Survey of General Topology. Part 2: Hausdorff Spaces

W. E. Aitken*

July 2023 version

You can think of general topology as a framework that considers generalizations of the familiar spaces encountered in mathematics such as the classical subspaces of Euclidean space $\mathbb{R}^n$. This general study unifies the study of many disparate spaces, and encourages mathematicians to seek out new examples of topological spaces within various fields of study in order to apply the insights of general topology to their investigations. In this process of generalization it took some time to settle on the now common standard definition of a topological space (as given in the first document of the series). There are more general notions, more specific notions that still cover most classical examples, and of course various equivalent formulations. One early framework for topology developed by Felix Hausdorff, an important pioneer in set theory and general topology, assumes the following property: given any two distinct points x, y there are disjoint open subsets U, V such that $x \in U$ and $y \in U$. This is called a "separation axiom" since it describes the separation of points with open sets. Note especially that this axiom holds for subspaces of $\mathbb{R}^n$, and it holds for the more general spaces that Hausdorff was interested in. This property is not required in the current conception of a topological space (formally defined in the previous document of this series), but such spaces are ubiquitous and worth study as an important type of topological space. Today we call such spaces "Hausdorff spaces".

The theory of Hausdorff spaces is a bit simpler than the theory of general spaces, and Hausdorff spaces possess some nice properties that generalize familiar properties of classical spaces (as we will see later in this document) that cannot be proved for more general spaces. There are, however, several spaces of interest to mathematicians that are not Hausdorff spaces. So it is generally accepted that one should admit more general spaces than Hausdorff spaces. However, any result that depends on the Hausdorff axiom, in other words, any result about Hausdorff spaces, is centrally important since most spaces of interest are in fact Hausdorff spaces. The goal of this document is to present some of the basic properties of Hausdorff spaces.

In a later document we will consider metric spaces in detail. For now, it is enough to know that such spaces include most spaces familiar to mathematicians, and all such spaces are Hausdorff spaces. In the optional sections of this document we will consider other separation axioms that hold of metric spaces. There is a well-known hierarchy of such axioms ordered from weaker to stronger: T_0, T_1, T_2, T_3, T_4, with others sometimes added. The axiom T_2 is the Hausdorff axiom metioned above. Informally, the stronger the separation axiom the more the space behaves like a metric space. So it is useful sometimes to see what separation axioms are required, if any, to generalize a familiar result from metric spaces to more general topological spaces. These separation axioms are used sometimes to describe conditions that guarantee when a topological space can in fact be given a metric space structure.[1] The general study of separation axioms is less central to the core study of topology than the study of Hausdorff spaces (with axiom T_2), so it is reasonable for the reader to pass over these for now, and return to them only if needed and only after the core concepts (metric spaces, compactness, connectedness, et cetera) are mastered. So these sections are labeled "optional".

The Series: Survey of General Topology

This document is the second of a series which surveys the basics of general topology. This document builds on the first document of the series which covers several of the fundamental topological concepts: topological spaces, open subsets, closed subsets, bases, open neighborhoods, limit points, interiors of subsets, boundaries of subsets, continuous functions, homeomorphisms, products of two spaces, and so on. Other topics such as metric spaces, connectedness, and compactness will be covered in later documents in the series. The documents in this series are focused on the logical structure of the subject, and on the most central results (less central results are sometimes considered in optional sections). Minor details of proof are left to the reader: I believe that working through these details is an excellent exercise which can be pleasant and not too arduous, and gives the reader an overall better experience than reading someone else's write-up of such details. Aside from leaving out such minor details, this series is intended to describe a full and rigorous developments of the essentials of general topology.[2]

1 Prerequisites

General topology is largely a self-contained subject, and this series of notes aims to be self-contained and rigorous with three caveats.

1. Each document in the series builds on results in the previous documents. In particular, this document assume the reader is familiar with the basic properties of open and closed sets in a general topological space, as well as open neighborhoods.

