General Topology. Part 4: Metric Spaces

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January 2020 version

This document introduces the concept of a metric space.[1] It is the fourth document in a series concerning the basic ideas of general topology, and it assumes the results from the earlier documents as well as basic facts about the ordered complete field $\mathbb{R}$.

This series is written for a reader with at least a rough familiarity with topology, including examples, who is ready to work through a systematic development of the subject. This series can also serve as a reference or a review of topology. I have attempted to give full and clear statements of the definitions and results, with motivations provided where possible, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So some of the proofs may be quite terse or missing altogether. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is straightforward. Supplied proofs are sometimes just sketches, but I have attempted to be detailed enough that the reader can supply the details without too much trouble. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant.

1 Logical Dependencies

The following section (Section 2) is logically independent from the earlier documents in the series. The section only uses basic set theory (including functions) and some facts about the field $\mathbb{R}$ of real numbers. More specifically it uses the fact that $\mathbb{R}$ is an ordered field such that every nonnegative element has a square root. Completeness (LUB and GLB properties) are also used, but only in the definitions at the end of the section starting with Definition 7. Other sections also require knowing that $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$. We assume a very basic knowledge of $\mathbb{R}^n$, with the operation of addition, and the inner product on $\mathbb{R}^n$.

The next two sections (Sections 3 and 4) consider the topology of metric spaces and so depend on the first two documents in the series including the notion of

[1]This material on metric spaces overlaps with Munkres 1975, Chapter 2 (Sections 9 and 10: pages 117 to 134).

Hausdorff space. The final section (Section 5) uses sequences and so depends also on the third document of the series.

2 Metrics

Metric spaces provide important examples of topological spaces. Metric spaces are simply sets equipped with distance functions. Since we have many intuitions build up from the notion of distance, metric spaces are conceptually more accessible than abstract topological spaces. It may therefore be advisable to learn about metric spaces before learning about topological spaces in general.

It turns out that we really only need four properties of classical distance in our topological arguments. So we define the general notion of a metric in terms of these key properties.

Definition 1 (Metric)**.** A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following four laws:

(i) $d(x, y) \geq 0$ for all $x, y \in X$.

(ii) $d(x, y) = 0$ if and only if $x = y$.

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (the triangle inequality).

Definition 2 (Metric Space)**.** A set equipped with a metric is called a *metric space.*

Definition 3 (Open and Closed Balls)**.** Let $r > 0$, and $x \in X$. The *open ball centered at x with radius r* is the set of all y such that $d(x, y) < r$. We write this as $B_{<r}(x)$, or $B_{d,<r}(x)$ if we wish to show its dependency on d.

For $r \geq 0$ we define the *closed ball* $B_{\leq r}(x)$ in a similar manner with the condition that $d(x, y) \leq r$.

We also write $B_{<r}(x)$ as $B_r(x)$, and $B_{\leq r}(x)$ as $\overline{B}_r(x)$. The later notation is potentially confusing since a closed ball is not necessarily the closure of the open ball.

Example 1. If $X = \mathbb{R}$, then $d(x, y) = |x - y|$ is a metric. This can be seen from basic facts about absolute values, or can be seen as a special case of the following proposition. Observe that for $\mathbb{R}$ with this metric, the open ball $B_{<r}(x)$ is the open interval $(x - r, x + r)$ and the closed ball $B_{\leq r}(x)$ is the closed interval $[x - r, x + r]$.

Proposition 1. *Consider the Euclidean distance* $d(x, y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{1/2}$ *on* $\mathbb{R}^n$*. This function is a metric.*

(Here $x, y \in \mathbb{R}^n$ and $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.)

Proof. We use basic facts about ordered fields, together with the fact that $\mathbb{R}$ is an ordered field closed under nonnegative square roots. Most properties follow immediately, so we focus on the triangle inequality.

