General Topology. Part 5: General Products

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This document concerns general products of topological spaces.[1] It is the fifth document in a series concerning the basic ideas of general topology. This series is written for a reader with at least a rough familiarity with topology, including examples, who is ready to work through a systematic development of the subject. This series can also serve as a reference or a review of topology. I have attempted to give full and clear statements of the definitions and results, with brief motivations provided where possible, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So some of the proofs may be quite terse or missing altogether. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is straightforward. Supplied proofs are sometimes just sketches, but I have attempted to be detailed enough that a prepared reader can supply the details without too much trouble. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant.

1 Required Background

This document is the fifth document in a series devoted to the basic ideas of general topology and we assume some of the results from the previous documents. Most of what we need is in the first document, *Part 1: First Concepts*. In the last main section (Section 5) we use the definition of a Hausdorff space from *Part 2: Hausdorff Spaces*, and the definition of a convergent sequence from *Part 3: Sequential Convergence*. We don't need anything from the fourth document, *Part 4: Metric Spaces*.

In the earlier document (*Part 1: First Concepts*) we considered Cartesian products of two sets, but here we need general Cartesian products. So officially we assume the reader is familiar with such Cartesian products in set theory. We review this set theoretical background in this section, which might be enough for many readers.

[1]This material on products overlaps with Munkres 1975, Chapter 2 (112 to 114).

We start with the idea of an indexed family. This generalizes a sequence, but where the index set is not assumed to be ordered in any way. Let I be a set which we call the *index set*. An *I-indexed family* $\mathbf{c}$ is just a function with domain I and some set as a codomain.[2] We usually write c_i instead of $\mathbf{c}(i)$ for the image of $i \in I$, using sequential notation in place of functional notation. We write the family as $(b_i)_{i\in I}$ or just (b_i) when the index set does not need to be mentioned.

We are interested in families of sets in this section but will be interested families of topological spaces in the following sections. If $(X_i)_{i\in I}$ is a family of sets we can define the union and intersection

$$\bigcup_{i\in I} X_i \qquad \text{and} \qquad \bigcap_{i\in I} X_i$$

in the usual way. It follows from the axioms of set theory that this union is a set, and if I is nonempty then so is the intersection. (The intersection of an empty family can be specified as the "universe" where the universe is usually clear from context). We write $\bigcup X_i$ and $\bigcap X_i$ where the index set does not need to be mentioned.

Let $(X_i)_{i\in I}$ be a family of sets. Then the *Cartesian product* is the following set:

$$\prod_{i\in I} X_i \;\stackrel{\text{def}}{=}\; \{(x_i)_{i\in I} \mid x_i \in X_i\}$$

where any $(x_i)_{i\in I}$ in the above definition is understood to be a family $I \to \bigcup U_i$. In particular, the Cartesian product is a subset of the set of such functions. This guarantees that the Cartesian product is a set in the sense of axiomatic set theory. We can write $\prod X_i$ when the index set I does not need to be specifically mentioned.

Observe that if X_i is the empty set $\emptyset$ for some i in the index set then $\prod X_i = \emptyset$. The converse holds and is a significant fact: it is equivalent to the axiom of choice. We also note that the axiom of choice will be needed in the proof of Proposition 7.

Definition 1 (Projection maps)**.** Let $\prod_{i\in I} X_i$ be a Cartesian product of sets, and let $j \in I$. Then the *jth projection map* π_j is the function

$$\pi_j : \prod_{i\in I} X_i \to X_j$$

defined by the rule $(x_i) \mapsto x_j$. Observe that this function is surjective.

Every function of the form $f : Z \to \prod_{i\in I} X_i$ can be described in terms of *component functions*:

Definition 2 (Component functions)**.** Let $\prod_{i\in I} X_i$ be a Cartesian product of sets, and let $f : Z \to \prod_{i\in I} X_i$ a a function. For each $j \in I$, the *j-component function* f_j of f is the composition $\pi_j \circ f$ where π_j is the j-projection map.

In particular, if $f(z) = (x_i)$ then $f_j(z) = x_j$. So we have

$$f(z) = (f_i(z))_{i\in I}.$$

[2]The codomain is general and can be even be taken to be the universal class V of set theory. By the axiom of replacement in ZF set theory, the image will always be a set. By choosing this image, or any set containing this image, we can always assume the codomain is a set.

The universal property for Cartesian products of sets allows us to define f in terms of component functions f_j:

Proposition 1 (Universal Property)**.** *Let $(X_i)_{i\in I}$ be an indexed collection of sets, let Z be set, and for each $i \in I$ let $f_i \colon Z \to X_i$ be a function. Then there is a unique function*

$$f : Z \to \prod_{i\in I} X_i$$

such that $f_j = \pi_j \circ f$ for each $j \in J$, where π_j is the j-projection map.

