. Since $A_{\mathfrak{p}}$ has finite order it is diagonalizable with eigenvalues all equal to roots of unity. So its characteristic polynomial factors as described by Artin.

Some of the convergence issues can be handled with the following well-known criterion: if an infinite series $\sum |a_i|$ converges then the corresponding infinite product $\prod(1 + a_i)$ converges, and the terms $1 + a_i$ can be reordered freely with convergence to the same result. Furthermore, if each term $1 + a_i$ is nonzero then the limit is nonzero. (See, for example, [17], Chapter 5, Proposition 3.1 for some justification.)

On the other hand, it might be convenient to wait on convergence issues until we have the formula for the logarithm given by Artin below.

One can now expand (9) in a Dirichlet series and express the coefficients in terms of the character χ. The resulting formulas are not very clear ("Die Formeln werden aber wenig übersichtlich"). On the other hand, we arrive at a simple formula for the logarithm of (9).

First we associate a conjugacy class $C_{\mathfrak{p}^\nu}$ to any power $\mathfrak{p}^\nu$ of a prime ideal $\mathfrak{p}$. We simply take the class consisting of $A_{\mathfrak{p}}^\nu$ where $A_{\mathfrak{p}} \in C_{\mathfrak{p}}$. It is easy to see that this forms a conjugacy class of G. We write

$$\chi\left(\mathfrak{p}^\nu\right) = \chi(\sigma) \tag{10}$$

where σ is any member of $C_{\mathfrak{p}^\nu}$.

Now let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_f$ be the roots of the equation $|Et - A_{\mathfrak{p}}| = 0$. Then

$$\chi\left(\mathfrak{p}^\nu\right) = \varepsilon_1^\nu + \varepsilon_2^\nu + \ldots + \varepsilon_f^\nu. \tag{11}$$

So we get for $|t| < 1$

$$-\log|E - tA_{\mathfrak{p}}| = -\sum_{i=1}^{f} \log(1 - t\varepsilon_i) = \sum_{i=1}^{f}\sum_{\nu=1}^{\infty} \frac{\varepsilon_i^\nu}{\nu} t^\nu = \sum_{\nu=1}^{\infty} \frac{\chi(\mathfrak{p}^\nu)}{\nu} t^\nu$$

Remark. Here we understand log as a multivalued functions, or equivalently we regard some of our equations as being valid modulo $(2\pi i)\mathbb{Z}$. So for example, the first equation above can be regarded as a congruence modulo $(2\pi i)\mathbb{Z}$.

This leads to the desired formula:

$$+\log L(s, \chi; k) = \sum_{\mathfrak{p}^\nu} \frac{\chi(\mathfrak{p}^\nu)}{\nu\left(N\mathfrak{p}^\nu\right)^s}, \tag{12}$$

where the sum varies over all powers of prime ideals of k not dividing the relative discriminant of K/k.

Remark. Associated convergence issues can be justified by the observation that

$$\sum_{\mathfrak{p}^\nu} \left| \frac{\chi(\mathfrak{p}^\nu)}{\nu\,(N\mathfrak{p}^\nu)^s} \right| \;<\; \sum_{M=1}^{\infty} m \frac{f}{M^{\sigma_0}} = mf\zeta(\sigma_0) < \infty$$

assuming that $\Re(s) \geq s_0 > 1$. Here $\zeta(s)$ is the classical Zeta function, the left sum is taken over all ideals of the form $\mathfrak{p}^\nu$, in any order, where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_k$ relatively prime to the relative discriminant of K/k, and ν is a positive integer. Also f is the degree of the representation Γ and m is the degree $[k\colon \mathbb{Q}]$, so that at most m ideals of the form $\mathfrak{p}^\nu$ can share the same norm.

In particular we have the absolute convergence of

$$\sum_{\mathfrak{p}^\nu} \frac{\chi(\mathfrak{p}^\nu)}{\nu\,(N\mathfrak{p}^\nu)^s}$$

which justifies the manipulations above. We also get uniform convergence on the set $\Re(s) \geq s_0$ for each $s_0 > 1$, and so the sum gives a homomorphic function on the set defined by $\Re(s) > 1$. By exponentiation we get the desired convergence properties for our Euler product expansion of L as well, including the invariance under reordering of terms with a product that defines a holomophic function in s with no zeros on the set defined by $\Re(s) > 1$.

Either from (9) or even better from (12) one sees that

$$L(s, \chi + \chi') = L(s, \chi) L(s, \chi') \tag{13}$$

for any two characters χ and χ'.

If χ is a simple character then we will call the associated L-series a *primitive L-series*. If χ is a general character expressed in terms of simple characters, as in (1) then (13) gives us

$$L(s, \chi) = \prod_{i=1}^{x} \left(L(s, \chi^i) \right)^{r_i}. \tag{14}$$

A brief remark about the dependence on the field K: suppose Ω is an extension of K that is Galois over k. Then $G = \mathrm{Gal}(K/k)$ is isomorphic to the quotient group $\mathrm{Gal}(\Omega/k)/\mathrm{Gal}(\Omega/K)$. If $\sigma \in \mathrm{Gal}(\Omega/k)$ is such that (8) is valid for all algebraic integers in Ω then it will of course be valid for all algebraic integers in K. Furthermore, (8) will be valid for algebraic integers A in K if we replace σ with any element of the coset $\sigma\,\mathrm{Gal}(\Omega/K)$. Next observe that every character of $\mathrm{Gal}(\Omega/k)/\mathrm{Gal}(\Omega/K)$ is a character of $\mathrm{Gal}(\Omega/k)$, and every simple character of $\mathrm{Gal}(\Omega/k)/\mathrm{Gal}(\Omega/K)$ is a simple character of $\mathrm{Gal}(\Omega/k)$. In particular every L-series using K as the extension will essentially be an L-series using Ω as the extension, and if the L series is primitive using K then it will be primitive using Ω. However, the relative discriminant of Ω/k may exclude a finite number of prime factors in the L-series that occur using the relative discriminant of K/k. But we will consider L-series that differ from each other by only a finite number of factors as being essentially the same. By the way, we will be able to normalize the L-series later to be truly invariant of K.

Remark. The above uses a fundamental compatibility principle for of the Frobenius element associated with two extensions Ω/k and K/k of a common base field k. This principle is needed in several places in this paper, so I will go ahead and codify it as a lemma. I will switch the roles of K and Ω here since in what follows Ω is often used to denote an intermediate field.

Lemma 1. *Suppose K/k is a Galois extension of number fields with Galois group G and let Ω be an intermediate field such that Ω/k is also Galois. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_k$ not dividing the relative discriminant of K/k, let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_\Omega$ above $\mathfrak{p}$, and let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_K$ above $\mathfrak{q}$. In other words we have a triple extension $K/\Omega/k$ with corresponding prime ideals $\mathfrak{P}, \mathfrak{q}, \mathfrak{p}$.*

Then if $\sigma \in G$ is the Frobenius element associated to $\mathfrak{P}$, then the restriction σ' of σ to Ω is the Frobenius element of $\mathfrak{q}$ in the Galois group of Ω/k. When we identify the Galois group of Ω/k with G/H where H is the Galois group of K/Ω, then this Frobenius element σ' is the coset $\sigma H \in G/H$.

Proof. By (8) we have that $\sigma A - A^{N\mathfrak{p}} \in \mathfrak{P}$ for all $A \in \mathcal{O}_K$. So

$$\sigma' A - A^{N\mathfrak{p}} \in \mathfrak{P} \cap \mathcal{O}_\Omega = \mathfrak{q}$$

for all $A \in \mathcal{O}_\Omega$. Hence

$$\sigma' A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{q}}$$

for all $A \in \mathcal{O}_\Omega$, and so σ' is the desired Frobenius element. By basic Galois theory, σ' corresponds to the coset $\sigma H \in G/H$. □

3 The Theorem on Induced Representations

Let H be a subgroup of G, let Ω be the subfield of K fixed by H, so H is the Galois group of K/Ω.

The first main theorem (Satz 1) concerns the following situation:

- Δ is a representation of H.
- Γ_Δ is the induced representation of G.
- ψ is the character of Δ, and χ_ψ is the character of Γ_Δ.
- Exclude as factors of $L(s, \psi; \Omega)$ any prime dividing the relative discriminant of K/k (considered as an ideal of the ring of integers of Ω).

Satz 1. *In the situation discussed above*

$$L(s, \psi; \Omega) = L(s, \chi_\psi; k). \tag{15}$$

Proof. We set up the following notation:

- Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_k$ not dividing the relative discriminant of K/k.
- Let $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_r$ be the prime ideals of $\mathcal{O}_\Omega$ dividing $\mathfrak{p}\mathcal{O}_\Omega$:

$$\mathfrak{p}\mathcal{O}_\Omega = \mathfrak{q}_1 \mathfrak{q}_2 \cdots \mathfrak{q}_r.$$

- Let l_i be the relative degree of $\mathfrak{q}_i$ over $\mathfrak{p}$. In other words, $(N\mathfrak{p})^{l_i}$ is the size $N\mathfrak{q}_i$ of the residue field $\mathcal{O}_\Omega/\mathfrak{q}_i$.
- For each $\mathfrak{q}_i$ choose a prime ideal $\mathfrak{P}_i$ of $\mathcal{O}_K$ dividing $\mathfrak{q}_i\mathcal{O}_K$.
- For each such $\mathfrak{P}_i$ let $\tau_i \in G$ be chosen so that $\mathfrak{P}_i = \tau_i\mathfrak{P}_1$. (Recall that the Galois group G acts transitively on the primes of $\mathcal{O}_K$ dividing $\mathfrak{p}\mathcal{O}_K$).
- Let $\sigma \in G$ be the Frobenius element associated with $\mathfrak{P}_1$ over k. In other words,
$$\sigma A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{P}_1}$$
for all $A \in \mathcal{O}_K$.
- Let $\sigma_i \stackrel{\text{def}}{=} \tau_i\sigma\tau_i^{-1}$. Observe that
$$\sigma_i A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{P}_i} \tag{16}$$
for all $A \in \mathcal{O}_K$ so σ_i is the Frobenius element associated with $\mathfrak{P}_i$. Thus σ_i generates the decomposition group of $\mathfrak{P}_i$.

Claim: *The Frobenius element associated with $\mathfrak{P}_i$ over Ω is equal to $\sigma_i^{l_i}$.* To see this first observe that from (16)

$$\sigma_i^{l_i} A \equiv A^{(N\mathfrak{p})^{l_i}} \equiv A^{N\mathfrak{q}_i} \pmod{\mathfrak{P}_i}, \tag{17}$$

So to establish that $\sigma_i^{l_i}$ is the Frobenius element we just need to show that $\sigma_i^{l_i} \in H$. In the special case where $A = \alpha \in \mathcal{O}_\Omega$ we have from (17) and Fermat's little theorem that

$$\sigma_i^{l_i}\alpha \equiv \alpha^{N\mathfrak{q}_i} \equiv \alpha \pmod{\mathfrak{P}_i}.$$

Since $\mathfrak{p}$ does not divide the relative discriminant of K/k, this means that $\sigma_i^{l_i}\alpha = \alpha$ for all $\alpha \in \mathcal{O}_\Omega$ and hence for all $\alpha \in \Omega$. So $\sigma_i^{l_i} \in H$ as desired.

Remark. Note that σ_i is in the decomposition group of $\mathfrak{P}_i$, and so $\sigma_i^{l_i}$ is, of course, in this decomposition group. Since $\mathfrak{P}_i$ is unramified over $\mathfrak{q}_i$, the canonical map from the decomposition group of $\mathfrak{P}_i$ to the Galois group of $\mathcal{O}_K/\mathfrak{P}_i$ is injective.

Next we observe that l_i is the smallest positive power ν of σ_i such that $\sigma_i^\nu \in H$. To see this observe that if $\sigma_i^\nu \in H$ then by (16)

$$\sigma_i^\nu\alpha = \alpha \equiv \alpha^{(N\mathfrak{p})^\nu} \pmod{\mathfrak{P}_i}$$

for all $\alpha \in \mathcal{O}_\Omega$. Thus $(N\mathfrak{p})^\nu \geq (N\mathfrak{p})^{l_i} = N\mathfrak{q}_i$ and so $\nu \geq l_i$.

Remark. The last step becomes clear when we observe that every element of the residue field has been shown to be a root of $X^{(N\mathfrak{p})^\nu} - X$ which is a polynomial in X of degree $(N\mathfrak{p})^\nu$. But the residue field has $(N\mathfrak{p})^{l_i} = N\mathfrak{q}_i$ elements.

Claim: *Consider cosets $H\sigma_\nu^a\tau_\nu$ and $H\sigma_\mu^b\tau_\mu$. These cosets are equal if and only if $\nu = \mu$ and $a \equiv b \pmod{l_\nu}$.*

One direction of this claim is straightforward since $\sigma_\nu^{l_\nu} \in H$, so if $a \equiv b$ modulo l_ν then $H\sigma_\nu^a = H\sigma_\nu^b$ and so $H\sigma_\nu^a\tau_\nu = H\sigma_\nu^b\tau_\nu$. For the other direction, assume that $H\sigma_\nu^a\tau_\nu = H\sigma_\mu^b\tau_\mu$, and so

$$\sigma_\nu^a\ \tau_\nu = \tau_0\ \sigma_\mu^b\ \tau_\mu$$

with $\tau_0 \in H$. So by the definition of σ_ν and σ_μ

$$\tau_0 = \sigma_\nu^a \, \tau_\nu \, \tau_\mu^{-1} \, \sigma_\mu^{-b} = \tau_\nu \, \sigma^{a-b} \, \tau_\mu^{-1}$$

and, since σ is in the decomposition group of $\mathfrak{P}_1$,

$$\tau_0 \mathfrak{P}_\mu = \tau_\nu \, \sigma^{a-b} \, \tau_\mu^{-1} \mathfrak{P}_\mu = \tau_\nu \, \sigma^{a-b} \mathfrak{P}_1 = \tau_\nu \mathfrak{P}_1 = \mathfrak{P}_\nu.$$

Since $\tau_0 \in H$, it is the identity map on $\mathfrak{P}_\mu \cap \mathcal{O}_\Omega = \mathfrak{q}_\mu$, but the image of $\mathfrak{P}_\mu \cap \mathcal{O}_\Omega$ is $\mathfrak{P}_\nu \cap \mathcal{O}_\Omega = \mathfrak{q}_\nu$. So $\mathfrak{q}_\mu = \mathfrak{q}_\nu$. Thus $\mu = \nu$. We then have $\sigma_\nu^a \, \tau_\nu = \tau_0 \, \sigma_\nu^b \, \tau_\nu$ so that $\sigma_\nu^{a-b} \in H$ which implies that $a \equiv b \pmod{l_\nu}$.

So we have identified $l_1 + \ldots + l_r$ distinct cosets of H. But we know that

$$l_1 + \ldots + l_r = [\Omega \colon k] = [G : H].$$

So we have identified all the right cosets of H.

Note that $H\sigma_\nu^a \tau_\nu = H\tau_\nu \sigma^a$, and so have coset representations, as in (4) with S_i varying in the sequence

$$\tau_1, \tau_1\sigma, \ldots, \tau_1\sigma^{l_1-1}, \tau_2, \tau_2\sigma, \ldots\ldots, \tau_r, \tau_r\sigma, \ldots, \tau_r\sigma^{l_r-1}.$$

In other words, each S_i is of the form $\tau_\nu \sigma^a$ with $0 \le \nu \le r$ and $0 \le a < l_\nu$.

According to (5), in the induced representation Γ_Δ of G, the element $\sigma \in G$ is represented by the matrix described in terms of blocks as follows:

$$B_\sigma = \left(A_{S_i \sigma S_k^{-1}} \right) = \left(A_{\tau_\nu \sigma^{a-b+1} \tau_\mu^{-1}} \right)$$

where, as above, $A_{\tau_\nu \sigma^{a-b+1} \tau_\mu^{-1}}$ is the zero block if $\tau_\nu \sigma^{a-b+1} \tau_\mu^{-1}$ is not in H.

Remark. Here the row blocks are indexed by (ν, a) and the column blocks are indexed by (μ, b).

Note that $\tau_\nu \sigma^{a-b+1} \tau_\mu^{-1} \in H$ if and only if $\tau_\nu \sigma^{a-b+1} \in H\tau_\mu$. But since $\tau_\nu \sigma^{a-b+1}$ is in the coset $H\sigma_\nu^{a-b+1}\tau_\nu$ we conclude that the block is zero unless both $\mu = \nu$ and $a - b + 1 \equiv 0 \pmod{l_\nu}$.