In what follows, we define the dot product of elements of $\mathbb{R}^n$ in the usual way. This dot product is clearly symmetric and bilinear. If $x \in \mathbb{R}^n$ we define the norm $|x|$ to be the (nonnegative) square root of the dot product $x \cdot x$. So in this notation

$$d(x, y) = |x - y|.$$

Our first goal is to see that $x \cdot y \leq |x| \cdot |y|$ for all $x, y \in \mathbb{R}^n$. To see this, observe that for all $s, t \in \mathbb{R}$ we have

$$|sx + ty|^2 = (sx + ty) \cdot (sx + ty) = s^2|x|^2 + 2st(x \cdot y) + t^2|y|^2$$

and this quantity is nonnegative. If x and y are non-zero then let $s = |x|^{-1}$ and $t = -|y|^{-1}$, then the nonnegativity of the above gives $-st(x \cdot y) \leq 1$. The result follows. (Choosing $t = |y|^{-1}$ as well, we could extend the result to $|x \cdot y| \leq |x| \cdot |y|$). The case where x or y is $(0, \ldots, 0)$ is immediate.

Now observe that

$$|x + y|^2 = (x + y) \cdot (x + y) \leq |x|^2 + 2x \cdot y + |y|^2 \leq |x|^2 + 2|x||y| + |y|^2 = (|x| + |y|)^2.$$

So $|x + y| \leq |x| + |y|$. Now replace x with $x - y$ and y with $y - z$. □

So $\mathbb{R}^n$ equipped with the Euclidean distance is a metric space. In fact, any subset of $\mathbb{R}^n$ is a metric space. More generally, we have the following.

Proposition 2. *If X is a metric space, and $Y \subseteq X$. Then the restriction of the metric to Y is a metric on Y.*

The finite Cartesian product of metric spaces can be made into a metric space. A simple way to define a metric on the product is as follows.

Definition 4 (Product Metric)**.** Let $X_1, \ldots, X_n$ be metric spaces. Let d_{prod} be the metric on the Cartesian product $X_1 \times \cdots \times X_n$ defined by the formula

$$d_{\text{prod}}(x, y) = \max d_i(x_i, y_i)$$

where $d_i : X_i \times X_i \to \mathbb{R}$ is the metric on X_i, where x_i is the ith coordinate of x, and where y_i is the ith coordinate of y. We call this the *product metric*.

Remark. If we impose a boundedness condition, we can define a product metric for infinite products. However, we have to take some care with the definition if we want it to be compatible with the product topology. We will not pursue metrics for infinite products in this document, but see Munkres (Munkres 1975, Chapter 2) for more information.

Proposition 3. *The product metric is a metric on $X_1 \times \cdots \times X_n$.*

Corollary 4. *The function $d'(x, y) = max\,|x_i - y_i|$ on $\mathbb{R}^n$ is a metric on $\mathbb{R}^n$.*
(Here $x, y \in \mathbb{R}^n$ and $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.)

The above corollary give us a second way of forming a metric space out of $\mathbb{R}^n$. We will see later that both metric spaces have the same topology due to the following result:

Proposition 5. *Let d be the Euclidean metric on $\mathbb{R}^n$, and let d' be the product metric on $\mathbb{R}^n$. Then for all $x, y \in \mathbb{R}^n$,*

$$d'(x,y) \leq d(x,y) \leq \sqrt{n} \cdot d'(x,y).$$

Corollary 6. *For each $r > 0$ and $x \in \mathbb{R}^n$, let $B_r(x)$ denote the open ball according to the Euclidean metric, and $B'_r(x)$ the open the open ball according to the product metric. Then, for any $r > 0$ and $x \in \mathbb{R}^n$,*

$$B_r(x) \subseteq B'_r(x) \subseteq B_{r\sqrt{n}}(x).$$

Open and closed balls for products are well-behaved:

Proposition 7. *Let $X_1, \ldots, X_n$ be metric spaces, and let $X_1 \times \cdots \times X_n$ be the metric space equipped with the product metric. If $x \in X_1 \times \cdots \times X_n$ then*

$$B_r(x) = B_r(x_1) \times \cdots \times B_r(x_n) \quad \textit{and} \quad B_{\leq r}(x) = B_{\leq r}(x_1) \times \cdots \times B_{\leq r}(x_n).$$

Definition 5. (The Derived Bounded Metric) If $d \colon X \times X \to \mathbb{R}$ is a metric on X, then define $d_1 \colon X \times X \to \mathbb{R}$ by the formula $d_1(x,y) = \min(1, d(x,y))$. Then d_1 is called the *derived bounded metric* formed from d.