We assume the reader is familiar with the above proposition, but if not the proof is straightforward.

2 Products of Spaces

In this section let $(X_i)_{i\in I}$ be an indexed family of topological spaces with Cartesian product $\prod_{i\in I} X_i$. In particular, each element of $\prod_{i\in I} X_i$ is an I-indexed family $(x_i)_{i\in I}$ where $x_i \in X_i$. Here I is allowed to be an infinite set.

Let $\mathcal{B}$ be the collection of subsets of $\prod_{i\in I} X_i$ of the form $\prod_{i\in I} Y_i$ where (1) each Y_i is open in X_i, and (2) $Y_i = X_i$ for all but a finite number of $i \in I$.

Lemma 2. *The collection $\mathcal{B}$ is closed under finite intersections. In particular, this collection is a potential basis for the set $\prod_{i\in I} X_i$.*

Definition 3. The *product topology* on $\prod_{i\in I} X_i$ is the topology generated by the above potential basis. In particular, $\mathcal{B}$ above is a basis for this topology.

Remark. The requirement (2) $Y_i = X_i$ for all but a finite number of $i \in I$ for $\mathcal{B}$ will seem unmotivated at first. However, Proposition 9 and its corollary, as well as Proposition 13, are clearly desired results, and requirement (2) is necessary for the proofs of these results.

Remark. By comparing bases, we can derive some natural homeomorphisms. For instance, the above definition gives the binary Cartesian product $X_1 \times X_2$ the same topology as defined in the first document of this series (where here $I = \{1, 2\}$). In the case where I is a singleton, the product is naturally homeomorphic to the space itself: $\prod_{i\in\{1\}} X_i$ is canonically homeomorphic to X_1.

The following proposition can be used to give natural homeomorphisms in other situations. For example, it shows that for triple products (where $I = \{1, 2, 3\}$, say), the resulting topological space is naturally homeomorphic to the topological space $(X_1 \times X_2) \times X_3$.

Proposition 3. *Let I and J be disjoint sets. If $X_I = \prod_{i\in I} X_i$ and $X_J = \prod_{i\in J} X_i$, then $X_I \times X_J$ is naturally homeomorphic to $\prod_{i\in I\cup J} X_i$.*

Proof. Take the natural bijection between the underlying sets. This bijection induces a bijection between subsets of the domain and subsets of the codomain. We can choose a basis for each topology so that they correspond via the induced bijection. This means that the map, and its inverse are continuous. $\square$

Remark. We can also use bases to show that products are well-behaved with respect to permutations of the spaces (i.e., reindexing by J where I and J are bijective). In particular, if $\sigma\colon I \to J$ is a bijection, and if $Y_j = X_{\sigma i}$ for each $i \in I$, then the natural bijection between $\prod_{i\in I} X_i$ and $\prod_{j\in J} Y_j$ is a homeomorphism.

If each component X_i has a specified basis, we can give an associated basis for the product:

Proposition 4. *Suppose (X_i) is a family of topological spaces and that $\mathcal{B}_i$ is a basis for X_i. Consider the collection $\mathcal{B}$ of sets $\prod Y_i$ such that each $Y_i \in \mathcal{B}_i$ and such that $Y_i = X_i$ for all but a finite number of indices i. Then $\mathcal{B}$ is a basis of the space $\prod X_i$.*

3 Products of Subspaces

In this section let $(X_i)_{i\in I}$ be a family of topological spaces.

Suppose Y_i is a subset of X_i for each $i \in I$. Then $\prod Y_i$ has two topologies: (i) the subspace topology as a subspace of $\prod X_i$, and (ii) the product topology on $\prod Y_i$ where each Y_i is considered a subspace of X_i. We observe that these topologies are the same; they consist of the same collection of open subsets:

Lemma 5. *Suppose that Y_i be a subset of X_i for each $i \in I$. Then the set $\prod_{i\in I} Y_i$ can be given a topology in two natural ways: (i) as a subspace of $\prod_{i\in I} X_i$ and (ii) as a product space where each Y_i has the subspace topology induced from X_i. These two topologies on $\prod_{i\in I} Y_i$ are in fact the same.*

Proof. It is enough to describe a common basis for the two topologies. We start with the set theoretic identity, for A_i and Y_i subsets of X_i:

$$\prod_{i\in I}(A_i \cap Y_i) = \prod_{i\in I} A_i \;\cap\; \prod_{i\in I} Y_i.$$

Consider the collection of sets of the form $\prod_{i\in I}(A_i \cap Y_i)$ where A_i is open in X_i and $A_i = X_i$ for all but a finite number of $i \in I$. Observe that this is a basis for both topologies. □

The product of open subsets is not necessarily open if the index set is infinite. However, the product of closed sets is closed:

Proposition 6. *Suppose Z_i is closed in X_i for each $i \in I$. Then the product $\prod Z_i$ is closed in $\prod X_i$*

Proposition 7. *For each $i \in I$ suppose that Y_i is a subset of X_i. Then*

$$\overline{\prod Y_i} = \prod \overline{Y_i}.$$

So closure "commutes" with products.