For a fixed ν we can consider the square matrix C_ν which is described as a block matrix whose (a, b) block is the δ by δ square matrix $\left(A_{\tau_\nu \sigma^{a-b+1} \tau_\nu^{-1}} \right)$. In particular the (a, b) block is zero unless $a - b + 1 \equiv 0 \pmod{l_\nu}$. Note that C_ν is an $l_\nu\delta$ by $l_\nu\delta$ square matrix. Then (for a suitable ordering of a basis) one can write B_σ in terms of blocks as follows:

$$B_\sigma = \begin{pmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_r \end{pmatrix}.$$

If $a = 0, 1, \ldots, l_\nu - 2$ then the (a, b) block of C_ν is zero unless $b = a + 1$, and when $b = a + 1$ the block is $A_{\tau_\nu \sigma^{a-b+1} \tau_\nu^{-1}}$ which is the δ by δ identity matrix E. If $a = l_\nu - 1$ then the (a, b) block of C_ν is zero unless $b = 0$ and the $(l_\nu - 1, 0)$ block

is given by $A_{\tau_\nu \sigma^{l_\nu} \tau_\nu^{-1}} = A_{\sigma_\nu^{l_\nu}} = A_{\sigma_\nu}^{l_\nu}$. So C_ν decomposes into blocks as follows:

$$C_\nu = \begin{pmatrix} 0 & E & 0 & \cdots & 0 \\ 0 & 0 & E & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & E \\ A_{\sigma_\nu^{l_\nu}} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The characteristic polynomial in t is then

$$|E - tB_\sigma| = \prod_{\nu=1}^{r} |E - tC_\nu| = \prod_{\nu=1}^{r} \begin{vmatrix} E & -tE & 0 & \cdots & 0 \\ 0 & E & -tE & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & -tE \\ -tA_{\sigma_\nu^{l_\nu}} & 0 & 0 & \cdots & E \end{vmatrix}.$$

Adding t times the first column to the second, then t times the (new) second to the third, and so on, one gets

$$|E - tB_\sigma| = \prod_{\nu=1}^{r} \begin{vmatrix} E & 0 & 0 & \cdots & 0 \\ 0 & E & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -tA_{\sigma_\nu^{l_\nu}} & -t^2 A_{\sigma_\nu^{l_\nu}} & -t^3 A_{\sigma_\nu^{l_\nu}} & \cdots & E - t^{l_\nu} A_{\sigma_\nu^{l_\nu}} \end{vmatrix}.$$

Thus

$$|E - tB_\sigma| = \prod_{\nu=1}^{r} \left| E - t^{l_\nu} A_{\sigma_\nu^{l_\nu}} \right|.$$

Note this last formula does not depend on the choice of σ since different choices of Frobenius elements gives the same characteristic polynomials.

The contribution of $\mathfrak{p}$ to $L(s, \chi_\psi; k)$ is

$$\frac{1}{|E - (N\mathfrak{p})^{-s} B_\sigma|} = \prod_{\nu=1}^{r} \frac{1}{|E - (N\mathfrak{p})^{-l_\nu s} A_{\sigma_\nu^{l_\nu}}|} = \prod_{\nu=1}^{r} \frac{1}{|E - (N\mathfrak{q}_\nu)^{-s} A_{\sigma_\nu^{l_\nu}}|}.$$

We have already seen that the Frobenius element associated with $\mathfrak{P}_i$ over Ω is equal to $\sigma_i^{l_i}$. So the right hand side of the above formula gives the product of the $\mathfrak{q}_\nu$ contributions to $L(s, \psi; \Omega)$. Thus Satz 1 is proved.

□

4 Factorization of Zeta Functions

Satz 1 gives us, for starters, a factorization of zeta functions of intermediate fields Ω in terms of primitive L-series associated to K/k

When we consider the trivial representation and the trivial character $\chi = 1$ (der Hauptcharakter χ_1) we get

$$L(s, \chi_1; k) = \prod_{\mathfrak{p}} \frac{1}{|E - (N\mathfrak{p})^{-s} A_{\mathfrak{p}}|} = \prod_{\mathfrak{p}} \frac{1}{1 - (N\mathfrak{p})^{-s}}$$

which is, up to a finite number of factors, just the zeta function $\zeta_k(s)$ of the base field.

More generally if Ω is an intermediate field between k and K, and if H is the Galois group of K/Ω with trivial character (Hauptcharakter) ψ_1 then

$$L(s, \psi_1; \Omega) = \zeta_\Omega(s),$$

at least up to a finite number of factors. Let Π_Ω be the induced representation associated with the trivial representation of H. Note that Π_Ω is simply the representation associated with the permutation of cosets of H in G (so if Ω is itself Galois over k, it corresponds to the regular representation of the Galois group of Ω over k). Thus the associated character χ_Ω has the property that, for any $\sigma \in G$, the value $\chi_\Omega(\sigma)$ is the number of cosets fixed by σ under this action, so is determined in a most simple manner. If we decompose χ_Ω in terms of primitive characters

$$\chi_\Omega(\sigma) = \sum_{i=1}^{x} g_i \chi^i(\sigma)$$

then g_i is obtained using (2):

$$g_i = \frac{1}{n} \sum_{\sigma} \chi_\Omega(\sigma) \chi^i(\sigma^{-1}) \tag{18}$$

(n is the order of G and so is $n = [K : k]$). So Satz 1 in combination with (14) implies

$$\zeta_\Omega(s) = \prod_{i=1}^{x} \big(L(s, \chi^i)\big)^{g_i} \tag{19}$$

which is the desired factorization (up to a finite number of factors).

Remark. From what Artin has said up to this point it is apparent that he regards G as acting on the left for its natural action on K, but regards G as acting on the right for linear representation. Under this convention Π_Ω is the permutation representation of the right action of G on the collection $H\backslash G$ of right cosets. However, the associated character $\chi_\Omega(\sigma)$ is the same whether we use left actions or right actions here (in other words, the number of left cosets fixed by $\sigma \in G$ is the same as the number of right cosets fixed by σ).

In the special case of $K = \Omega$ the induced representation is the regular representation and we get the simple formula

$$\zeta_K(s) = \prod_{i=1}^{x} \big(L(s, \chi^i)\big)^{f_i}. \tag{20}$$

Remark. Here f_i is the degree of character χ_i. So if K/k is Abelian, we have $f_i = 1$. In general, all the primitive L-series for K/k occur in the factorization.

Formula (19) gives all the relations between the zeta functions of intermediate fields. To get such a relation, one uses (19) for various Ω and eliminates the L-series factors $L(s, \chi^i)$. One is left with relations between zeta functions. So the equations (19) can be regarded as parameterizing relations. We will show later (Section 8) that this is the only way to get relations between zeta functions (when we reduce to the case $k = \mathbb{Q}$).[6] This essentially solves the problem of relations between zeta functions.

There is another way to formulate our results. Observe that the factorization (19) of $\zeta_\Omega(s)$ runs parallel to the factorization into irreducible polynomials of the group determinant (Gruppendeterminante) associated to the permutation representation Π_Ω. So one can say the following:

One gets all the relations between zeta functions of intermediate fields by finding the relations between the group determinants associated with transitive permutation actions of G, and replacing the group determinants with the corresponding zeta functions.

Remark. The "group determinants" that Artin mentions above are certain homogeneous polynomials associated to groups and their representations. They are not as familiar today as they were when Artin wrote this paper, so I will give some details. They are called "determinants" since they arise as determinants of matrices with entries that are homogeneous linear polynomials. These polynomials were studied by Dedekind and Frobenius, and their study led Frobenius to his theory of characters of non-Abelian groups in 1896 that is in fact the basis of the current paper (see [9]). They are easy enough to define: consider the polynomial ring $\mathbb{C}[X_{g_1}, \ldots, X_{g_n}]$ associated to a given finite group $G = \{g_1, \ldots, g_n\}$ where the X_{g_i} are independent variables. If $g \mapsto A_g$ is a representation of G by complex matrices, then the determinant associated to the representation is simply the determinant of the following matrix:

$$A_G \stackrel{\text{def}}{=} \sum_{g \in G} X_g A_g.$$

The matrix A_G has a particularly nice description if the representation is a permutation representation, and even more so for the regular representation (it is a good exercise to work these out). The determinant associated to the regular representation is called the "group determinant" of G and can be thought of as a fundamental algebraic invariant of G.

The determinant associated to a representation is an irreducible polynomial if and only if the representation is an irreducible representation, and the decomposition of a representation is reflected in the factorization of its associated determinant. Observe also that the degree of such a determinant polynomial is equal to the degree of the representation. Note that, historically speaking, the problem of factoring the group determinant proceeds, and in fact motives, the problem of

[6] See E. Artin, Über die Zetafunktionen gewisser algebraisher Zahlkörper (Concerning the zeta functions of certain algebraic number fields), Math. Ann Bd. 89, where the relations in special cases are obtained.

decomposing a representation into irreducible factors that is the starting point of modern representation theory (See [9]).

For a simple example, the group determinant of a two-element group $G = \{1, \sigma\}$ is just $X_1^2 - X_\sigma^2$ which factors as $(X_1 + X_\sigma)(X_1 - X_\sigma)$, reflecting the fact that the regular representation of G decomposes into two irreducible representations, each of degree 1.

For now these relations are only valid up to a finite number of factors. Because of the existence of functional equations for zeta functions, we can use the well-known methods of Herrn Hecke to show the relations are exactly valid.[7]

Remark. These methods of Hecke allow us to use functional equations of zeta functions to conclude that if a relation between zeta functions is valid up to a finite number of Euler factors, then the relation holds exactly. (See Lemma 2 below for an illustration of this phenomenon.)

Of course, similar considerations apply for relations between L-Series of intermediate fields.

Remark. We can use Artin's results to get an even more dramatic conclusion. Suppose ζ_Ω is a zeta function with base number field Ω, or more generally consider L functions with base field Ω. Then by result alluded to at the and of Section 2, we can take K to be an extension of Ω that is Galois over $\mathbb{Q}$. So the above considerations allow us to express ζ_Ω (or more general L-functions) in terms of primitive L-functions over $\mathbb{Q}$. Artin, in Section 8 below, will show that this decomposition is unique.

5 The Abelian Case

We now consider the case where G is Abelian. We investigate whether the primitive L-series defined in this document correspond to the usual L-series.

Remark. These earlier L-series were defined by Weber and generalize those defined by Dirichlet. They are defined in terms of characters of class groups (where characters are understood here in the traditional Dirichlet-Dedekind sense as a homomorphism from a finite Abelian group into $\mathbb{C}^\times$).

When G is Abelian, each conjugacy class has a single element. So for each prime $\mathfrak{p}$ of k not dividing the relative discriminant of K/k there is exactly one Frobenius element $\sigma \in G$, and (8) holds for all primes $\mathfrak{P}$ in K above $\mathfrak{p}$. One can replace (8) with the congruence

$$\sigma A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{p}}. \tag{21}$$

Further, the irreducible representations of G are all of degree 1, and they correspond to the ordinary Abelian characters $\chi^i(\sigma)$ of G. Hence

$$L(s, \chi^i) = \prod_{\mathfrak{p}} \frac{1}{1 - \frac{\chi^i(\sigma)}{N\mathfrak{p}^s}} \tag{22}$$

where σ denotes the Frobenius element associated to $\mathfrak{p}$.

[7] E. Hecke: Über eine neue Anwendung der Zetafunktion auf die Arithmetik der Zahlkörper (concerning a new application of zeta functions to the arithmetic of number fields). Göttinger Nachrichten 1917.

Remark. Equation (21) follows from the Chinese remainder theory. In (22) the Frobenius element σ depends on $\mathfrak{p}$. Artin makes this implicit, but a notation such as $\sigma(\mathfrak{p})$ could be used here if we wanted to make this explicit.

Now in this situation K is the class field of a certain class group $\{C_1, \ldots, C_n\}$ for a certain modulus $\mathfrak{m}$ (a certain ideal of $\mathcal{O}_k$) with the property that a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ prime to $\mathfrak{m}$ splits into prime ideals of the first degree in $\mathcal{O}_K$ if and only if $\mathfrak{p}$ is in C_1 where C_1 is the identity class (Hauptklasse).[8]

Remark. We can think of $\{C_1, \ldots, C_n\}$ as a certain quotient group of the multiplicative group of fractional ideals whose prime factors are prime to $\mathfrak{m}$. In other words, each C_i is a class of fractional ideals prime to $\mathfrak{m}$. There is a minimal ideal $\mathfrak{m}$ that we can use called the conductor, but we get well-defined version of the class group when we use multiplies of this minimal modulus. Replacing a modulus by a multiple gives a class group that is naturally isomorphic to the first, so we can often say "the ideal class group" associated to K/k is we are not concerned about the exact modulus. However replacing a modulus with a multiple can reduce the set of prime ideals of $\mathcal{O}_k$ prime to the modulus, but only by a finite number.

The identity between our new L-series and the usual L-series will be shown once we are able to prove the following:

Satz 2.

a) *The Frobenius element σ of $\mathfrak{p}$ depends only on the ideal class C_i containing $\mathfrak{p}$, (so we can assign a Frobenius element to each ideal class C_i by choosing any prime ideal in that class as a representative).*

b) *This Frobenius map gives an isomorphism between the ideal class group and the Galois group G.*

Remark. Observe that if Satz 2 holds for a certain modulus $\mathfrak{m}$ then it automatically holds for any multiple of $\mathfrak{m}$. So there are really two versions of Satz 2, the strong version and the weak version. The strong version asserts the result where the class group is taken with any valid modulus $\mathfrak{m}$, or equivalently with the conductor as the modulus. The weak version asserts the result for some modulus $\mathfrak{m}$, or equivalently asserts (a) for "almost all" prime ideals, i.e. all prime ideals of $\mathcal{O}_k$ outside a certain finite subset (and where we can then let $\mathfrak{m}$ be any valid modulus).

When we know that almost all prime ideals of a given ideal class C_i must have the same Frobenius element, we can conclude that all ideal classes containing infinitely many prime ideals can be assigned a well-defined Frobenius element. But note that every ideal class C_i contains an infinite number of prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ by a suitable generalization of Dirichlet's theorem concerning primes in arithmetic progressions. Thus we can assign a Frobenius element to any class. This is the content of the first part of Satz 2.

This result implies that every character of the Galois group G is then a character of the ideal class group and conversely. So any L-series in our sense is then a L-series in the usual sense. Conversely, if an ordinary L-series is given for an ideal

[8] See Teiji Takagi: *Über eine Theorie des relativ Abelschen Zahlkörpers* (concerning a theory of relative Abelian number fields), Journal of the College of Science, Tokyo 1920 [18]. Further reference to Takagi will generally be from this paper.

class group then it will be an L-series for the character of the Galois group of the associated class field. So Satz 2 implies that our new definition is indeed a generalization of the old definition, agreeing with the old definition in the case where K/k is Abelian.

Remark. Satz 2 is called "Artin reciprocity". It is the culmination of classical class field theory, and will be proved by Artin in an article [4] appearing a few years later in 1927. When Artin wrote the current article in 1923, Teiji Takagi had already developed class field theory to a very high degree, and Artin builds on this here. Takagi's results give the following. If K/k is an Abelian extension of degree n then K is the class field of a class group $\mathcal{C} = \{C_1, \ldots, C_n\}$ defined with respect to a modulus $\mathfrak{m}$ for some ideal $\mathfrak{m}$ of $\mathcal{O}_k$. What this means is that $\{C_1, \ldots, C_n\}$ partitions the collection of ideals, and even fractional ideals, of $\mathcal{O}_k$ prime to $\mathfrak{m}$. Furthermore, the set $\mathcal{C} = \{C_1, \ldots, C_n\}$ of these classes is a group where $C_i C_j$ is defined as the class containing $I_i I_j$ for any choice $I_i \in C_i$ and $I_j \in C_j$. The modulus $\mathfrak{m}$ is such that all prime ideals $\mathfrak{p} \in C_i$ prime to $\mathfrak{m}$ are unramified in $\mathcal{O}_K$ in the sense that $\mathfrak{p}\mathcal{O}_K$ factors into distinct prime ideals. Furthermore, for such $\mathfrak{p}$ prime to $\mathfrak{m}$, we have that $\mathfrak{p}$ is in the identity class C_1 if and only if $\mathfrak{p}$ splits in $\mathcal{O}_K$ (in the sense that $\mathfrak{p}\mathcal{O}_K$ factors into n distinct primes of relative degree 1).

Another very important result of Takagi is that $\mathcal{C} = \{C_1, \ldots, C_n\}$ is isomorphic to the Galois group G of K/k. Interestingly, Takagi showed the isomorphism abstractly and did not supply a particular isomorphism. What Artin reciprocity does is gives a explicit canonical isomorphism $\mathcal{C} \to G$.

Satz 2 is also of interest in itself. It gives an explicit description of the isomorphism between the Galois group G and the ideal class group. In the case where G is cyclic, Satz 2 is completely identical with the general reciprocity law, assuming the base field k has the associated roots of unity. And indeed the agreement is so obvious that Satz 2 has to be interpreted as as the general reciprocity law (even when k does not have the associated roots of unity) even if the formulation seems a bit strange (fremdartig) at first as a reciprocity law.

Remark. The general reciprocity referred here, and in the next paragraph, seems to be a version developed by Takagi mentioned in special case 5. below. This law is less familiar today than other reciprocity laws, but the important take-away is that Takagi's law generalizes the classical reciprocity laws. Since Artin reciprocity generalizes Takagi's reciprocity law it automatically generalizes all the more familiar classical reciprocity laws.