Lemma 8. *If $d \colon X \times X \to \mathbb{R}$ is a metric on X then the derived bounded metric is also a metric on X.*

Proof. We concentrate on the triangle inequality since the other properties are clear. Suppose otherwise that $d_1(x,z) > d_1(x,y) + d_1(y,z)$. Since $d_1(x,z) \leq 1$ we would then have both $d_1(x,y) < 1$ and $d_1(y,z) < 1$. So the following holds:

$$d(x,z) \leq d(x,y) + d(y,z) = d_1(x,y) + d_1(y,z) < d_1(x,y) \leq 1.$$

Thus $d(x,y), d(y,z)$, and $d(x,z)$ are all strictly less than 1. So the supposition can be written as $d(x,z) > d(x,y) + d(y,z)$, a contradiction. □

Example 2. Suppose X is the disjoint union of metric spaces. Replace each metric with the derived bounded metric. Then if we define the distance of two points in distinct spaces of the disjoint union to be 1, then the result is a metric space.

Proposition 9. *Let $y \in B_r(x)$ in a metric space. There is an open ball $B_{r'}(y)$ with center y such that $B_{r'}(y) \subseteq B_r(x)$.*

Proof. Let $r' = r - d(x,y)$. □

Corollary 10. *Let $y \in B_{r_1}(x_1) \cap B_{r_2}(x_2)$ in a metric space. There is an open ball $B_{r'}(y)$ with center y such that $B_{r'}(y) \subseteq B_{r_1}(x_1) \cap B_{r_2}(x_2)$.*

Proposition 11. *Let $x, y \in X$ and $r \geq 0$. Suppose $y \notin B_{\leq r}(x)$. Then there is an open ball $B_{r'}(y)$ disjoint from $B_{\leq r}(x)$. In particular, $B_r(x)$ and $B_{r'}(y)$ are disjoint.*

Proof. Let $r' = d(x,y) - r$. □

As we have observed $\mathbb{R}^n$ can be made into a metric space using the Euclidean distance or the product metric based on the usual metric on $\mathbb{R}$. There is a third common way of defining a metric on $\mathbb{R}^n$ called the *taxicab metric* since it can be seen to be inspired by the driving distance between two points in a city that is nicely laid out with a rectangular grid of streets.

Definition 6. The *taxicab metric* on $\mathbb{R}^n$ is the defined by the formula

$$d(a,b) = \sum_{i=1}^{n} |a_i - b_i|$$

where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, n_n)$.

More generally if $X_1, \ldots, X_n$ are metric spaces with metrics $d_1, \ldots, d_n$, then the *taxicab metric* on $X_1 \times \cdots \times X_n$ is defined by the formula

$$d(a,b) = \sum_{i=1}^{n} d_i(a_i, b_i)$$

where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are points of $X_1 \times \cdots \times X_n$.

Proposition 12. *Suppose $X_1, \ldots, X_n$ are metric spaces with metrics $d_1, \ldots, d_n$. Then the taxicab metric is a metric on $X_1 \times \cdots \times X_n$.*

Proposition 13. *Suppose $X_1, \ldots, X_n$ are metric spaces with metrics $d_1, \ldots, d_n$. Let d_Σ be the taxicab metric on $X = X_1 \times \cdots \times X_n$, and let d_Π be the product metric on X. Then, for all $a, b \in X$,*

$$d_\Pi(a,b) \le d_\Sigma(a,b) \le n \cdot d_\Pi(a,b).$$

Corollary 14. *Let $X = X_1 \times \cdots \times X_n$ where $X_1, \ldots, X_n$ are metric spaces with metrics $d_1, \ldots, d_n$. For each $x \in X$ and positive $r \in \mathbb{R}$, let $B_r^\Sigma(x)$ denote the open ball according to the taxicab metric, and $B_r^\Pi(x)$ an open the open ball according to the product metric. Then, for each $x \in X$ and positive $r \in \mathbb{R}$,*

$$B_r^\Sigma(x) \subseteq B_r^\Pi(x) \subseteq B_{n\,r}^\Sigma(x).$$

So far we have not used the full completeness property of of $\mathbb{R}$, but only the fact that $\mathbb{R}$ is an ordered field where every nonnegative element has a square root. However, the completeness of $\mathbb{R}$ is needed to define concepts of diameter and set distance.