Proof. One direction is a corollary of the previous proposition. Note that the other direction uses the axiom of choice. □

4 Functions for Products

We suppose $\prod_{i \in I} X_i$ is a product of topological spaces.

Proposition 8. *The projection maps* $\pi_j \colon \prod X_i \to X_j$ *are continuous.*

Proposition 9. *A function* $f : Z \to \prod X_i$ *is continuous if and only if the component function* $f_j = \pi_j \circ f$ *is continuous for each* $j \in I$*.*

Proof. One direction is straightforward since the composition of continuous functions is continuous. For the other direction, we first derive an intersection formula for the preimage of a subset of the form $\prod Y_i$ where each $Y_i \subseteq X_i$. By this the inverse image of a basis element is the finite intersection of open sets, so is open. □

As a corollary, we see that a topological product is a product in the sense of category theory. The following is the topological version of Proposition 1 (and its proof should use Proposition 1 as well as the above proposition):

Corollary 10. *Let* $(X_i)_{i \in I}$ *be an indexed collection of spaces, let* Z *be space, and for each* $i \in I$ *let* $f_i \colon Z \to X_i$ *be a continuous function. Then there is a unique continuous function* $f : Z \to \prod_{i \in I} X_i$ *such that* $f_i = \pi_i \circ f$ *for each* i*.*

We can use the above to prove the following:

Proposition 11. *If each* $f_i \colon X_i \to X'_i$ *is continuous then so is the function*

$$\prod_{i \in I} X_i \to \prod_{i \in I} X'_i$$

whose j*-coordinate function is* $(x_i) \mapsto f_j x_j$ *for each* $j \in I$*. In other words, the* j*th coordinate function is* $f_j \circ \pi_j$*.*

5 Other Properties

Let $\prod_{i \in I} X_i$ be a product of topological spaces.

Proposition 12. *If each* X_i *is Hausdorff, then* $\prod_{i \in I} X_i$ *is Hausdorff.*

Proposition 13. *Let* $(p_j)_{j \in J}$ *be a sequence of points in* $\prod_{i \in I} X_i$*. For each* $j \in J$ *and* $i \in I$ *let* $p_{j,i}$ *be the* i*-coordinate of* p_j*, so* $p_j = (p_{j,i})_{i \in I}$*. Let* $(x_i) \in \prod X_i$*. Then the sequence* (p_j) *converges to the point* (x_i) *if and only if the sequence* $(p_{j,i})_{j \in J}$ *in* X_i *converges to the point* x_i *in* X_i *for each* $i \in I$*.*

Remark. When speaking of the convergence of sequences we usually require that the index set be order-isomorphic to $\mathbb{N}$, but the proof of the above is valid for any totally ordered J.

In future documents we will define the concept of compactness. After that we will take up Tychnonoff's theorem: the product of compact topological spaces is compact.

Appendix: Restricted Products

There is a variant of products that sometimes arise, for example in number theory (in topological rings called "adeles").

Definition 4. Let $(X_i)_{i \in I}$ be a family of topological spaces, and for each $i \in I$ let U_i be a designated open subspace of X_i. The the *restricted Cartesian product* is defined to be the subset of $\prod X_i$ consisting of the elements (x_i) with the following property: $x_i \in U_i$ for all but finitely many $i \in I$.

Our aim is to give the restricted Cartesian product a topology. Let $(X_i)_{i \in I}$ and $(U_i)_{i \in I}$ be as above, and let P be the restricted Cartesian product. Let $\mathcal{B}$ be the collection of subsets of P the form $\prod W_i$ such that W_i is open in X_i and such that $W_i = U_i$ for all but a finite number of $i \in I$.

Lemma 14. *The collection $\mathcal{B}$ is closed under finite intersections. In particular, $\mathcal{B}$ is a potential basis for P.*

Definition 5. Let $(X_i)_{i \in I}$ be a family of topological spaces, and for each $i \in I$ let U_i be a designated open subspace of X_i. Let P be the associated restricted Cartesian product. Then the topology on P is defined to be the unique topology with basis $\mathcal{B}$ where $\mathcal{B}$ is as defined above.