The situation is, however, that our provisional proof of Satz 2 only really succeeds in the cases where the general reciprocity law is accessible to us, that is for K of prime degree over k or composite fields of such extensions. For general fields we must, for the time being, just postulate Satz 2. We will do so in future sections which will allow us to regard all purely Abelian matters as being settled.

In this section we will prove Satz 2 in the cases accessible to us. We will proceed in stepwise fashion where we give the most general results possible in in order to make the relationships stand out more clearly.

1. *A prime ideal $\mathfrak{p}$ is in the identity class C_1 (the "Hauptklasse") if and only if the corresponding Frobenius element σ is the identity in G.*

Remark. Of course here we are only interested in prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ prime to the modulus $\mathfrak{m}$. As we will see in the proof, this result holds for any valid modulus.

Proof. If the Frobenius of $\mathfrak{p}$ is the identity element of G then $A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{P}}$ holds for all $A \in \mathcal{O}_K$ and all primes $\mathfrak{P}$ of $\mathcal{O}_K$ dividing $\mathfrak{p}\mathcal{O}_K$. This implies that the residue field $\mathcal{O}_K/\mathfrak{P}$ has $N\mathfrak{p}$ elements and so the degree $[\mathcal{O}_K/\mathfrak{P} : \mathcal{O}_k/\mathfrak{p}]$ is 1. Thus $\mathfrak{p}\mathcal{O}_K$ factors into primes of relative degree 1, which means that $\mathfrak{p} \in C_1$ by Takagi Satz 31.

Conversely, if $\mathfrak{p}$ is in the identity class C_1 then $\mathfrak{p}\mathcal{O}_K$ factors into primes of relative degree 1. Thus $A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{P}}$ holds for all prime ideals $\mathfrak{P}$ dividing $\mathfrak{p}\mathcal{O}_K$ and all $A \in \mathcal{O}_K$. This means that $\sigma = 1$ works as the Frobenius element. □

Remark. Note that $A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{P}}$ holds for all $A \in \mathcal{O}_K$ if and only if every element of the residue field $\mathcal{O}_K/\mathfrak{P}$ is a root of $x^{Np} - x$. Lagrange's theorem on the number of roots of a polynomial of a given degree and by Fermat's little theorem, this holds in turn if and only if $\mathcal{O}_K/\mathfrak{P}$ is equal to its subfield $\mathcal{O}_k/\mathfrak{p}$.

Remark. The above, when combined with Takagi's class field theory, allows us to jump from homomorphisms to isomorphisms. Suppose in fact that we have a homomorphism $\mathcal{C} \to G$ from the class group $\mathcal{C}$ associated to K/k to the Galois group G of K/k. Suppose also that the class of any prime ideal $\mathfrak{p}$ maps to the associated Frobenius element (perhaps even with a finite number of exceptions). Assume $C \in \mathcal{C}$ is a class in the kernel. Then 1. implies that C is the identity class (using a density result via Weber L-functions). Thus $\mathcal{C} \to G$ is injective. From Takagi's class field theory, $\mathcal{C}$ and G have the same size (in fact Takagi showed they are isomorphic), thus $\mathcal{C} \to G$ is surjective as well.

Remark. The following result is one where we have to be careful about the distinction between the strong and weak versions of Satz 2. The proof seems to give the following: any modulus for which Satz 2 holds for K/k will also yield Satz 2 for Ω/k where Ω is an intermediate field.

2. *If Satz 2 is valid for an Abelian extension K/k then it is valid for Ω/k for any intermediate field Ω.*

Proof. Let $G_\Omega \subseteq G$ be the Galois group of K/Ω. Let r be the order of G_Ω and let s be the index of G_Ω in G. As usual the quotient G/G_Ω will be identified with the Galois group of Ω/K.

We assume Satz 2 for the extension K/k so there is a class group $\{C_1, \ldots, C_n\}$ relative to some modulus $\mathfrak{m}$, and a Frobenius isomorphism $\mathcal{C} \to G$ sending $C_i \in \mathcal{C}$ to the Frobenius element $\sigma \in G$ associated to any prime ideal in C_i.

Since Ω is an intermediate field, by Takagi's results Ω is the class field for a class group $\mathcal{H} = \{H_1, \ldots, H_s\}$, and moreover $\mathcal{H}$ can be chosen to come from a quotient group of $\mathcal{C}$. In other words, we can use the same modulus $\mathfrak{m}$ for $\mathcal{H}$ as for $\mathcal{C}$, and we can write the identity class (Hauptklasse) of $\mathcal{H}$ as the union of classes of $\mathcal{C}$

$$H_1 = C_1 \cup C_2 \cup \cdots \cup C_r$$

(where we reindex the elements of $\mathcal{C}$ as necessary).[9] By 1. (above) if $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_k$ not dividing $\mathfrak{m}$ then the Frobenius element of $\mathfrak{p}$ in G/G_Ω (relative to the extension Ω/k) is equal to the identity coset $G_\Omega \in G/G_\Omega$ if and only if $\mathfrak{p} \in H_1$. So by the compatibility of the Frobenius for K/k compared to Ω/k we have that the Frobenius element of C_i in G is in G_Ω if and only if $C_i \subseteq H_i$.[10]

We now show that all primes in a given class H_i have the same Frobenius element in G/G_Ω. Since $\mathcal{H}$ comes from a quotient group of $\mathcal{C}$, we can write H_i as $C_i'H_1$ for some $C_i' \in \mathcal{C}$. So if $\mathfrak{p} \in H_i$ is a prime ideal we have $\mathfrak{p} \in C_i'C_j$ for some $1 \leq j \leq r$ (by the decomposition of H_1). By Satz 1 for K/k we have that $\mathfrak{p}$ has Frobenius element $\sigma_i\tau_j \in G$ where $\sigma_i \in G$ is the Frobenius element of the class C_i' and $\tau_j \in G_\Omega$ is the Frobenius element of C_j (recall $C_j \subseteq H_1$ so $\tau_j \in G_\Omega$). Observe that $\sigma_i\tau_j$ is in the coset $\sigma_i G_\Omega$, and so by the compatibility of the Frobenius elements for K/k compared to Ω/k we have that the Frobenius element of $\mathfrak{p}$ in G/G_Ω is $\sigma_i G_\Omega$. Thus all primes $\mathfrak{p}$ in H_i have the same Frobenius element in G/G_Ω. This proves the first part of Satz 1 for Ω/k.

Now we have a well-defined Frobenius function $\mathcal{H} \to G/G_\Omega$, and we must show it is a homomorphism. This follows from the fact that the following commutes, where the horizontal maps are the Frobenius maps and the vertical maps are the natural quotient maps:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & G \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & G/G_\Omega \end{array}$$

Since the vertical map $\mathcal{C} \to \mathcal{H}$ is surjective, and since the top three maps are homomorphisms, the bottom map must also be a homomorphism.

This Frobenius map $\mathcal{H} \to G/G_\Omega$ is surjective since if σG_Ω is in G/G_Ω, then σ is the Frobenius in G for some $\mathfrak{p}$, which means σG_Ω is the corresponding Frobenius in G/G_Ω. Since $\mathcal{H}$ and G/G_Ω have the same order, the map is in fact an isomorphism. □

Remark. Artin's proof original is a bit terse, so I expanded it a bit in my translation (and even snuck in a commutative diagram not in the original). I will add extra explanatory details to other proofs as we proceed. One thing Artin did not need to do, however, is to argue that the map $\mathcal{H} \to G/G_\Omega$ is surjective since as pointed in a remark after result 1. above, we know such a Frobenius map must be an isomorphism once we know it is a homomorphism. Alternatively, we can see surjectivity right away from the commutative diagram and the fact that the top and right maps are obviously surjective.

[9] Let C_1 be the identity class (die Hauptklasse) of $\mathcal{C}$ and let $\mathcal{I}_\mathfrak{m}$ be the full group of fractional ideals of $\mathcal{O}_k$ relatively prime to the modulus $\mathfrak{m}$. Then there is a principle of class field theory similar to what we find in Galois theory: the intermediate fields of K/k are in bijective correspondence with subgroups of $\mathcal{I}_\mathfrak{m}$ containing C_1. This correspondence reverses inclusion. Given such a subgroup H_1 of fractional ideals, the Galois group of the corresponding intermediate extension Ω/k is isomorphic to $\mathcal{I}_m/H_1$, and the cosets of H_1 in $\mathcal{I}_m$ give the class group associated to Ω/k. Note also that the prime ideals of H_1 are exactly the prime ideals in $\mathcal{I}_\mathfrak{m}$ that split in Ω.

[10] See Lemma 1.

Remark. In several places in the above proof we used the the compatibility of the Frobenius for K/k compared to Ω/k. This was addressed at the end of Section 2 (and is summarized in Lemma 1 in the commentary). Note that this compatibility is what justifies the commutative diagram that I inserted into the above proof.

Remark. The next result can also be regarded as a justification for either the strong or the weak versions of Satz 2. In other words, if the strong version of Satz 2 holds for K_1 and K_2 then the proof yields the strong version for K_1K_2. If, however, only the weak version of Satz 2 holds for K_1 and K_2 then the proof can be regarded as a proof for the weak version of K_1K_2. This is based on the observation that any modulus valid for an Abelian extension is valid for any subextension.

3. *Suppose Satz 2 holds for two Abelian extension K_1 and K_2 of k whose intersection is k, then it holds for the composite field $K = K_1K_2$. .*

Proof. Let $\mathfrak{m}$ be common modulus such that Satz 2 holds for K_1 and K_2 with modulus $\mathfrak{m}$, and let $C_1, \dots, C_n; D_1, \dots, D_m$ be classes taken for the modulus $\mathfrak{m}$ where the C_i form the class group for K_1 and D_i form the class group for K_2. Let G_1 be the Galois group of K_1/k and G_2 be the Galois group of K_2/k. Suppose that C_i has Frobenius $\sigma_i \in G_1$ and D_j has Frobenius $\tau_j \in G_2$.

As we know from Galois theory, the Galois group of K_1K_2/k can be identified with $G_1 \times G_2$. Note that K_1K_2 is the class field associated to the class group described by the partition of ideals prime to $\mathfrak{m}$ given by the intersections $C_r \cap D_s$.[11] The product of classes for this class group is described by the following equation:

$$(C_r \cap D_s)(C_u \cap D_v) = C_rC_u \cap D_sD_v.$$

Now let $A_1 \in \mathcal{O}_{K_1}$ and $A_2 \in \mathcal{O}_{K_2}$ be generators fo K_1/k and K_2/k respectively. Let $A = \varphi(A_1, A_2)$ be in $\mathcal{O}_{K_1K_2}$. Let $\mathfrak{p}$ be in $C_r \cap D_s$. Then

$$A^{N\mathfrak{p}} \equiv \varphi(A_1^{N\mathfrak{p}}, A_2^{N\mathfrak{p}}) \equiv \varphi(\sigma_r A_1, \tau_s A_2) \equiv (\sigma_r, \tau_s)A \pmod{}p.$$

Suppose A is an integral element of K_1K_2 of the form A_1A_2 with $A_1 \in \mathcal{O}_{K_1}$ and $A_2 \in \mathcal{O}_{K_2}$. If $\mathfrak{p}$ is a prime ideal in $C_r \cap D_s$ then

$$A^{N\mathfrak{p}} = A_1^{N\mathfrak{p}}A_2^{N\mathfrak{p}} \equiv (\sigma_r A_1)(\tau_s A_2) = (\sigma_r, \tau_s)A \pmod{\mathfrak{p}}$$

Thus the Frobenius of any prime ideal in $C_r \cap D_s$ is (σ_r, τ_s), which is independent of the choice of $\mathfrak{p}$. So the first part of Satz 2 holds. The second part follows as well based on what we have shown. □

[11] This is not too difficult to show. In fact we can appeal the the principle of footnote 9. Let $\mathcal{I}_\mathfrak{m}$ be the group of fractional ideals of $\mathcal{O}_k$ prime to $\mathfrak{m}$, and let $\mathcal{P}_\mathfrak{m}$ be the subgroup of principal ideals with totally positive generators congruent to 1 modulo $\mathfrak{m}$. Then $\mathcal{P}_\mathfrak{m}$ corresponds to the ray class field $L_\mathfrak{m}$ that clearly contains K_1K_2 since it contains both K_1 and K_2. Under the correspondence between subfields of $L_\mathfrak{m}$ containing k and subgroups of $\mathcal{I}_\mathfrak{m}$ containing $\mathcal{P}_\mathfrak{m}$, the group C_1 corresponds to K_1 and D_1 corresponds to K_2. So $C_1 \cap D_1$ corresponds to the smallest subfield of $L_\mathfrak{m}$ containing both K_1 and K_2, which is just K_1K_2. The cosets of $C_1 \cap D_1$ in $\mathcal{I}_\mathfrak{m}$ can be seen to be the sets $C_r \cap D_s$ as desired. To see this consider the injective homomorphism $\mathcal{I}_\mathfrak{m}/(C_1 \cap D_1) \to \mathcal{I}_\mathfrak{m}/C_1 \times \mathcal{I}_\mathfrak{m}/D_1$ which must be an isomorphism since $[K_1K_2 : k] = [K_1 : k][K_2 : k]$ (or equivalently, since C_1D_1 corresponds to k, the smallest common subfield of K_1 and K_2, and so must be all of $\mathcal{I}_\mathfrak{m}$).

Remark. In the above proof, Artin does not describe explicity what $\varphi(x, y)$ is, but from context it seems to be a polynomial in $k[x, y]$. Furthermore, to support the congruences, the coefficients should be expressible as fractions of integral elements with denominators not in $\mathfrak{p}$. Artin does not address the existence of such a polynomial. Fortunately, there is straightforward way to prove the result that does not rely such a polynomial $\varphi(x, y)$:

As in the above proof, let $\mathfrak{p}$ be a prime ideal of $C_r \cap D_s$ with Frobenius element $(\sigma, \tau) \in G_1 \times G_2$. By Lemma 1, and thinking of the Galois group of K_1/k as the quotient $G_1 \times G_2/G_2$ (with G_2 embedded in $G_1 \times G_2$ in the usual way) then the Frobenius element of $\mathfrak{p}$ for the extension K_1/k is the coset

$$(\sigma, \tau)G_2 = (\sigma, 1)G_2.$$

Under the identification of $G_1 \times G_2/G_2$ with G_1, which identifies the two descriptions of the Galois group of K_1/k, this element is σ. Thus $\sigma = \sigma_r$ since σ_r is the Frobenius element of $\mathfrak{p}$ for K_1/k. Similarly, $\tau = \tau_s$. Thus the Frobenius of $\mathfrak{p}$ is (σ_r, τ_t) as claimed.

Note that because of 3. (and the structure theorem of finite Abelian groups) we can reduce the proof of Satz 2 to cyclic extensions of prime power degree. However, in this paper we will only fully succeed in proving Satz 2 in the case of cyclic extensions of prime degree.

4. *Satz 2 holds for $K = k(\zeta)$ where $\zeta = e^{\frac{2\pi i}{m}}$ is an mth root of unity.*[12]

Proof. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a class group associated to the field extension K/k where, as usual, C_1 is the identity class (die Hauptklasse). For now we allow any modulus $\mathfrak{m}$ for $\mathcal{C}$, valid for K/k, that at least satisfies the following condition: every prime ideal dividing $m\mathcal{O}_k$ also divides $\mathfrak{m}$ (we will later show that $\mathfrak{m} = m\mathcal{O}_k$ is in fact valid). The first step is to identify the prime ideals in C_1 by determining a splitting law. In other words, we wish to describe which prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ prime to $\mathfrak{m}$ have the property that $\mathfrak{p}\mathcal{O}_K$ factors into distinct primes of relative degree one.

Given such a prime ideal $\mathfrak{p}$, we know that $\mathfrak{p}$ is unramified in K/k and that the distinct mth-roots of unity in $\mathcal{O}_K$ map to distinct mth roots of unity in the residue field $\mathcal{O}_K/\mathfrak{P}$ for any prime $\mathfrak{P}$ above $\mathfrak{p}$. So $\mathcal{O}_K/\mathfrak{P}$ contains all the mth roots of unity. Since $\mathfrak{p}$ splits in $\mathcal{O}_K$, the residue field $\mathcal{O}_k/\mathfrak{p}$ is isomorphic to $\mathcal{O}_K/\mathfrak{P}$ and so itself contains all the mth root of unity. In this case the order $N\mathfrak{p}-1$ of the multiplicative group $(\mathcal{O}_k/\mathfrak{p})^\times$ is divisible by m. In other words, $N\mathfrak{p} \equiv 1 \pmod{m}$.

Conversely, suppose $N\mathfrak{p} \equiv 1 \pmod{m}$ where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_k$ not dividing $\mathfrak{m}$. Then for each algebraic integer $A = \alpha_0 + \alpha_1\zeta + \ldots$ in $\mathcal{O}_K$ (with $\alpha_i \in \mathcal{O}_k$)

$$A^{N\mathfrak{p}} \equiv A \pmod{\mathfrak{p}}.$$

So the residue field $\mathcal{O}_K/\mathfrak{P}$ has size bounded by $N\mathfrak{p}$, and so equal to $N\mathfrak{p}$, for all primes $\mathfrak{P}$ above $\mathfrak{p}$. Thus $\mathfrak{p}$ splits in $\mathcal{O}_K$.