Definition 7 (Bounded, Diameter)**.** Let X be a metric space with metric d. A nonempty subset $Y \subseteq X$ is said to be *bounded* if there is a B such that $d(x,y) \le B$ for all $x, y \in Y$. The supremum of the set of such bounds is called the *diameter* of Y.

Definition 8 (Distance to Set)**.** Let X be a metric space with metric d. Let S be a nonempty subset of X, and let $x \in X$. Then the infimum of $d(x,s)$ with $s \in S$ is called the *distance of x to S*. We can write this as $d(x,S)$.

Definition 9 (Distance between Sets)**.** Let X be a metric space with metric d. Let Y and Z be nonempty subsets of X. Then the infimum of $d(y,z)$ with $y \in Y$ and $z \in Z$ is called the *distance between* Y *and* Z. We can write this as $d(Y,Z)$.

3 The Topology of a Metric Space

The previous section developed metric spaces without reference to the general concepts of topology developed in earlier documents. In this section we begin to explore the topology associated with a metric space X.

Definition 10 (Topology of a Metric Space)**.** Let X be a metric space. We say that a subset $U \subseteq X$ is *open* if for all $x \in U$ there is an open ball $B_r(x)$ such that $B_r(x) \subseteq U$.

Such open sets satisfy the requirements for open sets in general topology:

Lemma 15. *The empty set $\emptyset$ and the whole space X are open. The (arbitrary) union of open sets is open. The intersection of two open sets is open.*

Lemma 16. *Every open ball is an open set.*

Proof. See Proposition 9. □

Proposition 17. *A metric space equipped with open sets (as defined above) is a topological space. The collection of open balls is a basis for this topological space.*

Remark. Given a metric space, the topology described above is sometimes called the *topology generated by the metric.*

Proposition 11 gives us the following two propositions.

Proposition 18. *Every metric space is a Hausdorff space.*

Proposition 19. *Every closed ball in a metric space is a closed subset in the topological sense.*

Definition 11 (Metrizable)**.** A topological space X is *metrizable* if there is a metric on X which generate the given topology on X.

Not every topological space is metrizable. In fact, metrizable spaces have some distinctive properties:

Proposition 20. *If a space is X metrizable then X is Hausdorff and every point $x \in X$ has a countable local basis.*

It is important to note that distinct metrics can generate the same topology. The following describes when this occurs:

Lemma 21. *Let $d\colon X \times X \to \mathbb{R}$ and $d'\colon X \times X \to \mathbb{R}$ be metrics on X. Then d and d' generate the same topology if and only if both the following occur: (i) for every open ball $B_{d,r}(x)$ using metric d there is an $r' > 0$ such that $B_{d',r'}(x) \subseteq B_{d,r}(x)$, (ii) for every ball $B_{d',r'}(x)$ using metric d' there is an $r > 0$ such that $B_{d,r}(x) \subseteq B_{d',r'}(x)$.*

This lemma together with Corollary 6 and Corollary 14 yields the following two results:

Proposition 22. *Suppose $X_1, \ldots, X_n$ are metric spaces. Then the product metric and the taxicab metric generate the same topology on the product $X_1 \times \cdots \times X_n$.*

Proposition 23. *For each $n > 0$, the Euclidean metric, the taxicab metric, and the product metric on $\mathbb{R}^n$ generate the same topology.*

Proposition 24. *A metric on a set X and the derived bounded metric generate the same topology on X.*

Proof. Start with the fact that for radius $r < 1$, the two metrics give the same open balls. □

Subset topologies and finite Cartesian product topologies are well behaved with respect to metrics:

Proposition 25. *Let X be a metric space and let Y be a subset of X. Then the topology on Y generated by the restricted metric is the subspace topology on Y.*

Proof. The collections of sets of the form $B_r(x) \cap Y$ with $x \in Y$ is a basis for both topologies of Y. □

Proposition 26. *Let $X_1, \ldots, X_n$ be metric spaces. Then the topology on the metric space $X_1 \times \cdots \times X_n$ which uses the product metric is the product topology.*