We have now established our desired splitting law: for prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ prime to $\mathfrak{m}$, then $\mathfrak{p}$ splits if and only if $N\mathfrak{p} \equiv 1 \pmod{m}$. So by a fundamental

[12] An analogous proof can be produced for class fields of complex multiplication. This shows how the reciprocity laws can be obtained through transcendental generators of the class fields.

result of class field theory, for prime ideals $\mathfrak{p}$ of $\mathcal{O}_k$ prime to $\mathfrak{m}$, we have $\mathfrak{p} \in C_1$ if and only if $N\mathfrak{p} \equiv 1 \pmod{m}$.

We wish to extend this to showing that C_1 consists the of the fractional ideals $\mathfrak{a}$ prime to $\mathfrak{m}$ such that $N\mathfrak{a} \equiv 1 \pmod{m}$, and in fact that all fractional ideals in a given class C_i have the same norm modulo m. It turns out that we can do this by showing that K is contained in the ray class field of k for modulus $m\mathcal{O}_k$, which will allow us to choose $\mathfrak{m}$ to be $m\mathcal{O}_k$. So let $\mathcal{C}_m$ be the ray class group of k modulo m.[13]

Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathcal{O}_k$ in the same class in $\mathcal{C}_m$. Then $\mathfrak{a} = \alpha\mathfrak{b}$ for some $\alpha \in k$ positive in all embeddings of k into $\mathbb{R}$ and such that $\alpha \equiv 1 \pmod{m}$. For such α we have

$$N(\alpha\mathcal{O}_k) = |N\alpha| = N\alpha \equiv 1 \pmod{m},$$

so

$$N\mathfrak{a} \equiv N\alpha N\mathfrak{b} \equiv N\mathfrak{b} \pmod{m}.$$

So all the ideals in a given class of $\mathcal{C}_m$ have the same norm, and we have a homomorphism $\mathcal{C}_m \to (\mathbb{Z}/m\mathbb{Z})^\times$. Combining classes of norm 1 yields a subgroup $\mathcal{K}_m$ of $\mathcal{C}_m$ (the kernel of this norm homomorphism), and the quotient $\mathcal{C}_m/\mathcal{K}_m$ determines a class group with the property that two fractional ideals (prime to m) are in the same class if and only if they have the same norm.

In particular, $\mathcal{C}$ and $\mathcal{C}_m/\mathcal{K}_m$ both have the property that (with at most finitely many exceptions) a prime ideal $\mathfrak{p}$ is in the identity class if and only if $N\mathfrak{p} = 1$. According to class field theory this means that the class fields of $\mathcal{C}$ and $\mathcal{C}_m/\mathcal{K}_m$ have the same primes that split (with a finite number of possible exceptions), and so must be equal. So K is the class field of the class group $\mathcal{C}_m/\mathcal{K}_m$, where this class group is taken to have modulus $m\mathcal{O}_k$. We can now fix $\mathfrak{m}$ to be $m\mathcal{O}_k$, and identify $\mathcal{C}$ with $\mathcal{C}_m/\mathcal{K}_m$. In particular all fractional ideals of a given class C_i have the same norm modulo m.

Since $K = k(\zeta)$, we can view the Galois group G of K/k to be a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$ where σ is identified with the integer t modulo m for which $\sigma\zeta = \zeta^t$.

Let C_i be a class of $\mathcal{C}$, and assume that the fractional ideals of C_i have norm congruent to n_i modulo m. Let $\sigma \in G$ be the Frobenius element of some prime $\mathfrak{p}$ of C_i. Since $\sigma A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{P}}$ for all $A \in \mathcal{O}_K$ and all primes $\mathfrak{P}$ in $\mathcal{O}_K$ above $\mathfrak{p}$, we have in particular that

$$\sigma\zeta \equiv \zeta^{N\mathfrak{p}} \equiv \zeta^{n_i} \pmod{\mathfrak{P}}.$$

[13] The ray class group modulo $\mathfrak{m}$ can be defined as $\mathcal{I}_\mathfrak{m}/\mathcal{P}_\mathfrak{m}$ where $\mathcal{I}_\mathfrak{m}$ is the group of fractional ideals prime to $\mathfrak{m}$ and $\mathcal{P}_\mathfrak{m}$ is the subgroup of principal ideals generated by elements $\alpha \in k$ such that $\alpha \equiv 1 \pmod{\mathfrak{m}}$ and such that α is positive in all real embeddings of k. It is a basic result that every class of the ray class group contains integral ideals, and in fact prime ideals (by a generalization of Dirichlet's theorem).

The condition $\alpha \equiv 1 \pmod{\mathfrak{m}}$ can be interpreted as saying that α is the quotient β/γ of algebraic integers such that β and γ are prime to $\mathfrak{m}$ and such that $\beta \equiv \gamma \pmod{\mathfrak{m}}$. In the current proof we are concerned with the ideal $\mathfrak{m} = m\mathcal{O}_k$, and so we have $\sigma\beta \equiv \sigma\gamma \pmod{m}$ for all σ in the Galois group of K/k. In particular, $N\beta \equiv N\gamma \pmod{m}$, which we can express as saying that $N\alpha \equiv 1 \pmod{m}$. This is the norm of α as an element of $\mathbb{Q}$; the norm of the associated principal fractional ideal is the absolute value of the norm of its generator α. Since we assume that α is positive in all real embeddings of k in $\mathbb{R}$, its norm is positive, and so we get that $N(\alpha\mathcal{O}_k) \equiv 1 \pmod{m}$ where here we mean the norm of the associated principal fractional ideal.

However $\sigma\zeta = \zeta^t$ for some integer t. So $\zeta^{n_i} \equiv \zeta^t \pmod{\mathfrak{P}}$. As mentioned above, distinct m-roots of unity in $\mathcal{O}_K$ map to distinct mth roots of unity in the residue field $\mathcal{O}_K/\mathfrak{P}$. We conclude that $\zeta^{n_i} = \zeta^t$, and so, identifying G with a subgroup of $(\mathbb{Z}/m\mathbb{Z})^\times$, we see that the Frobenius element is just $n_i \in (\mathbb{Z}/m\mathbb{Z})^\times$. In particular all primes of C_i share the same Frobenius element, proving the first part of Satz 2.

By the multiplicativity of the norm map, the Frobenius map is a homomorphism. Observe that the Frobenius map has kernel consisting of the class C_1 alone since only ideals in C_1 have norm conguent to 1 modulo m. Since $\mathcal{C}$ and G have the same number of elements (according to Takagi's theory), the induced map $\mathcal{C} \to G$ is an isomorphism. □

Remark. As mentioned above, it is not really necessary to prove the map is an isomorphism since it being a homomorphism is enough. (See remark after claim 1.).

Remark. This gives Satz 2 specifically for modulus $m\mathcal{O}_k$.

Remark. At this point we know that at least a weak form of Satz 2 holds when $k = \mathbb{Q}$ (using 2., 4. and the Kronecker-Weber theorem that every finite Abelian extension of $\mathbb{Q}$ is a subfield of $\mathbb{Q}(\zeta)$ for suitable ζ).

Remark. As with other proofs in this translation, the above proof is much expanded and somewhat modified from Artin's original proof in order to make the argument more accessible to the modern reader. Here we provide more commentary for the proof. Let $\mathfrak{p}$ be prime to $\mathfrak{m}$. We can use the factorization of the polynomial $x^m - 1$ in $\mathcal{O}_K[x]$ into linear polynomials and its reduction modulo $\mathfrak{p}$ to get a factorization into linear polynomials $(\mathcal{O}_K/\mathfrak{P})[x]$. Since the deriviative mx^{m-1} is relatively prime to $x^m - 1$, the roots in $(\mathcal{O}_K/\mathfrak{P})[x]$ must be distinct (we know that m is not in $\mathfrak{P}$ by our assumption on $\mathfrak{m}$). This explains why distinct mth roots of unity map to distinct roots of unity in the residue field $\mathcal{O}_K/\mathfrak{P}$.

We also used the fact that $(A_1 + A_2)^{N(\mathfrak{p})} \equiv A_1^{N(\mathfrak{p})} + A_2^{N(\mathfrak{p})} \pmod{\mathfrak{p}}$ for all $A_1, A_2 \in \mathcal{O}_K$. This follows from the fact that $N(\mathfrak{p})$ is a power of the characteristic p of $\mathcal{O}_k/\mathfrak{p}$.

The next result gives Satz 2 for a class of Kummer extensions:

5. *Suppose k contains the root of unity $\zeta = e^{\frac{2\pi i}{m}}$ where $m = l^n$ is a power of a prime l. Then Satz 2 holds for all cyclic extensions K of k of degree $m = l^n$.*

Proof. It is a standard result of Galois theory[14] that any such extension K is of the form $k\left(\mu^{1/m}\right)$ for some $\mu \in k$ and some fixed choice $\mu^{1/m}$ of mth root. Observe also that the Galois group G can be identified with the group of mth roots of unity: the action of $\sigma \in G$ is determined by image of $\mu^{1/m}$ which must be of the form $c(\sigma)\mu^{1/m}$ for some mth root of unity $c(\sigma)$. The map $\sigma \mapsto c(\sigma)$ is our desired isomorphism of G with the group of mth roots of unity. For convenience we can take μ to be in $\mathcal{O}_k$ so that $\mu^{1/m} \in \mathcal{O}_K$.

[14] See for instance Aluffi [1], Chapter VII, Proposition 6.19. In fact, this result is so central to Galois theory that it was essentially stated by Galois himself in the case that m is prime, but Galois' argument has a gap (essentially he fails to show $\mu \neq 0$). See Edwards [8], §46, Page 63 for a discussion of the gap in Galois' manuscript and a simple way to fix it in a manner that would have been accessible to Galois himself.

Suppose $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_k$ prime to l. Since $\zeta \in \mathcal{O}_k$, the residue field $\mathcal{O}_k/\mathfrak{p}$ has a primitive mth root of unity. In other words m divides $N\mathfrak{p}-1$, the order of the multiplicative group of the residue field $\mathcal{O}_k/\mathfrak{p}$. In other words, $N\mathfrak{p} \equiv 1 \pmod{m}$.

Therefore,

$$\left(\mu^{1/m}\right)^{N\mathfrak{p}} \equiv \mu^{(N\mathfrak{p}-1)/m}\mu^{1/m} \equiv \left(\frac{\mu}{\mathfrak{p}}\right)\mu^{1/m} \pmod{\mathfrak{p}}$$

where $\left(\frac{\mu}{\mathfrak{p}}\right)$ is the mth power character, whose values are mth roots of unity.[15] In particular, the Frobenius element for $\mathfrak{p}$ is the element of G identified with the mth root of unity $\left(\frac{\mu}{\mathfrak{p}}\right)$.

The essential statement of the general reciprocity law, as given by Takagi, is exactly that $\left(\frac{\mu}{\mathfrak{p}}\right)$ only depends on the class containing $\mathfrak{p}$ (in fact, this holds for any ideal $\mathfrak{a}$ prime to μ).[16] So let $\mathcal{C}$ be a class group for K/k with modulus $\mathfrak{m}$ (containing μ and l, say) for which we are certain that $\left(\frac{\mu}{\mathfrak{a}}\right)$ depends only on the class of $\mathfrak{a}$ in $\mathcal{C}$ for all integral ideals relatively prime to $\mathfrak{m}$.[17] It follows now that the first part of Satz 2 holds for such a modulus $\mathfrak{m}$. The multiplicativity of the power character implies that the Frobenius map is a homomorphism. From this we get the rest of Satz 2.[18] □

Remark. The above proof is perhaps the most challenging for the modern reader to verify since it relies on results of Takagi that Artin does not spell out in detail (nor do the modern sources I have consulted). The power character is well-known though and is easy to define. Following Section 4.1 of [11], let k be a number field containing all the mth roots of unity where m is a positive integer. Recall that the reduction mod $\mathfrak{p}$ map sends distinct mth roots of unity to distinct mth roots of unity when $\mathfrak{p}$ is prime to m. By Fermat's little theorem we have

$$\alpha^{N\mathfrak{p}-1} \equiv 1 \pmod{\mathfrak{p}}$$

for all $\alpha \in \mathcal{O}_k$ outside of $\mathfrak{p}$, and so $\alpha^{(N\mathfrak{p}-1)/m}$ reduces to an mth root of unity in the residue field $\mathcal{O}_k/\mathfrak{p}$. The power character $\left(\frac{\alpha}{\mathfrak{p}}\right)_m$ for such an α and $\mathfrak{p}$ is defined to be the unique m root of unity such that

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_m \equiv \alpha^{(N\mathfrak{p}-1)/m} \pmod{\mathfrak{p}}.$$

[15] See for instance Lemmermeyer [11], Section 4.1, or Ireland and Rosen [10], Section 14.2.

[16] Takagi [19].

[17] Artin's original proof does not specify what modulus $\mathfrak{m}$ will work here. Perhaps it is $l\mu\mathcal{O}_k$ or $m\mu\mathcal{O}_k$. In any case, it should be clear by looking at Takagi's paper [19]. Until I have the opportunity to consult Takagi's paper, I will just use any modulus that gets the job done here.

In fact, Artin does not mention the modulus at all in the proof. He also does not specify what l is, but it is pretty clear from context that l must at least be a prime. It could be that l is restricted to odd primes. Again, it might require digging into Takagi's paper.

[18] We get at least the weak form of Satz 2. Artin also mentions that $\left(\frac{\mu}{\mathfrak{p}}\right)$ can take on any mth root of unity as a value and uses this to justify surjectivity of the Frobenius map, but as mentioned after result 1. above this follows already from what we have done.

This can be extended from $\mathfrak{p}$ to other ideals prime to m by defining the symbol to be multiplicative with respect to ideal multiplication. We can even define $\left(\frac{\alpha}{\beta}\right)_m$ for relatively prime elements nonzero elements $\alpha, \beta \in \mathcal{O}_k$, with β prime to m, by considering the ideal generated by β.

The power character is central to the study of various reciprocity laws. For example, the Eisenstein reciprocity law ([10], Section 14.2, or [11] Section 11.2) can be elegantly expressed using the power character for the field $k = \mathbb{Q}(\zeta_l)$:

Theorem (Eisenstein reciprocity). *Suppose l is an odd prime, suppose ζ_l is a primitive lth root of unity, and suppose $a \in \mathbb{Z}$ is not divisible by l. If $\alpha \in \mathbb{Z}[\zeta_l]$ is relatively prime to a, and if α is a primary element (meaning that α is not a unit, is prime to l, and is congruent to an element of $\mathbb{Z}$ modulo $(1-\zeta_l)^2$) then*

$$\left(\frac{\alpha}{a}\right)_l = \left(\frac{a}{\alpha}\right)_l.$$

According to [11] (Section 11.4), Takagi [19] generalized this result from $\mathbb{Q}(\zeta_l)$ to any number field containing ζ_l. Apparently Takagi connected such reciprocity laws with his class field theory, which then opened the door to the above result of Artin and to Artin's general reciprocity law.

Remark. Because of the close connection between the Frobenius element and the power character illustrated in the above proof, it is common to introduce reciprocity-like symbols for the Frobenius. The expression

$$\left(\frac{K/k}{\mathfrak{p}}\right)$$

denotes the Frobenius element associated to $\mathfrak{p}$. As usual, here K/k is an Abelian extension of number fields, and $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_k$ unramified in K/k. This symbol is called the *Artin symbol* in honor of the ideas introduced in this paper. When K/k is Galois but not Abelian, the Frobenius element depends on the choice of prime above $\mathfrak{p}$, and this leads to the *Frobenius symbol* (introduced by Hasse)

$$\left[\frac{K/k}{\mathfrak{P}}\right]$$

where $\mathfrak{P}$ is a prime ideal of $\mathcal{O}_K$ unramifield in K/k (See Section 3.2 of [11]).

6. *Suppose $K = k(\alpha)$ is cyclic of degree $r = l^n$ over k where l is a prime, and suppose $\Omega = k(\zeta)$ is an extension of k of degree m where $\zeta = e^{2\pi i/l}$. So m divides $l-1$ and is necessarily prime to l. If Satz 2 holds for $K^* = \Omega(\alpha)$ over Ω then Satz 2 holds for K over k.*

Remark. This claim can be generalized, with essentially the same proof, to the following:

Suppose K/k is an an Abelian extension of degree r, and suppose Ω/k is an Abelian extension of degree m where m and r are relatively prime. Let $K^* = K\Omega$ be the composite field. If Satz 2 holds for K^*/Ω then Satz 2 holds for K/k as well.

In the following translation, Artin's original argument has been adapted to support this more general statement.

Proof. We write our Galois groups as $G(K/k), G(K^*/K), G(\Omega/k), G(K^*/\Omega)$, and $G(K^*/k)$, where K^* is the composite field $K\Omega$. Since r and l are relatively prime, the intersection of K and Ω is k, and $G(K^*/k)$ can be identified with

$$G(K/k) \times G(\Omega/k)$$

using the usual isomorphisms from Galois theory. This identification also allows us to identify $G(K/k)$ with $G(K^*/\Omega)$, and $G(\Omega/k)$ with $G(K^*/K)$. We will write G for both $G(K/k)$ and $G(K^*/\Omega)$, and we will write H for both $G(\Omega/k)$ with $G(K^*/K)$. Thus, for example, if $\sigma \in G$ then σ can be regarded as an automorphism of K fixing k, or as the unique extension of this automorphism to an automorphism K^* that fixes Ω.

Fix an ideal $\mathfrak{m}$ of $\mathcal{O}_k$ that gives a valid modulus for the class group of K^*/k. So $\mathfrak{m}$ is also a valid modulus for the subextensions K/k and Ω/k. Let $\mathcal{C}(K^*/k)$, $\mathcal{C}(K/k)$, and $\mathcal{C}(\Omega/k)$ be the respective class groups all using modulus $\mathfrak{m}$. By replacing $\mathfrak{m}$ by a multiple if necessary we can also choose $\mathfrak{m}$ so that Satz 2 holds for K/Ω with modulus $\mathfrak{m}\mathcal{O}_\Omega$

Step 1. The first step of the proof is to construct a class group $\mathcal{C}(K^*/\Omega)$ with modulus $\mathfrak{m}\mathcal{O}_\Omega$ together with an explicit isomorphism $\mathcal{C}(K^*/\Omega) \to \mathcal{C}(K/k)$. We begin by considering the relative norm map $\mathcal{I}_\Omega \to \mathcal{I}_k$ where $\mathcal{I}_\Omega$ is the group of fractional ideals of Ω prime to $\mathfrak{m}\mathcal{O}_\Omega$ and where $\mathcal{I}_k$ is the group of fractional ideals of k prime to $\mathfrak{m}$. Note that $\mathcal{C}(K/k)$ can be described as a quotient group of $\mathcal{I}_k$ and so the composition

$$\mathcal{I}_\Omega \to \mathcal{I}_k \to \mathcal{C}(K/k)$$

is a homomorphism. Let $\mathfrak{C}_0$ be the kernel of this composition. Observe that if $\beta \in \Omega$ is prime to $\mathfrak{m}\mathcal{O}_\Omega$ and satisfies $\beta \equiv 1 \pmod{\mathfrak{m}\mathcal{O}_\Omega}$ then the relative norm $N\beta$ in k must satisfy the congruence $N\beta \equiv 1 \pmod{\mathfrak{m}}$ (since $\mathfrak{m}$ is the intersection of $\mathfrak{m}\mathcal{O}_\Omega$ with k). Furthermore, if β is also positive in all real embeddings of Ω into $\mathbb{R}$ then the relative norm $N(\beta) \in k$ is totally positive as well. In particular, the principal ideal generated by such β must be in the kernel $\mathfrak{C}_0$. This means that the quotient group $\mathcal{I}_\Omega/\mathfrak{C}_0$ yields a class group for modulus $\mathfrak{m}\mathcal{O}_\Omega$.

Let $\mathfrak{q}$ be a prime ideal of $\mathcal{O}_\Omega$ prime to $\mathfrak{m}\mathcal{O}_\Omega$, and let $\mathfrak{p}$ be the intersection of $\mathfrak{q}$ with the subfield k. In particular the relative norm $N\mathfrak{q} \subseteq \mathcal{O}_k$ is $\mathfrak{p}^f$ where f divides the relative degree m. Observe that $\mathfrak{q}$ splits in K^* if and only if $\mathfrak{p}$ splits in K, since f does not divide r. But $\mathfrak{p}$ splits in K if and only it is in the identity class of $\mathcal{C}(K/k)$. Since f is prime to the order of $\mathcal{C}(K/k)$ this occurs if and only if $\mathfrak{p}^f$ in in the identity class of $\mathcal{C}(K/k)$. In other words, $\mathfrak{q}$ splits in K^* if and only if $\mathfrak{q} \in \mathfrak{C}_0$. By Takagi's results this means that K^* is the class field extension of Ω corresponding to $\mathcal{I}_\Omega/\mathfrak{C}_0$. So we write $\mathcal{C}(K^*/\Omega)$ for $\mathcal{I}_\Omega/\mathfrak{C}_0$. Furthermore, the homomorphism $\mathcal{I}_\Omega \to \mathcal{C}(K/k)$ induces an injective homomorphism $\mathcal{C}(K^*/\Omega) \to \mathcal{C}(K/k)$. Since both groups have order r, this map $\mathcal{C}(K^*/\Omega) \to \mathcal{C}(K/k)$, induced by the relative norm map, is an isomorphism.

Step 2. The second step is to use the isomorphism of step 1, and the assumption of Satz 2 for K^*/Ω, to define a Frobenius isomorphism on the class group $\mathcal{C}(K/k)$. For the isomorphism we will try the composition

$$\mathcal{C}(K/k) \to \mathcal{C}(K^*/\Omega) \to G(K^*/\Omega) \to G(K/k)$$

where the first map is the inverse of the isomorphism of step 1, the second is the Frobenius isomorphism that exists by assumption of Satz 2 for K^*/Ω, and the third map is the natural isomorphism given by restrictions of automorphisms. This composition is an isomorphism, so to prove Satz 2 for K/k and modulus $\mathfrak{m}$ we just need to show that this maps the class of a prime ideal $\mathfrak{p}$ to its corresponding Frobenius element in $G(K/k)$.

So fix a prime ideal $\mathfrak{p}$ of $\mathcal{O}_k$ prime to $\mathfrak{m}$ and let $\mathfrak{q}$ be a prime ideal in $\mathcal{O}_\Omega$ above $\mathfrak{p}$. Let (σ, τ) be the Frobenius element in $G(K^*/k) = G \times H$ corresponding to $\mathfrak{p}$. Identifying G and H with subgroups of $G \times H$, we can write this element as $\sigma\tau$ and

$$\sigma\tau A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{p}}$$

for all $A \in \mathcal{O}_{K^*}$. So $\sigma A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{p}}$ for $A \in \mathcal{O}_K$, and $\tau A \equiv A^{N\mathfrak{p}} \pmod{\mathfrak{p}}$ for $A \in \mathcal{O}_\Omega$ (where here G is identified with $G(K^*/\Omega)$ and H is identified with $G(K^*/K)$). Thus σ is the Frobenius element of $\mathfrak{p}$ in $G = G(K/k)$, and τ is the Frobenius element of $\mathfrak{p}$ in $H = G(\Omega/k)$. Note that the relative norm $N\mathfrak{q}$ is $\mathfrak{p}^f$ where f is the order of τ in H. So

$$\sigma^f A \equiv \sigma^f \tau^f A \equiv (\sigma\tau)^f A \equiv A^{N\mathfrak{p}^f} \equiv A^{N\mathfrak{q}} \pmod{\mathfrak{q}}$$

for all $A \in \mathcal{O}_{K^*}$ (where here $N\mathfrak{q}$ is the absolute norm). Thus σ^f is the Frobenius element associated with $\mathfrak{q}$. Note that f divides $m = [\Omega : k]$, so f is relatively prime to $r = [K : k]$. This means that $uf \equiv 1 \pmod{r}$ for some u, and $(\sigma^f)^u = \sigma$.

The isomorphism $\mathcal{C}(K^*/\Omega) \to \mathcal{C}(K/k)$ from step 1 sends the class of $\mathfrak{q}^u$ to the class of its relative norm $(\mathfrak{p}^f)^u$. The class of $\mathfrak{p}^{fu}$ is the class of $\mathfrak{p}$ since $fu \equiv 1$ modulo r. Thus the inverse isomorphism $\mathcal{C}(K/k) \to \mathcal{C}(K^*/\Omega)$ maps the class of $\mathfrak{p}$ to the class of $\mathfrak{q}^u$. Since the class of $\mathfrak{q}$ maps to its Frobenius σ^f under the next map $\mathcal{C}(K^*/\Omega) \to G(K^*/\Omega)$, the class of $\mathfrak{q}^u$ maps to

$$(\sigma^f)^u = \sigma^{fu} = \sigma.$$

Finally, σ maps to σ under the map $G(K^*/\Omega) \to G(K/k)$ (here we are identifying $G = G(K/k)$ with $G(K^*/\Omega)$).

In conclusion the above composition $\mathcal{C}(K/k) \to G(K/k)$ sends the class of a prime ideal to its Frobenius element. □

Remark. This shows that the weak version of Satz 2 for K^*/Ω implies the weak version of Satz 2 for K/k.

Remark. We now see that Satz 2 holds for any Abelian extension of degree equal to the product of distinct primes. To see this first assume that K/k has prime degree l. Using 5. we have Satz 2 for $K(\zeta)/k(\zeta)$ where ζ is a primitive lth root of unity. By 6. we have Satz 2 for K/k as well. Finally 3. extends Satz 2 to K/k when $[K : k]$ the product of distinct primes, or more generally when the Galois group is the product of cyclic groups of prime order.

Remark. So Artin has proved the following:

Theorem. *Suppose K/k is an Abelian extension of number fields with Galois group G. If G can be factored into cyclic groups of prime order then the weak form of Satz 2 holds for K/k.*

6 Continuation of $L(s,\chi)$ to $\Re(s) \leq 1$

We return to the general case, assuming Satz 2 holds for the Abelian case. We write $m(\sigma)$ for the order of an element $\sigma \in G$ of the Galois group of K/k. For each $\sigma \in G$, let $\mathfrak{g}^\sigma$ be the subgroup generated by σ, and let Ω_σ be the subfield of K of elements fixed by σ. So $\mathfrak{g}^\sigma$ is the Galois group of K/Ω_σ.

Let $\psi_i^{(\sigma)}$ for $i = 1, \ldots, m(\sigma)$ be the irreducible characters of the Abelian group $\mathfrak{g}^\sigma$ where $\psi_1^{(\sigma)}$ is the trivial character (the "Hauptcharakter" or the "principal character"). If we denote by $\chi_{\psi_i^{(\sigma)}}$ the induced character of G then, by Satz 1, equation (15), we have the following which is valid up to a finite number of factors in the Euler products:

$$L\left(s, \psi_i^{(\sigma)} \,;\, \Omega_\sigma\right) = L\left(s, \chi_{\psi_i^{(\sigma)}} \,;\, k\right).$$

As in (7), we decompose each $\chi_{\psi_i^{(\sigma)}}$ and obtain the factorizations

$$L\left(s, \psi_i^{(\sigma)}\right) = \prod_{\nu=1}^{x} \left(L(s, \chi^\nu)\right)^{r_{i\nu}^{(\sigma)}} \qquad (i = 1, 2, \ldots, m(\sigma)) \tag{23}$$

where each $r_{i\nu}^{(\sigma)}$ is a nonnegative integer, and again with validity up to a finite number of factors in the Euler product. By Satz 2, the left-hand side of (23) corresponds to a traditional L-series whose extension to $\mathbb{C}$ and functional equation was established by Hecke. We can use the equations (23) to solve for $L(s, \chi^\nu)$ in order to prove the continuation of each $L(s, \chi^\nu)$. We can focus on the case $\nu > 1$ since $L(s, \chi^1) = \zeta_k(s)$ is a Dedekind zeta function whose meromorphic continuation is known.[19]

For $\nu > 1$ we will show that $L(s, \chi^\nu)$ can be expressed in terms of a product of rational powers of the $L\left(s, \psi_i^{(\sigma)}\right)$ where σ varies in G and i varies in $\{2, \ldots, m(\sigma)\}$, avoiding the trivial character $\psi_1^{(\sigma)}$.

Because of (23) it suffices to show that the system of x linear equations

$$\sum_{\sigma \neq 1} \sum_{i=2}^{m(\sigma)} r_{i\nu}^{(\sigma)} x_i^\sigma = \delta_{k\nu} \qquad \nu = 1, 2, \ldots, x \tag{24}$$

can be solved for each given k in the sequence $2, \ldots, x$.[20]

Remark. Suppose $x_i^\sigma \in \mathbb{Q}$ is a solution to the above system of linear equations (for

[19] Actually for any χ^i of degree 1 we have the continuation since that is the case that Hecke considered (assuming Satz 2).

[20] Here Artin is using k as an index. Once we show (24) can be solved, k will return to its role as denoting the base field.

a fixed k), then

$$\begin{aligned}\prod_{\sigma\neq 1}\prod_{i=2}^{m(\sigma)} L\left(s,\psi_i^{(\sigma)}\right)^{x_i^\sigma} &= \prod_{\sigma\neq 1}\prod_{i=2}^{m(\sigma)}\prod_{\nu=1}^{x} L(s,\chi^\nu)^{r_{i\nu}^{(\sigma)}x_i^\sigma}\\ &= \prod_{\nu=1}^{x}\prod_{\sigma\neq 1}\prod_{i=2}^{m(\sigma)} L\left(s,\chi^\nu\right)^{r_{i\nu}^{(\sigma)}x_i^\sigma}\\ &= \prod_{\nu=1}^{x} L(s,\chi^\nu)^{\delta_{k\nu}}\\ &= L(s,\chi^k).\end{aligned}$$

There is a subtlety here: the x_i^σ are allowed to be rational and so the above quantities are dependent on how the various rational powers are chosen. Depending on the choices it is possible that the calculation is valid only up to a dth root of unity where d is a common denominator for the x_i^σ. So we should think of this equality as holding up to a dth root of unity, and as usual up to a finite number of Euler factors.

What we can safely say is that $L(s,\chi^k)^d$ can be expressed in terms of a product of integral powers of the $L\left(s,\psi_i^{(\sigma)}\right)$ (ignoring a finite number of Euler factors), and so $L(s,\chi^k)^d$ has a meromorphic continuation to $\mathbb{C}$. Another way to say this is that there is a meromorphic continuation of $L(s,\chi^k)$ on a Riemann surfaces $\mathcal{L}$ mapping onto $\mathbb{C}$ with fibers of size bounded by d. Or we can take the old point of view that $L(s,\chi^k)$ is a "multivalued function" that has an analytic continuation outside a discrete set of singularities.

As we will see, Artin suspects this continuation is single valued (that is, $\mathcal{L}$ can be taken to be $\mathbb{C}$). In other words, Artin hoped that $L(s,\chi^k)$ itself, and not a power, has a meromorphic continuation. This was first proved by R. Brauer [6] in 1947. Artin's deeper conjecture that this continuation is actually analytic when $\chi^k \neq 1$ is still open.

Now $r_{i1} = 0$ for each $i > 1$, so equation (24) with $\nu = 1$ automatically holds.[21] Thus we only need to consider $\nu \geq 2$. So in order for (24) to be solvable, it is sufficient that the matrix

$$\left(r_{i\nu}^{(\sigma)}\right) \qquad \sigma \neq 1, \qquad i = 2,\ldots,m(\sigma); \qquad \nu = 2,\ldots,x$$

has rank $x-1$. Here we regard the columns as being indexed by (σ,i) and rows as being indexed by ν. For this matrix to have rank $x-1$ it is necessary and sufficient that the $x-1$ rows of this matrix be linearly independent. So we just need to to show that the only solution to the system of linear equations

$$\sum_{\nu=2}^{x} r_{i\nu}^{(\sigma)} y_\nu = 0 \tag{25}$$

is the zero solution with $y_\nu = 0$ (where the system contains an equation for each (σ,i) where $\sigma \neq 1$ and $2 \leq i \leq m(\sigma)$.) So assume $y_2,\ldots,y_x$ is a solution

[21] This follows from (6).

to the system. Fix σ and $\tau \in \mathfrak{g}^\sigma$ (where $\tau = 1$ is allowed), and for each i from 2 to $m(\sigma)$ multiply (25) by $\psi_i^{(\sigma)}(\tau)$. Now sum the resulting equations as i varies:

$$\sum_{i=2}^{m(\sigma)} \sum_{\nu=2}^{x} r_{i\nu}^{(\sigma)} \psi_i^{(\sigma)}(\tau)\, y_\nu = 0.$$

Using (6) we can simplify this equation, giving the following equation for each choice of $\sigma \in G$ and $\tau \in \mathfrak{g}^\sigma$:

$$\sum_{\nu=2}^{x} \big(\chi^\nu(\tau) - r_{1\nu}^{(\sigma)}\big) y_\nu = 0$$

or

$$\sum_{\nu=2}^{x} \chi^\nu(\tau) y_\nu = \sum_{\nu=2}^{x} r_{1\nu}^{(\sigma)} y_\nu.$$

The right hand side does not depend on τ, so the left hand side has the same value for all $\tau \in \mathfrak{g}^\sigma$. In particular,

$$\sum_{\nu=2}^{x} \chi^\nu(\tau) y_\nu = \sum_{\nu=2}^{x} \chi^\nu(1) y_\nu$$

for all $\tau \in \mathfrak{g}^\sigma$. Note the right hand side of this equation does not depend on σ, and so the left hand side has the same value for all $\tau \in G$. Call this value $-y_1$, so

$$\sum_{\nu=1}^{x} \chi^\nu(\tau) y_\nu = 0$$

for all $\tau \in G$. Using (2), we see that for each $i \in \{1, \ldots, x\}$

$$\begin{aligned} 0 = 0 \cdot \sum_{\tau \in G} \chi^i(\tau^{-1}) &= \left(\sum_{\nu=1}^{x} \chi^\nu(\tau) y_\nu \right) \sum_{\tau \in G} \chi^i(\tau^{-1}) \\ &= \sum_{\nu=1}^{x} \left(\sum_{\tau \in G} \chi^\nu(\tau) \chi^i(\tau^{-1}) \right) y_\nu \\ &= \sum_{\nu=1}^{x} n \delta_{\nu i} y_\nu = n y_i. \end{aligned}$$

So $y_i = 0$ for all $i \in \{1, \ldots, x\}$, establishing the linear independence claim, and so the solvability of (24).