Proof. See Proposition 7. Argue that respective bases for the two topologies generate the same topology on $X_1 \times \cdots \times X_n$. □

Proposition 27. *The order topology and the metric topology on $\mathbb{R}$ coincide.*

Proof. Compare bases. (See Example 1 for a description of balls.) □

Recall that in Definition 8 we defined the distance $d(x, S)$ from a point x to a subset S. We can use this concept to give a appealing characterization of closure:

Proposition 28. *Let X be a metric space with distance function d. Let S be a nonempty subset of X. Then the closure of S is the set of points of distance zero from S:*

$$\overline{S} = \{x \in X \mid d(x, S) = 0\}.$$

Proof. If x is in the closure of S, then for any $\varepsilon > 0$, the ball $B_\varepsilon(x)$ has a point s of S. Thus $d(x, S) \leq d(x, s) < \varepsilon$.

Conversely, if x has zero distance from S, then for any $r > 0$ the ball $B_r(x)$ must contain a point of S since otherwise $d(x, S) \geq r$. □

4 Continuous Function on Metric Spaces

The traditional δ-ϵ definition of continuity is valid for metric spaces:[2]

[2]We defined continuity at a point in the first document of the series.

Proposition 29. *Let $f\colon X \to Y$ be a map between two metric spaces. Then f is continuous at $x_0 \in X$ if and only if the following occurs: for every $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$f\left[B_\delta(x_0)\right] \subseteq B_\varepsilon\left(fx_0\right).$$

In particular, f is continuous if and only if the above holds for each $x_0 \in X$.

Remark. Observe that the condition of the above proposition is equivalent to the more traditional criterion for continuity of f at a point $x_0 \in X$: for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$ with $d(x, x_0) < \delta$ we have $d(fx, fx_0) < \varepsilon$.

So functions $\mathbb{R}^m \to \mathbb{R}^n$ continuous under the common δ-ϵ definition are continuous in the topological sense, and vice versa. For general metric spaces there is a fundamental class of continuous functions called *nonexpansive maps.*

Definition 12. A map $f\colon X \to Y$ between two metric spaces is said to be *nonexpansive* if

$$d_Y(fa, fb) \le d_X(a, b).$$

for all $a, b \in X$.

We have the following (using δ equal to ε):

Proposition 30. *Every nonexpansive map is continuous.*

Remark. Obviously constant functions and identity functions for metric spaces are nonexpansive. The composition of nonexpansive maps is nonexpansive.

Since metrics are symmetric, we get the following:

Lemma 31. *Let X be a metric space with metric d. A function $f\colon X \to \mathbb{R}$ is nonexpansive if and only if*

$$fa - fb \le d(a, b)$$

for all $a, b \in X$.

Proposition 32. *Let X be a metric space with metric d. If $x_0 \in X$ then the function $X \to \mathbb{R}$ defined by $y \mapsto d(x_0, y)$ is nonexpansive, so is continuous.*

Proof. This follows from the triangle inequality and the above lemma. □

Proposition 33. *Let X be a metric space with metric d. Consider $X \times X$ as a metric space with the taxicab metric. Then the X-metric*

$$d\colon X \times X \to \mathbb{R}$$

is a nonexpansive map $X \times X \to \mathbb{R}$ for the taxicab metric. Hence the metric on X is a continuous map $X \times X \to \mathbb{R}$ (where $X \times X$ has the product topology).

Proof. Let d_X be the given metric on X and let d' be the taxicab metric on $X \times X$. We use Lemma 31 applied to $f = d_X$. So fix $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $X \times X$. Then by the triangle inequality (applied twice),

$$d_X(a_1, a_2) - d_X(b_1, b_2) \le d_X(a_1, b_1) + d_X(b_1, b_2) + d_X(b_2, a_2) - d_X(b_1, b_2)$$

The right hand side simplifies to

$$d_X(a_1, b_1) + d_X(b_2, a_2) = d'((a_1, a_2), (b_1, b_2)).$$

Thus

$$f(a) - f(b) \leq d'(a, b)$$

as needed in order to use Lemma 31. $\square$

Lemma 34. *Let X be a metric space. Suppose $\mathcal{F}$ is a nonempty collection of nonexpansive functions $X \to \mathbb{R}$. For each $x \in X$ let $\mathcal{F}_x$ be the set $\{fx \mid f \in \mathcal{F}\}$.*

Suppose $\mathcal{F}_x$ is bounded from above for each $x \in X$. Then the function mapping $x \in X$ to the supremum of $\mathcal{F}_x$ is nonexpansive. Suppose $\mathcal{F}_x$ is bounded from below for each $x \in X$. Then the function mapping $x \in X$ to the infimum of $\mathcal{F}_x$ is nonexpansive.