We can now express each $L(s, \chi^\nu)$ in terms of Abelian L-series, which gives us a way to extend $L(s, \chi^\nu)$ with properties similar to those of traditional L-series. For example, if χ^ν is not the identity character ($\nu > 1$) the expression only involves $L\big(s, \psi_i^{(\sigma)}\big)$ with $i \neq 1$, so $L(s, \chi^j)$ is regular and nonvanishing at $s = 1$.

Now we change our initial definition of L-functions. A solution to (24) expresses $L(s, \chi^\nu)$ in terms of a product of rational powers of traditional L-series but only up to a finite number of factors. We can modify the definition of $L(s, \chi^\nu)$ so

that this expression is an exact equality, and then use (14) to define $L(s,\chi)$ for general characters. This modified definition changes $L(s,\chi)$ up to a finite number of factors, so all our results that are valid up to a finite number of factors will continue to hold with the modified definition. The resulting $L(s,\chi)$ will analytically continue as a multivalued function on the whole plane $\mathbb{C}$ minus possibly a discrete set of branch points, and going around a branch point will only change the value by a root of unity. The functional equation of Hecke holds for the Abelian L-series, so will yield a functional equation for our new L-series. This functional equation can be used to show that the definition of our L-series is independent of the solution to (24) used to build our new L-series.

Remark. Let $x_i^\sigma \in \mathbb{Q}$ be the numbers occurring in a solution to (24) (where we change k to j in what follows), then Artin proposes to use the resulting relation, originally valid only up to a finite number of Euler factors, as a new, modified definition:

$$L(s,\chi^j) \stackrel{\text{def}}{=} \prod_{\sigma\neq 1}\prod_{i=2}^{m(\sigma)} L\left(s,\psi_i^{(\sigma)}\right)^{x_i^\sigma}.$$

This makes $L(s,\chi^j)$ a multivalued function on $\mathbb{C}$ minus a discrete set of branch points, that is to say it is a meromorphic function on a Riemann surface covering $\mathbb{C}$. If d is the common denominator of the x_i^σ then

$$L(s,\chi^j)^d = \prod_{\sigma\neq 1}\prod_{i=2}^{m(\sigma)} L\left(s,\psi_i^{(\sigma)}\right)^{dx_i^\sigma}$$

gives an exact equation between meromorphic functions, where the functions on the right satisfy nice functional equations established by Hecke. From this we see that Artin's definition actually gives $L(s,\chi^j)$ as a meromorphic function on a Riemann surface $\mathcal{L}$ which covers $\mathbb{C}$ with degree bounded by d.

If we want to derive a functional equation for this meromorphic function $L(s,\chi^j)^d$ we need to observe that we can use the same solution to (24) for writing $L(s,\overline{\chi}^j)^d$ in terms of Abelian L-series:

$$L\left(s,\overline{\chi}^j\right)^d = \prod_{\sigma\neq 1}\prod_{i=2}^{m(\sigma)} L\left(s,\overline{\psi}_i^{(\sigma)}\right)^{dx_i^\sigma}$$

where $\overline{\chi}^j$ denotes the complex conjugate of χ^j, and $\overline{\psi}_i^{(\sigma)}$ denotes the complex conjugate of $\psi_i^{(\sigma)}$. The validity of this can be seen by oberving that (6) and (7) are well-behaved under complex conjugation, and the induced character of $\overline{\psi}_i^{(\sigma)}$ satisfies

$$\chi_{\overline{\psi}_i^{(\sigma)}} = \overline{\chi_{\psi_i^{(\sigma)}}}.$$

This gives us a version of (23) for conjugate characters using the same integers $r_{i\nu}^{(\sigma)}$ as the original (23), and so a solution to (24) will work for both $L(s,\chi^j)^d$ and $L\left(s,\overline{\chi}^j\right)^d$.

As we will see, the functional equation for Abelian L series is of a form that is closed under products, so gives a nice functional equation for $L(s,\chi^j)^d$. Artin

further observes that the functional equation forces $L(s,\chi^j)$, or better $L(s,\chi^j)^d$, to be independent of the solution to (24). In other words, there can be only one definition for $L(s,\chi^j)^d$ that satisfies such a functional equation and agrees with the earlier definition up to a finite number of Euler factors.

The functional equations for the Abelian L-series, and hence our new L-series, has the following form:[22]

$$L(1-s,\overline{\chi}^i) = a_i A^s \left(\Gamma(s)\right)^{l_i^{(1)}} \left(\cos\frac{s\pi}{2}\right)^{l_i^{(2)}} \left(\sin\frac{s\pi}{2}\right)^{l_i^{(3)}} L(s,\chi^i).$$

Here $l_i^{(1)}, l_i^{(2)}, l_i^{(3)}$ are rational, and A is a positive real number. Note that a_i depends on a choice of branch, and a_i may change by a root of unity if we change the branch.[23]

Remark. In verifying these claims it might be best to work with a power $L(s,\chi^j)^d$ that is meromorphic. As mentioned above, the transformation from χ^j and $\overline{\chi}^j$ is well-behaved and we can use the same solution to (24) for both χ^j and $\overline{\chi}^j$ to get compatible decompositions. So since the above functional equation has a form that is closed under powers and products, we get a functional equation for $L(s,\chi^j)^d$, and so for $L(s,\chi^j)$ for a choice of branch.

Remark. The form of the functional equation for Abelian L-series used here by Artin is a bit different than the form it is usually given today, so it is worth a few comments. (I have not consulted Hecke's original paper, nor the paper of Landau cited by Artin. Instead I consulted Tate's thesis. Tate was a student of Artin in the 1940s who showed how to replace Hecke's approach with an approach using harmonic analysis on the idèles.)

Suppose χ is an Abelian character with conductor $\mathfrak{f}$. Then Tate's thesis gives a form of the functional equation (see [7] pages 342–346) that leads naturally to the version used by Artin. To describe this, let S be a finite set of places of k including all divisors of the conductor $\mathfrak{f}$ and all Archimedean places. Tate shows that

$$L(1-s,\chi^{-1}) = \left(\prod_{\mathfrak{p}\in S}\rho_{\mathfrak{p}}(s)\prod_{\mathfrak{p}\notin S}(N\mathfrak{d}_{\mathfrak{p}})^{s-1/2}\,\chi^{-1}(\mathfrak{d}_{\mathfrak{p}})\right) L(s,\chi)$$

where $\rho_{\mathfrak{p}}(s)$ denotes certain meromorphic functions related to χ and the place $\mathfrak{p}$, which are explicitly calculated in Tate's thesis ([7], Pages 317, 319, 322). Here $\mathfrak{d}_{\mathfrak{p}}$ denotes the local different ideal. Recall that the absolute norm of the product $\mathfrak{d} = \prod \mathfrak{d}_{\mathfrak{p}}$ of these ideals gives the absolute discriminant $|\Delta_k|$ of the field k (where $\mathfrak{p}$ includes the non-Archimedean primes in S). The functions $\rho_{\mathfrak{p}}(s)$ are as follows:

- Suppose $\mathfrak{p}$ is a real place. Consider a principal ideals generated by $\alpha \in k^\times$ such that (1) $\alpha \equiv 1 \pmod{\mathfrak{f}}$, (2) $\alpha < 0$ at $\mathfrak{p}$, and (3) $\alpha > 0$ for all real places

[22] E. Landau, Über Ideale und Primideale in Idealklassen (concerning ideals and prime ideals in ideal classes). Math. Zeitschrift Bd. 2, Seite 104, Satz LXVI.

[23] We can take a_i to be a true constant and we can take $l_i^{(j)}$ to be integers if we raise both sides of the equation to an appropriate integral power.

not equal to $\mathfrak{p}$. (Weak approximations assures such an α exists). Then if χ has value 1 on such a principal ideal $\alpha\mathcal{O}_k$ then

$$\rho_{\mathfrak{p}}(s) = \left(\frac{2}{(2\pi)^s}\right)\cos\left(\frac{\pi s}{2}\right)\Gamma(s),$$

but if χ has value -1 on $\alpha\mathcal{O}_k$ then

$$\rho_{\mathfrak{p}}(s) = -i\left(\frac{2}{(2\pi)^s}\right)\sin\left(\frac{\pi s}{2}\right)\Gamma(s).$$

- If $\mathfrak{p}$ is a complex place then

$$\begin{aligned}\rho_{\mathfrak{p}}(s) &= (2\pi)^{1-2s}\frac{\Gamma(s)}{\Gamma(1-s)}\\ &= 2(2\pi)^{-2s}\sin(\pi s)\Gamma(s)^2\\ &= \left(\frac{2}{(2\pi)^s}\right)^2\sin\left(\frac{\pi s}{2}\right)\cos\left(\frac{\pi s}{2}\right)\Gamma(s)^2\end{aligned}$$

(Note this is just i times the product of the two formulas for real ρ). The first equation is essentially the formula from Tate's thesis. The other equations are justified by the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$$

and the double angle identity for the sine function.

- If $\mathfrak{p}$ is a ramified non-Archimedean place with conductor component $\mathfrak{f}_{\mathfrak{p}}$ then

$$\rho_{\mathfrak{p}}(s) = N(\mathfrak{d}_{\mathfrak{p}}\mathfrak{f}_{\mathfrak{p}})^{s-1/2}W_{\mathfrak{p}}(\chi)$$

where $W_{\mathfrak{p}}(\chi)$ is a certain root of unity called the root number.

When we substitute these formulas for $\rho_{\mathfrak{p}}(s)$ and simplify we obtain the formula

$$L(1-s,\chi^{-1}) = w\left(\frac{2}{(2\pi)^s}\right)^n(N(\mathfrak{f})|\Delta_k|)^{s-1/2}\sin\left(\frac{\pi s}{2}\right)^{n_1}\cos\left(\frac{\pi s}{2}\right)^{n_2}\Gamma(s)^nL(s,\chi)$$

where $n = [k:\mathbb{Q}]$, where n_1 and n_2 are two nonnegative integers with $n_1+n_2 = n$, where $N(\mathfrak{f})$ is the norm of the conductor of χ, and where w is a root of unity.

Remark. Artin makes the observation that the functional equation fixes the L-series exactly, not just up to a finite number of factors. In other words, the functional equation picks out a canonical representation of a class of L-series up to "finite Euler factor equivalence". Undoubtably this principal was well-known when Artin wrote this article, but it might be helpful to supply the details. The following Lemma helps make this clear and can be proved just by considering the location of zeros and poles. Note this lemma can be generalized to a much broader class of admissible functional equations, but we will stick to the concrete form given in the paper.

Lemma 2. *Suppose $L_1(s), \widetilde{L_1}(s), L_2(s), \widetilde{L_2}(s)$ are nonzero meromorphic functions on $\mathbb{C}$ and assume the following two conditions. (i)*

$$\frac{L_i(s)}{\widetilde{L_i}(1-s)} = a_i\, A_i^s\, \Gamma(s)^{n_i^{(1)}} \left(\cos\frac{s\pi}{2}\right)^{n_i^{(2)}} \left(\sin\frac{s\pi}{2}\right)^{n_i^{(3)}}$$

where $a_i \in \mathbb{C}^\times$, where $n_i^{(j)} \in \mathbb{Z}$, and where A_i is a positive real constant. And (ii)

$$L_2(s) = P(s)L_1(s), \qquad \widetilde{L_2}(s) = \widetilde{P}(s)\,\widetilde{L_1}(s)$$

where

$$P(s) = \prod_{i=1}^{e}(1-\varepsilon_i p_i^{-s})^{u_i}, \qquad \widetilde{P}(s) = \prod_{i=1}^{e}(1-\overline{\varepsilon}_i p_i^{-s})^{u_i}$$

where the product is over distinct pairs (ε_i, p_i) composed of a prime $p_i \in \mathbb{Z}$ and a root of unity $\varepsilon_i \in \mathbb{C}^\times$, and where $u_i \in \mathbb{Z}$. Then necessarily

$$L_1(s) = L_2(s)$$

as meromorphic functions.

Proof. Consider the function

$$R(s) = \frac{L_2(s)}{\widetilde{L_2}(1-s)} \cdot \frac{\widetilde{L_1}(1-s)}{L_1(s)} = \frac{P(s)}{\widetilde{P}(1-s)}$$

which, according to the functional equations of assumption (i), has the form

$$R(s) = a\, A^s\, \Gamma(s)^{n^{(1)}} \left(\cos\frac{s\pi}{2}\right)^{n^{(2)}} \left(\sin\frac{s\pi}{2}\right)^{n^{(3)}}$$

for some $a \in \mathbb{C}^\times$, $n^{(j)} \in \mathbb{Z}$, and A a positive real constant. Note that all the zeros and poles of $P(s)$ occur on the line $\Re(s) = 0$ (because p_i^s can equal ε_i only on this line), and all the zeros and poles of $\widetilde{P}(1-s)$ occur on the line $\Re(s) = 1$. So the only possible real zeros and poles of $R(s)$ occur when $s = 0$ or $s = 1$; in particular the number of real zeros and poles of $R(s)$ is finite. Since $\Gamma(s)$ has no zeros and poles for real $s > 0$, this forces $n^{(2)} = n^{(3)} = 0$ in order to avoid an infinite number of real zeros or poles for $R(s)$. Since $\Gamma(s)$ has an infinite number of real poles (at nonnegative integers), we can conclude that $n^{(1)} = 0$ as well. So $R(s) = aA^s$ has no zeros or poles.

As mentioned above, $P(s)$ and $\widetilde{P}(1-s)$ have disjoint sets of zeros and poles. Since the quotient $R(s)$ has no zeros or poles, this forces both $P(s)$ and $\widetilde{P}(1-s)$ to have no zeros or poles. Consider

$$P(s) = \prod_{j=1}^{e}(1-\varepsilon_j p_j^{-s})^{u_j}$$

and the zero sets of the factors $1-\varepsilon_j p_j^{-s}$. We see that s is in the zero set of the jth factor if and only if $p_j^s = \varepsilon_j$. If $\varepsilon_j = \exp(2\pi r_j i)$ with $r_j \in \mathbb{Q}$, then s is in the zero set if and only if $s = 0 + ti$ with

$$t = 2\pi\,\frac{r_j + k}{\log p_j}$$

for some $k \in \mathbb{Z}$. Observe that if $p_j = p_l$ but $\varepsilon_j \neq \varepsilon_l$, then there can be no common root to $1 - \varepsilon_j p_j^{-s}$ and $1 - \varepsilon_l p_l^{-s}$ simply because $p_j^s = p_l^s$ cannot be equal to both ε_j and ε_l. So consider the case where $p_i \neq p_l$. A common zero of $1-\varepsilon_j p_j^{-s}$ and $1-\varepsilon_l p_l^{-s}$ would yield a real t with

$$t = 2\pi \frac{r_j + k}{\log p_j} = 2\pi \frac{r_l + k'}{\log p_l}$$

where $k, k' \in \mathbb{Z}$. If, in addition, t is no zero, then we would be able to find two nonzero integers $a, b \in \mathbb{Z}$ where

$$\frac{\log p_l}{\log p_j} = \frac{a}{b}$$

and so $\log p_l^b = \log p_j^a$, or more simply $p_l^b = p_j^a$, a contradiction. So the only possible common zero of $1-\varepsilon_j p_j^{-s}$ and $1-\varepsilon_l p_l^{-s}$ is $s = 0$ (and that occurs only if $\varepsilon_j = \varepsilon_l = 1$).

Thus each factor $1 - \varepsilon_j p_j^{-s}$ of $P(s)$ has a zero that is not a zero of any other factor. Since $P(s)$ has no zeros or poles this implies that each $u_j = 0$. So $P(s) = 1$ as desired. □

Corollary 3. *Aside from a possible multiplication by a root of unity, the definition of $L(\chi^j, s)$ is independent of the solution to (24).*

Proof. Let $\{x_i^\sigma\}$ and $\{\tilde{x}_i^\sigma\}$ be two solutions to (24), where we use j for k in (24). Fix a positive integer d such that each $x_i^\sigma d$ and $\tilde{x}_i^\sigma d$ is in $\mathbb{Z}$ and so $L(\chi^j, s)^d$ is meromorphic whether we use the x_i^σ or the $\tilde{x}_i^\sigma$ to define $L(\chi^j, s)$. Consider $L_1(s), L_2(s)$ be equal to $L(\chi^j, s)^d$ according to the two expressions given by x_i^σ and $\tilde{x}_i^\sigma$ respectively. Note that $L_1(s)$ and $L_2(s)$ agree up to a finite number of Euler factor, and the Euler factors where they differ are powers of terms of the form

$$\frac{1}{1 - \varepsilon N(\mathfrak{p})^{-s}}$$

for some prime ideal $\mathfrak{p}$ in some number field and some root of unity ε. So $N(\mathfrak{p}) = p^l$ for some prime $p \in \mathbb{Z}$. By factoring the polynomial $1 - \varepsilon X^l$ into linear factors, we get the following:

$$\frac{1}{1 - \varepsilon N(\mathfrak{p})^s} = \prod_{\mu=1}^{l} \frac{1}{1 - \varepsilon_\mu p^{-s}}$$

where each ε_μ is a root of unity.