Proof. Suppose $\mathcal{F}_x$ is bounded from above for each $x \in X$, and let $h\colon X \to \mathbb{R}$ be the function mapping $x \in X$ to the supremum of $\mathcal{F}_x$. Let $a, b \in X$. Our goal is to show $ha - hb \leq d(a, b)$ (see Lemma 31).

Let $\delta > 0$. Then $ha - \delta < fa$ for some $f \in \mathcal{F}$, and of course $fb \leq hb$. So

$$ha - hb < fa - hb + \delta \leq fa - fb + \delta \leq d(a, b) + \delta.$$

Since $ha - hb < d(a, b) + \delta$ for all $\delta > 0$, we get $ha - hb \leq d(a, b)$.

The argument for the infimum is similar. $\square$

The following is a corollary (or can be proved directly since the proof is a bit simpler in this case).

Corollary 35. *Let X be a metric space. If f and g are nonexpansive functions $X \to \mathbb{R}$ then so are $x \mapsto \max\{fx, gx\}$ and $x \mapsto \min\{fx, gx\}$.*

Now we give examples of how the infimum and supremum of nonexpansive maps can be used to establish continuity of some key functions. Recall that in Definition 8 we defined the distance $d(x, S)$ from a point x to a subset S.

Proposition 36. *Let X be a metric space with distance function d. Let S be a nonempty subset of X. Then the distance function*

$$x \mapsto d(x, S)$$

is nonexpansive, so is continuous.

Proof. Let $\mathcal{F}$ be the collection of functions $x \mapsto d(x, s)$ with $s \in S$. By Proposition 32, $\mathcal{F}$ is a collection of nonexpansive functions $X \to \mathbb{R}$. For each $x \in X$ let $\mathcal{F}_x$ be the set of values $\{fx \mid f \in \mathcal{F}\} = \{d(x, s) \mid s \in S\}$. By Definition 8, the distance $d(x, S)$ is the infinimum of $\mathcal{F}_x$. In particular, the function

$$x \mapsto d(x, S) = \inf \mathcal{F}_x$$

is nonexpansive by Lemma 34. $\square$

Sometimes it will be useful to have a bounded version of the distance function to a set, for example in defining the bounded interior radius below. In the following we select 1 as our default upper bound:

Proposition 37. *Let X be a metric space with distance function d. Let S be a nonempty subset of X. If S is nonempty, let $d^1_S(x)$ be equal to the minimum of* 1 *and $d(x,S)$. If S is empty, let $d^1_S(x) = 1$. Then d^1_S is a nonexpansive function on X, so is continuous.*

Proof. See Corollary 35. □

Definition 13. Let U be an open set of a metric space X. If $x \in U$ then let $\mathrm{Rad}_U(x)$ be the set of $r \in (0,\infty)$ such that $B_r(x) \subseteq U$.

We also have a bounded version: let $\mathrm{Rad}^1_U(x)$ be $\mathrm{Rad}_U(x) \cap (0,1]$.

For $x \in U$, we define the *bounded interior radius* $\rho^1_U(x)$ to be the supremum of the set $\mathrm{Rad}^1_U(x)$. We define $\rho^1_U(x) = 0$ if $x \notin U$.

Proposition 38. *Let U be an open set of a metric space X, and let Z be the complement of U in X. Then for all $x \in X$*

$$\rho^1_U(x) = d^1_Z(x)$$

where d^1_Z is as in Proposition 37. In particular, the bounded interior radius function ρ^1_U is nonexpansive, and hence continuous.