With these ideas we can verify that L_1 and L_2 satisfy the requirements of the above lemma. So $L_1 = L_2$. This implies that the two definitions of $L(\chi^j, s)$ differ only by a dth root of unity factor. □

Remark. Note that the above lemma also implies that (23) is an exact equation (up to root of unity), a fact that Artin uses in the calculation of $l_i^{(1)}$.

To determine $l_i^{(1)}$ explicitly in the functional equation we use (23) and examine the exponent of $\Gamma(s)$ appearing in the functional equation. This is carried out in the following lemma. In the Abelian case the exponent of Γ is $[k : \mathbb{Q}]$, and the following lemma shows how to generalize this to the non-Abelian case.

Lemma 4. *The exponent of $\Gamma(s)$ appearing in the functional equation of $L(s,\chi^i)$ is equal to $f_i[k:\mathbb{Q}]$ where f_i is the degree of the representation associated to χ^i. In other words $f_i = \chi^i(1)$.*

Proof. By equation (23) we see that the exponent of $\Gamma(s)$ appearing in the functional equation of $L\big(s,\psi_i^{(\sigma)}\big)$ is equal to

$$\sum_{\nu=1}^{x} r_{i\nu}^{(\sigma)} l_\nu^{(1)}$$

but from the functional equation for Abelian L-series we know that this exponent should be the degree $[\Omega_\sigma:\mathbb{Q}]$. Thus, for each $\sigma \in G$ and $i = 1,\dots,m(\sigma)$,

$$\sum_{\nu=1}^{x} r_{i\nu}^{(\sigma)} l_\nu^{(1)} = [\Omega_\sigma:\mathbb{Q}] = [k:\mathbb{Q}]\frac{|G|}{m(\sigma)}.$$

Multiply by $\psi_i^{(\sigma)}(\tau)$ with $\tau \in \mathfrak{g}^\sigma$, and sum over i:

$$\sum_{i=1}^{m(\sigma)}\sum_{\nu=1}^{x} l_\nu^{(1)} r_{i\nu}^{(\sigma)} \psi_i^{(\sigma)}(\tau) = [k:\mathbb{Q}]\frac{|G|}{m(\sigma)}\sum_{i=1}^{m(\sigma)} \psi_i^{(\sigma)}(\tau).$$

Using (6) on the left and (3) on the right, this equation simplifies as

$$\sum_{\nu=1}^{x} l_\nu^{(1)} \chi^\nu(\tau) = [k:\mathbb{Q}]|G|\,\varepsilon_\tau$$

where ε_τ is 1 or 0 depending on whether $\tau = 1$ or $\tau \neq 1$.

The above equation is independent of σ, and so applies to all $\tau \in G$. Now multiply by $\chi^i(\tau^{-1})$ and sum over $\tau \in G$:

$$\sum_{\tau\in G}\sum_{\nu=1}^{x} l_\nu^{(1)}\chi^\nu(\tau)\chi^i(\tau^{-1}) = \sum_{\tau\in G}[k:\mathbb{Q}]|G|\,\varepsilon_\tau\chi^i(\tau^{-1}) = [k:\mathbb{Q}]|G|\,\chi^i(1) = [k:\mathbb{Q}]|G|f_i$$

but the left-hand simplifies by (2) to give $|G|\,l_i^{(1)}$. So $l_i^{(1)}|G| = f_i[k:\mathbb{Q}]|G|$. In other words, $l_i^{(1)} = f_i[k:\mathbb{Q}]$. □

We can determine some of the other constants in the functional equation in a similar manner.

Remark. In the above proof Artin regards equation (23) not as an equation valid up to a finite number of Euler factors but as an exact equation (or at least up to multiplication by a root of unity). This is justified based because both sides of (23) satisfy the right type of functional equations (see Lemma 2).

Here is another proof of the above lemma that might be of interest; it does not use the strong version of (23). We begin with a special case of (7):

$$\chi_{\psi_i^{(\sigma)}}(1) = \sum_{\nu=1}^{x} r_{i\nu}^{(\sigma)}\chi^\nu(1) = \sum_{\nu=1}^{x} r_{i\nu}^{(\sigma)} f_\nu$$

for each σ and each $i = 1, \ldots, m(\sigma)$. But the degree of the induced representation is just the index $[G : \mathfrak{g}^\sigma]$ so

$$\sum_{\nu=1}^{x} r_{i\nu}^{(\sigma)} f_\nu = \chi_{\psi_i^{(\sigma)}}(1) = [G : \mathfrak{g}^\sigma] = \frac{|G|}{m(\sigma)}.$$

In what follows let x_i^σ a solution to (24), (where we use j for k in that equation). Then by the previous equation and (24)

$$\begin{aligned}
\sum_{\sigma\neq 1}\sum_{i=2}^{m(\sigma)} x_i^\sigma \frac{|G|}{m(\sigma)} &= \sum_{\sigma\neq 1}\sum_{i=2}^{m(\sigma)}\sum_{\nu=1}^{x} x_i^\sigma r_{i\nu}^{(\sigma)} f_\nu \\
&= \sum_{\nu=1}^{x}\left(\sum_{\sigma\neq 1}\sum_{i=2}^{m(\sigma)} x_i^\sigma r_{i\nu}^{(\sigma)}\right) f_\nu \\
&= \sum_{\nu=1}^{x} \delta_{j\nu} f_\nu \\
&= f_j.
\end{aligned}$$

With this identity we can easily calculate the exponent of the expression

$$\left(\frac{2}{(2\pi)^s}\right)\Gamma(s)$$

in the functional equation for $L(s, \chi^j)$.[24] From Hecke's functional equation for Abelian L-series we have that the contribution from each $L\big(s, \psi_i^{(\sigma)}\big)$ is equal to

$$[\Omega_\sigma : \mathbb{Q}] = [k : \mathbb{Q}]\ \frac{|G|}{m(\sigma)},$$

and so the total contribution for $L(s, \chi^j)$ is

$$\sum_{\sigma\neq 1}\sum_{i=2}^{m(\sigma)} x_i^\sigma [\Omega_\sigma : \mathbb{Q}] = [k : \mathbb{Q}] \sum_{\sigma\neq 1}\sum_{i=2}^{m(\sigma)} x_i^\sigma \frac{|G|}{m(\sigma)} = [k : \mathbb{Q}] f_j.$$

We can argue similarly for the part of the function equation of $L(s, \chi^j)$ coming from factors of the type $(N(\mathfrak{f})|\Delta_k|)^{s-1/2}$ from the Abelian L-series factors. For each $L\big(s, \psi_i^{(\sigma)}\big)$ we can write this factor as $(B_i^\sigma |\Delta_\sigma|)^{s-1/2}$ where $|\Delta_\sigma|$ is the absolute discriminant of the field Ω_σ and B_i^σ is a positive integer. But

$$|\Delta_\sigma| = N_\sigma |\Delta_k|^{[\Omega_\sigma : k]}$$

where Δ_k is the discriminant of k and N_σ is some positive integer.[25] So we can write

$$(B_i^\sigma |\Delta_\sigma|)^{s-\frac{1}{2}} = (B_i^\sigma N_\sigma)^{s-\frac{1}{2}} \left(|\Delta_k|^{s-\frac{1}{2}}\right)^{[\Omega_\sigma : k]}.$$

[24] As usual, if it makes matters clearer take a power $L(s, \chi_i)^d$ that is meromorphic on $\mathbb{C}$ instead of dealing with branches. It is clear how to adapt this argument to such a power.

[25] This is a standard result in algebraic number theory. See, for instance, Neukirch [13], Corollary 2.10, page 202.

In the functional equation of $L(s,\chi^j)$ the terms $(B_i^\sigma N_\sigma)^{s-1/2}$ combine to give

$$\left(\prod_{\sigma\neq 1}\prod_{i=2}^{m(\sigma)}(B_{i,\sigma}N_\sigma)^{x_i^\sigma}\right)^{s-\frac{1}{2}}$$

which in Artin's notation is $\alpha_j^{s-\frac{1}{2}}$. Observe that α_i is the product of rational powers of positive integers.

The exponent of $|\Delta_k|^{s-1/2}$ in the functional equation of $L(s,\chi^j)$ will be given by

$$\sum_{\sigma\neq 1}\sum_{i=2}^{m(\sigma)} x_i^\sigma[\Omega_\sigma : k] = \sum_{\sigma\neq 1}\sum_{i=2}^{m(\sigma)} x_i^\sigma \frac{|G|}{m(\sigma)} = f_j$$

where we have used the formula for f_j established above. Thus we get the discriminant factor $\left(|\Delta_k|^{f_j}\right)^{s-\frac{1}{2}}$.

All in all, we get the following:

Satz 3. *The primitive L-series $L(s,\chi^i)$ can be analytically continued to the whole plane aside from a possibly discrete set of branch points. The orders of the branch points are (unformly) bounded.*[26] *For $i>1$ the continuation of (each branch of) $L(s,\chi^i)$ is holomorphic and nonzero in a neighborhood of $s=1$. There are zero-free neighborhoods of the line $\Re(s)=1$, including a region on the plane defined by $\sigma \geq 1 - c/\log t$ for some constant $c>0$ (here we write a complex number as $s=\sigma+it$ with $\sigma,t\in\mathbb{R}$). These L-functions satisfy a functional equation of the form:*

(26)

$$\frac{L(1-s,\overline{\chi}^i)}{L(s,\chi^i)} = \varepsilon_i\left(\frac{2}{(2\pi)^s}\right)^{mf_i}\left(\alpha_i|\Delta_k|^{f_i}\right)^{s-\frac{1}{2}}\left(\cos\frac{s\pi}{2}\right)^{l_i^{(2)}}\left(\sin\frac{s\pi}{2}\right)^{l_i^{(3)}}(\Gamma(s))^{mf_i}.$$

where Δ_k is the discriminant of k, α_i is a product of rational powers of (rational) positive integers, ε_i are algebraic integers that depends only on the branch under consideration with $|\varepsilon_i|=1$, $m=[k:\mathbb{Q}]$, and $f_i=\chi^i(1)$ is the degree of the representation associated to the character χ^i. Furthermore, $l_i^{(2)}$ and $l_i^{(3)}$ are rational numbers.

With these types of methods ("Auf demselben Wege") it should also be possible to establish the single-valuedness of our functions, of which one can easily convince oneself in special cases. At least one can prove that the branching orders are divisible only by primes dividing $|G|$.

Completely new methods will probably be needed to show that our L-Series are analytic on all of $\mathbb{C}$ (aside from the L-series associated to the trivial character (Hauptcharakter)).

[26] In other words, $L(s,\chi^i)$ can be meromorphically continued on some Riemann surface covering $\mathbb{C}$ of finite degree d. In fact, $L(s,\chi^i)^d$ can be meromorphically continued on $\mathbb{C}$ itself for some positive power d.

Remark. As mentioned above, the methods of this paper show that $L(s,\chi^i)^d$ is entire for some positive integer d. In other words, $L(s,\chi^i)$ can be regarded as a d-valued function. Artin mentions here that it should be possible to prove that $d=1$ is possible ("die Eindeutigkeit unseren Funktionen"), in other words that $L(s,\chi^i)$ is meromorphic on the whole plane. His next sentence means that we should at least be able to find a d such that the only primes dividing d are divisors of $|G|$. The former claim would have to wait until 1947 when it was proved by R. Brauer [6], but later claim is, as Artin says, fairly easy to show: see the following remark.

It is still an open problem however on whether the Artin L-series is analytic in general. It has been shown in some cases by Langlands and Tunnell, and these cases were used by Wiles in his proof of Fermat's Last Theorem.

Remark. We now outline an argument for Artin's claim on branching orders for primes p not dividing the order $|G|$. Recall that as part of the proof of the solvability of (24), Artin shows that the matrix $(r_{i\nu}^{(\sigma)})$ has linearly independent rows. We can reduce this matrix modulo p, and by working in a suitable extension $\mathbb{F}_q$ of $\mathbb{F}_p$ (containing roots of unity of order $|G|$) we can mimic the proof given above for $\mathbb{Q}$ and show that it also works over $\mathbb{F}_p$ as long as $|G|$ is not zero modulo p.

Once we know that the matrix $(r_{i\nu}^{(\sigma)})$ has linearly independent rows modulo p, we can find an $x-1$ by $x-1$ submatrix whose mod p reduction is nonsingular. In other words, we can find a $x-1$ by $x-1$ submatrix whose determinant is an integer not divisible by p. We can then find a solution to (23) in terms of rational numbers whose denominators are not divisible by p. This gives a d_p sheeted cover of $\mathbb{C}$ such that $L(s,\chi^i)$ is meromorphic on the cover.

In particular, if one goes around a branch point of $L(s,\chi^i)$ then the value will change value by a multiplicative factor that is a d_p-root of unity. In other words, the order of the branch is relatively prime to p. This applies to all primes not dividing $|G|$ as one goes around a branch point. Let d be the GCD of all the d_p. Going around any branch point changes the value by a d-th root of unity, so $L(s,\psi^i)^d$ descends to a meromorphic function on $\mathbb{C}$, and at the same time the only primes dividing d are primes dividing $|G|$.

7 Conjecture of Frobenius (now called the Chebotaryov Density Theorem)

With the the result just derived one can easily confirm a conjecture of Frobenius using Formula (12).[27]

Remark. This density conjecture of Frobenius that Artin proves here is what we today call the *Chebotaryov (or Chebotarev) density theorem.*[28] Unbeknownst to Artin, Nikolai Chebotaryov had already proved this result about a year earlier in 1922 without using these new L-series. Artin gives a proof here, but it is requires

[27] See §5, Formulas (16) and (18) of the 1896 work of Frobenius cited in footnote 5.

[28] Nikolai Chebotaryov (1894–1947) was a mathematician from Ukraine and Russia. The spelling "Chebotaryov" is a transliteration of the Ukrainian version of his name, while "Chebotarev" is a transliteration of the Russian version. He was born in Ukraine and was educated at Kyiv University. He later became a professor at Kazan University in Russia in 1928 where he spent the remainder of his career.

Artin's reciprocity (Satz 2) in order to be assured that Satz 3 holds. Satz 2 was not fully proved until 1927 when Artin proved his reciprocity law. It is interesting to note that Artin's 1927 proof of his reciprocity law was inspired by the 1925 German versions of Chebotaryov's proof of this density theorem that Artin read only after he completed the current paper.

Not only can you derive the conjecture results, but you can also sharpen them without effort. From formula (12) for $\log L(s,\chi)$ it follows from known methods that

$$\sum_{N\mathfrak{p}\le x} \chi^i(\mathfrak{p}) = \delta_{1i}\mathrm{Li}(x) + O\left(xe^{-a\sqrt{\log x}}\right), \tag{27}$$

where $\delta_{11}=1$, but otherwise $\delta_{1i}=0$.

Remark. By "known methods", Artin is presumably referring to a combination of methods used to prove Dirichlet's theorem together with those needed for the the prime number theorem generalized to number fields. As usual, the error term can be greatly improved if one assumes the generalized Riemann hypothesis.

For example, a classical form of the prime number theorem is that

$$\pi(x) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\log x}}\right),$$

for some $a>0$. (See for instance Theorem 6.9, page 179 of [12]). Here $\pi(x)$ is the number of primes in $\mathbb{Z}$ less than x and

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t}dt.$$

The proof of the prime number theorem uses a zero-free region for $\zeta(s)$ similar to that described in Satz 3 for Artin L-functions. (See Theorem 6.6, page 172 of [12] for the classical zero-free region).

For a real number x and a conjugacy class C of G, let $\pi(x,C)$ be the number of prime ideals $\mathfrak{p}$ of k with $N\mathfrak{p}\le x$ whose Frobenius class is C.

We multiply (27) by $\chi^i(\sigma^{-1})$ where $\sigma\in C$, and sum over i. From (3) we get

$$\frac{|G|}{|C_r|}\pi(x,C_r) = \sum_{i=1}^{x}\sum_{N\mathfrak{p}\le x}\chi^i(\sigma^{-1})\chi^i(\mathfrak{p}) = \mathrm{Li}(x) + O\left(xe^{-a\sqrt{\log x}}\right).$$

Satz 4. *For a real number x and a conjugacy class C of G, let $\pi(x,C)$ be the number of prime ideals $\mathfrak{p}$ of k with $N\mathfrak{p}\le x$ whose Frobenius class is C. Then*

$$\pi(x,C) = \frac{|C|}{|G|}\mathrm{Li}(x) + O\left(xe^{-a\sqrt{\log x}}\right). \tag{28}$$

So the density of prime ideals in the class C is equal to the density of C in G. In particular, in each class C there is an infinite number of prime ideals whose Frobenius class is C.

This theorem is a generalization of Dirichlet's theorem concerning primes in an arithmetic progression, which (with the help of our general reciprocity law) can be seen to be a special case.[29] Its true meaning has yet to be clarified ("Seine wahre Bedeutung harrt noch der Aufklärung").

8 Multiplicative Relations Between L-Series

Satz 5. *If the base field k is $\mathbb{Q}$ then there are no multiplicative relations between the primitive L-Series.*

Proof. Suppose x_i are integers such that

$$\prod_{i=1}^{x} \left(L(s,\chi^i)\right)^{x_i} = 1.$$

Then by (12), with $k = \mathbb{Q}$,

$$\log L(s,\chi^i) = \sum_{p^\nu} \frac{\chi^i(p^\nu)}{\nu p^{\nu s}}$$

where the sum is over all prime powers $p^\nu > 1$. So when we sum over the χ^i we get

$$\sum_{p^\nu} \left(\sum_{i=1}^{x} x_i \chi^i(p^\nu) \right) \frac{1}{\nu p^{\nu s}} = 0.$$

Remark. One can tentavely think of the above equality as holding modulo $2\pi i\mathbb{Z}$. But in any case the right hand side is a constant on the connected set $\Re(s) > 1$. Since the left hand side is a Dirichlet series with constant term 0 this forces the right hand side to be 0 as asserted (by the uniqueness of coefficients of a Dirichlet series).

By the uniqueness of the coefficients of a Dirichlet series we have

$$\sum_{i=1}^{x} x_i \chi^i(p) = 0$$

for all primes p (and in fact, the prime power p^ν coefficients vanish as well). Recall from (10) that $\chi^i(p)$ denotes to the value of χ^i at the Frobenius of p.

By Satz 4, each conjugacy class C of G is the Frobenius class for an infinite number of primes p. So

$$\sum_{i=1}^{x} x_i \chi^i(\tau) = 0$$

for all $\tau \in G$. This implies, in the usual way, that $x_i = 0$ for all i. □

[29] Although Satz 2 is not fully proved in this paper, it is proved in special cases including that of $\mathbb{Q}(\zeta)/\mathbb{Q}$. When we work out the class field theory for $\mathbb{Q}(\zeta)/\mathbb{Q}$, we find that Artin's reciporocity gives a correspondence between the set of primes of $\mathbb{Q}$ with a fixed Frobenius in the Galois group of $\mathbb{Q}(\zeta)/\mathbb{Q}$ and the set of primes of $\mathbb{Q}$ in a certain arithmetic progression. So Satz 5 applied to the fields $\mathbb{Q}(\zeta)/\mathbb{Q}$ is really just Dirichlet's theorem.

Remark. The last step is just due to the linear independence of characters. This can be shown using (2).

Remark. We assumed x_i were integers in the above proof since that is the main case under consideration, but we can let x_i be complex and use the above argument to show that the functions $\log L(s,\chi^1),\dots,\log L(s,\chi^x),1$ are linearly independent over $\mathbb{C}$.

Satz 5 is not valid for general algebraic number fields k since conjugate prime ideals can undermine the result. In fact, one can easily construct examples (even with $[k:\mathbb{Q}]=2$) in which conjugate characters give rise to the same L-series. (In fact, we will see some examples in Section 9 where different characters of a given Galois group can give rise to the same L-series).

Based on Satz 5 we see how to find all the relationships between any finite collection of ζ-functions or L-series. Find a Galois extension E of $\mathbb{Q}$ that contains all the field extensions K/k used to define the zeta and L-series that you are interested in. We can consider all of our given functions as being defined using characters for $E/\mathbb{Q}$, and all of these can be expressed in terms of primitive L-series for $E/\mathbb{Q}$, which are independent by Satz 5. We can use elimination to find all the relations between our functions because any additional relations are ruled out by Satz 5. The remark at the end of Section 2 shows we do not necessarily have to transition to a common E to get our decompositions since extending the field does not change the decomposition. (The common field was mainly used to prove uniqueness). So we have reached a conclusion to the problem of determining multiplicative relations.

Remark. Here Artin to the independence of the common Galois extension E. If you wish to break the dependency on a common Galois $E/\mathbb{Q}$, you would want a way to identify when two primitive L-series for $E_1/\mathbb{Q}$ and $E_2/\mathbb{Q}$ are equal. One way is to agree to classify each primitive L-series by the minimal Galois extension $E/\mathbb{Q}$ for which it arises. In other words, consider only irreducible *faithful* representations of Galois groups with base field $\mathbb{Q}$. Note that we have independence for the infinite collection of such primitive L-series over $\mathbb{Q}$ (and the $\mathbb{C}$-linear independence for their repective logarithmic functions). When we combine this section with the results of Section 2 we have the following result:

Corollary 5. *Every Artin L-series factors uniquely as the product of primitive L-series defined over $\mathbb{Q}$.*

Remark. Above Artin mentions using elimination to find relations. This essentially means using commonplace matrix manipulations on integral matrices. We describe this in more detail.

Suppose $\ell_1,\dots,\ell_m$ gives a collection of L-series, and $L_1,\dots,L_t$ are all the primitive L-series defined over $\mathbb{Q}$ that arises in the decompositions of $\ell_1,\dots,\ell_m$. Then we can identify each ℓ_i with an element of $\mathbb{Z}^t$ through the exponents of its decomposition. Consider the $\mathbb{Z}$-module homomorphism

$$\Phi\colon \mathbb{Z}^m \to \mathbb{Z}^t$$

sending $(c_1,\dots,c_m)$ to the element of $\mathbb{Z}^t$ associated to $\ell_1^{c_1}\cdots\ell_m^{c_m}$. Then the kernel of Φ is a free $\mathbb{Z}$-module of rank bounded by m. The elements in this kernel give us our relations between $\ell_1,\dots,\ell_m$, and Satz 5 assures us that these are all the relations.

We can concretely calculate a basis for the kernel, i.e. identify all fundamental relations for $\ell_1, \ldots, \ell_m$, by using row and column operations on the matrix representing Φ to identify the kernel. The matrix representing Φ can be written down as soon as we have decomposed each ℓ_i: its jth column is the exponents occurring in the decomposition of ℓ_i (assuming we multiply the matrix on the left). For example, using (29) below, the fundamental relations among $\zeta, \zeta_5, \zeta_6, \zeta_{10}, \zeta_{12}, \zeta_{15}, \zeta_{20}, \zeta_{30}, \zeta_{60}$ discussed there can be calculated from calculating the kernel of the following:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 & 0 & 1 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 & 1 & 2 & 1 & 3 & 5 \end{bmatrix}.$$

For example, the column vector $(2, 0, -2, 0, 0, 0, -1, 1, 0)$ is in the kernel and so gives the relation $\zeta^2 \zeta_6^{-2} \zeta_{20}^{-1} \zeta_{30} = 1$.

9 Applications to Icosahedral Fields

Finally we apply these results to icosahedral extensions, the simplest extensions that cannot be obtained through a series of Abelian extensions. Let K/k be a Galois extension of number fields with Galois group G isomorphic to the icosahedral group. Observe that Satz 2 holds for intermediate Abelian extensions K'/k'. To see this observe that Abelian groups of the form H_1/H_2, where H_1 is a subgroup of G and H_2 is a normal subgroup of H_1, have order dividing $60 = 2^2 \cdot 3 \cdot 5$. The p-power part of such a group is a cyclic group of order dividing p for $p = 3, 5$, and the 2-power part of such a group is either cyclic of order dividing 2, or is the (Klein) four groups ("Vierergruppe") since G has no elements of order 4. (Satz 2 has been proved for Abelian Galois groups that are products of cyclic groups of prime order.)

Remark. The group G of symmetries of the icosahedron is isomorphic to the alternating group A_5, which is a simple group of order 60. The subgroups of A_5 include cyclic subgroups of the following orders: 1, 2, 3, 5. There are also Klein four groups, and dihedral subgroups of order 6 and 10. Finally there are subgroups isomorphic to A_4, and of course A_5 itself. So there are intermediate fields of degree $1, 5, 6, 10, 12, 15, 20, 30$ and 60 over k. Note that two subgroups A_5 of the same order are actually conjugate, and so are isomorphic. This implies that two intermediate subfields of K/k of the same degree over k must be isomorphic, and so have equal zeta functions. Artin uses the notation Ω_n for a field of degree n over k, and ζ_n for its zeta function. We let ζ be the zeta function of the base field k, so $\zeta = \zeta_1$.

In G we have 5 conjugacy classes C_1, C_2, C_3, C_4, C_5 with $1, 15, 20, 12, 12$ elements respectively. The densities of prime ideals in these classes must be

$$\frac{1}{60}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}.$$

Furthermore, by the theory of characters developed by Frobenius, we have five simple characters of G and their degrees are $1, 3, 3, 4, 5$. We call the associated primitive L-series ζ, $L_3^{(1)}, L_3^{(2)}, L_4, L_5$.

We easily get the following factorizations using our methods (where the index refers to the degree of the field over k):

$$
\begin{aligned}
\zeta_5 &= \zeta\, L_4 \\
\zeta_6 &= \zeta\, L_5 \\
\zeta_{10} &= \zeta\, L_4 L_5 \\
\zeta_{12} &= \zeta\, L_3^{(1)} L_3^{(2)} L_5 \\
\zeta_{15} &= \zeta\, L_4\, (L_5)^2 \\
\zeta_{20} &= \zeta\, L_3^{(1)} L_3^{(2)}\, (L_4)^2\, L_5 \\
\zeta_{30} &= \zeta\, L_3^{(1)} L_3^{(2)}\, (L_4)^2\, (L_5)^3 \\
\zeta_{60} &= \zeta\, (L_3^{(1)} L_3^{(2)})^3\, (L_4)^4\, (L_5)^5
\end{aligned}
\tag{29}
$$

Remark. Verifying (29) is an exercise. One way to verify it is to do the following:

- Identify all subgroups of A_5, and the size of the intersections with each of the conjugacy classes $C_1, \ldots, C_5$.
- Derive explicit formulas for the five simple characters of A_5.
- Use Frobenius reciprocity to calculate the induced characters of trivial characters in terms of the irreducible characters of A_5.
- Use (14) and Satz 1.

Proposition 6. *If $G = \mathrm{Gal}(K/k)$ is the icosahedral group, then all the L-series associated to representations of G are meromorphic. In other words, they are single valued (outside of poles) when extended to $\mathbb{C}$.*

Proof. It is enough to verify this for irreducible representatives. The function ζ and each ζ_n are already known to be meromorphic (Hecke). Note that the first two equations of (29) show that L_4 and L_5 are meromorphic.

Observe that K is cyclic of degree 5 over an intermediate field Ω_{12} of degree 12 over k. There are four primitive L-series (in addition to ζ_{12}) associated to K/Ω_{12} and these can all be expressed in terms of our primitive L-series. They are also known to be entire (Hecke). Note that ζ_{60} factors as ζ_{12} times the product of these four L-series. Comparing the expressions for ζ_{12} and ζ_{60} in (29), we see that the product of these four L-series for K/Ω_{12} is

$$(L_3^{(1)} L_3^{(2)})^2\, (L_4)^4\, (L_5)^4\,.$$

Each of these L-series is based on a one-dimensional representative of $\mathrm{Gal}(K/\Omega_{12})$, and so can be expressed in terms of a degree 12 induced representation of G. So each decomposition gives 12 as the sum in terms of the integers 3, 4, 5. We conclude that 12 is decomposed as $3+4+5$ for each (note that 5 cannot occur twice in any one of the sums, so 5 must occur exactly once in each sum). We conclude that two

of these L-series factor as $L_3^{(1)}L_4L_5$ and the other two as $L_3^{(2)}L_4L_5$. (By the way, this gives an example of a field with identical primitive L-series where conjugate characters give the same L-series)[30]. So $L_3^{(1)}$ and $L_3^{(2)}$ are meromorphic. □

The above proof also shows the following:

Proposition 7. *The functions $L_3^{(1)}L_4L_5$ and $L_3^{(2)}L_4L_5$ are entire.*

We can identify other entire function:

Proposition 8. *The function L_5 is entire.*

Proof. We have an intermediate extension Ω_5/k of degree 5, and an intermediate extension Ω_{15}/k of degree 15 such that Ω_{15} is a cyclic cubic extension of Ω_5. This gives us two nontrivial degree 1 characters of $\mathrm{Gal}(K/\Omega_5)$ of order 3, whose induced characters are degree 5 characters of G. The associated L-series are entire since they are Abelian L-series with nontrivial characters. The product of these series is $\zeta_{15}/\zeta_5 = (L_5)^2$. Since the associated induced characters of G are of degree 5, both L-series must be equal to L_5. So L_5 is entire. (And this gives another example where distinct representations gives the same L-series). □

Proposition 9. *The products $L_3^{(1)}L_3^{(2)}$ and $L_3^{(1)}L_3^{(2)}L_4$ are entire.*

Proof. The proof is similar to the last proof. The first comes from using the quadratic extension Ω_{12}/Ω_6 where one produces an entire L-series equal to ζ_{12}/ζ_6. The second comes from using the quadratic extension Ω_{20}/Ω_{10} where one produces an entire L-series equal to ζ_{20}/ζ_{10}. □

Remark. We do not get anything essentially new from the other Abelian extensions. So L_5 and the combinations $L_3^{(1)}L_4L_5, L_3^{(2)}L_4L_5, L_3^{(1)}L_3^{(3)}$, and $L_3^{(1)}L_3^{(2)}L_4$ (and their products) are the only L-series we can prove are entire.

Remark. Let's look at the other Abelian subextensions in addition to those treated in the above two Propositions:

- Ω_{30}/Ω_{15} gives the entire function $\zeta_{30}/\zeta_{15} = L_3^{(1)}L_3^{(2)}L_4L_5$.
- K/Ω_{30} gives the entire function $\zeta_{60}/\zeta_{30} = \left(L_3^{(1)}L_3^{(2)}L_4\right)^2 (L_5)^2$.
- K/Ω_{20} gives the entire function $\zeta_{60}/\zeta_{20} = \left(L_3^{(1)}L_3^{(2)}L_4\right)^2 (L_5)^4$ which must factor into two Abelian (and so entire) L-functions corresponding to the two nontrivial characters ψ and ψ^{-1} of the corresponding cyclic Galois group H_3 of order 3. It turns out that the L-series associated to ψ and ψ^{-1} are equal and so are both $L_3^{(1)}L_3^{(2)}L_4\,(L_5)^2$. To see this, note that the decomposition

[30] Note that any five cycle and its inverse are in the same conjugacy class of A_4, which by Frobenius reciprocity implies that L-series for conjugate characters for K/Ω_{12} will have the same decomposition.

depends on the the multiplicities of the simple characters χ^i in the corresponding induced representations, and these can be calculated using Frobenius reciprocity: the multiplicities are respectively

$$\left\langle \psi, \operatorname{res} \chi^i \right\rangle_{H_3}, \quad \text{and} \quad \left\langle \psi^{-1}, \operatorname{res} \chi^i \right\rangle_{H_3}$$

where in these inner products we restrict χ^i to H_3. However, a three cycle in A_5 and its inverse are in the same conjugate class of A_5 and so have the same value under χ^i, which means that these two inner products are actually given by the same sum. This shows the multiplicities are the same.

- K/Ω_{15} gives the entire function $\zeta_{60}/\zeta_{15} = (L_3^{(1)} L_3^{(2)})^3 \, (L_4)^3 \, (L_5)^3$ that factors into three entire functions coming from Ω_{30}/Ω_{15} extensions. Looking at the earlier case Ω_{30}/Ω_{15}, we see these three functions are each $L_3^{(1)} L_3^{(2)} L_4 L_5$.
- K/Ω_{12} was treated above (Proposition 6).

Observe that these entire functions are all just products of the entire products already considered; nothing new.

Remark. Of the zeta functions from (29), we see that the following are divisible by ζ with entire quotient: $\zeta_6, \zeta_{12}, \zeta_{30}, \zeta_{60}$. (This leaves the other half in question, namely $\zeta_5, \zeta_{10}, \zeta_{15}, \zeta_{20}$). On can also verify immediately the relations between zeta functions from my earlier article [2].

Remark. This shows Artin's interest in the following question: if K/k then is ζ_K/ζ_k entire? This helps motivate Artin's conjecture that primitive L-series that are not zeta functions are entire.

Artin's earlier article [2], published in 1923, has some interesting relationships between these zeta functions in the current case of G isomorphic to A_5, the icosahedral group. Some of these include

$$\zeta_{20} \, \zeta^2 = \zeta_5^2 \, \zeta_{12}, \qquad \text{and} \quad \zeta_{30} \, \zeta^2 = \zeta_6^2 \, \zeta_{20}.$$

These are immediate given (29) above.

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