Here is a version of bounded interior radius for covers:

Definition 14. Let $\mathcal{U}$ be a collection of open subsets of a metric space and let W be the union of the open sets, so $\mathcal{U}$ is a cover of W. Given a point $x \in W$ we define $\tilde{\rho}^1_{\mathcal{U}}(x)$, the *bounded interior radius at x*, to be the least upper bound of the set of all $r \in (0,1]$ such that the ball $B_r(x)$ is as subset of an open set of the cover $\mathcal{U}$. In other words, $\tilde{\rho}^1_{\mathcal{U}}(x)$ is the supremum of the set

$$\{r \in (0,1] \mid \exists\, U \in \mathcal{U}\, (B_r(x) \subseteq U)\}\,.$$

If $x \notin W$ we set the bounded interior radius $\tilde{\rho}^1_{\mathcal{U}}(x)$ to be 0.

Lemma 39. *Let $\mathcal{U}$ be a nonzero collection of open subsets of a metric space and let W be the union of the open sets, so $\mathcal{U}$ is a cover of W. Then, for all $x \in X$, the bounded $\mathcal{U}$ interior radius $\tilde{\rho}^1_{\mathcal{U}}(x)$ is the supremum of the set $\{\rho^1_U(x) \mid U \in \mathcal{U}\}$.*

From this and Lemma 34 we get the following (the special case where $\mathcal{U}$ is empty is separate, but immediate):

Proposition 40. *Let X be a metric space with distance function d. Let $\mathcal{U}$ be a nonzero collection of open subsets of X. Then the bounded interior radius function $\tilde{\rho}^1_{\mathcal{U}}$ is nonexpansive, so is continuous.*

Definition 15. Let X and Y be metric spaces. An *isometry* is a map $f\colon X \to Y$ such that $d_Y\big(fx, fy\big) = d_X(x,y)$ for all $x, y \in X$.

Proposition 41. *Let $f\colon X \to Y$ by an isometry. Then f is a continuous injective function. If Y' is the image of f endowed with restricted metric, then the corresponding function $X \to Y'$ is a homeomorphic isometry.*

Proof. The map f is continuous since it is nonexpansive. Injectivity follows from the definition of metric.

The restricted function $X \to Y'$ is also an isometry, so is continuous and injective. Since this function is a bijection it has an inverse g. Observe that g is also an isometry, so is nonexpansive and continuous. □

5 Metric Spaces and Sequences

Since every metric space is Hausdorff, we get the following (applying results from Part 3 of the series):

Proposition 42. *Suppose X is a metric space. Then a convergent sequence has a unique limit. Furthermore, such a sequence has a unique accumulation point, and this accumulation point is the limit.*

Since every every point of a metric space has a countable local basis, we get the following two results (as in Part 3 of the series):

Proposition 43. *Let $S \subseteq X$ where X is a metric space. Then x is in the closure of S if and only if there is a sequence $(x_i)_{i\in\mathbb{N}}$ in S with limit x.*

Proposition 44. *Let X be a metric space and let Y be a topological space. Suppose that $f\colon X \to Y$ is a function. Then f is continuous at $x \in X$ if and only if f is continuous for sequences at x. So f is continuous if and only if it is continuous for sequences.*

Definition 16. Let $(x_i)_{i\in I}$ be a sequence in a metric space X. We say that (x_i) is *Cauchy* if for every $\varepsilon > 0$ there is an N such that if $i, j \geq N$ then $d(x_i, x_j) < \varepsilon$.

Proposition 45. *Every convergent series in a metric space is Cauchy.*

Definition 17. A metric space X is said to be *complete* if every Cauchy sequence converges.

We take the following theorem as given:

Theorem 46. *The metric space $\mathbb{R}$ is complete.*

Remark. We assume the reader is familiar with the basic properties of $\mathbb{R}$ which certainly includes the above theorem. We remind the reader that this can be derived from the LUB and GLB properties of $\mathbb{R}$ as follows. Suppose (x_i) is a Cauchy sequence. Then $\{x_i\}$ must have an upper and lower bound. For each k, let B_k be the supremum of $\{x_i \mid i \geq k\}$. Observe that B_k is a decreasing sequence with a lower bound. Let B be the infimum of $\{B_k\}$. Then without too much trouble, one can show that B is the limit of (x_i).