[1] For example, a compact space with a countable basis is metrizable if and only if it satisfies the T_2 axiom. A topological group is metrizable if and only if it is T_0 and the identity element has a countable basis of neighborhoods. A topological space with a countable basis is metrizable if and only if it statisfies this T_3 axiom (the Urysohn metrization theorem, see [5, Theorem 34.1]), and so on.

[2] This series is not intended to replace the many excellent textbooks that exists. Such textbooks provide more examples, motivation, exercises, and references; they also cover some less essential topics that are not included in my series. This series is more of a survey, also serving as a good review or reference for readers with some prior exposure to topology. The hope is that it will provide an efficient but fairly complete view of the essentials.

2. I do assume proficiency with logic, the conventions of mathematical proof, and basic set theory. I also assume some knowledge of the real numbers and the other basic number systems of mathematics, and in some examples I may assume other common results from mathematics, for example properties of polynomial and trigonometric functions.

3. I assume that the reader will want to fill in the routine details, so I do not need to give proofs if they are straightforward. I do try to provide at least enough details of proofs so that filling in the rest of the details is reasonable straightforward.

2 The Notion of a Hausdorff Space

Definition 1. A topological space X is called a *Hausdorff space* if for each $x, y \in X$ with $x \neq y$ there are disjoint open subsets U, V such that $x \in U$ and $y \in V$.

Being Hausdorff is a topological property:

Proposition 1. *Every space homeomorphic to a Hausdorff space is also a Hausdorff space.*

Proposition 2. *Every one-point subset of a Hausdorff space is closed. Thus every finite subset of a Hausdorff space is closed.*

Recall that we defined a *limit point* of a subset S of a topological space to be a point x such that every open neighborhood of x has at least one point of S not equal to x. In Hausdorff spaces a limit point x of S has the stronger property that each neighborhood of x contains an *infinite number* of points in S:

Proposition 3. *Let S be a subset of a Hausdorff space X. A point $x \in X$ is a limit point of S if and only if every open neighborhood of x has an infinite number of points of S.*

Remark. Earlier we used the term *accumulation point* of S for a point x with the property that every open neighborhood of x contains an infinite number of points of S. The above proposition shows that there is no need to introduce this distinction in Hausdorff spaces: limit points are accumulation points and vice versa.

Proposition 4. *Let X be totally ordered set considered as a topological space using the order topology. Then X is a Hausdorff space. In particular, $\mathbb{R}$ with its order topology is a Hausdorff space.*

Proof. Consider distinct points x, y with $x < y$. Consider separately the case where there is or is not a point z between x and y. □

Basic properties of the subspace topology gives the following:

Proposition 5. *A subspace of a Hausdorff space is Hausdorff.*

The only Hausdorff space structure you can put on a finite set is the discrete topology:

Proposition 6. *A finite Hausdorff space is discrete in the sense that every subset is open and closed.*

To check if a space is Hausdorff we only need to use basic open sets:[3]

Proposition 7. *Let X be a topological space with a chosen basis $\mathcal{B}$. Then X is Hausdorff if and only if the following occurs: for all $x, y \in X$ there are disjoint basic open sets U and V such that $x \in U$ and $y \in V$.*

The class of Hausdorff spaces is closed under products. For now we stick to binary products since this was the only type of product considered the previous document of the series:[4]

Proposition 8. *The product of two Hausdorff spaces is Hausdorff.*

Remark. From the above propositions we see that $\mathbb{R}^n$, considered as $\mathbb{R}^{n-1} \times \mathbb{R}$ when $n \geq 1$, is a Hausdorff space, as is any subspace of $\mathbb{R}^n$. This already covers a huge swath of examples. Later we will see that any metric space is a Hausdorff space as well. Since the specific topological spaces considered by most mathematicians are Hausdorff, non-Hausdorff spaces are considered somewhat artificial to many. However, the Zariski topology of algebraic geometry is a natural example of a non-Hausdorff topology. We won't consider the definition of the Zariski topology in this document, but the following example gives an accessible, somewhat natural, example.

Example 1. Here is a simple example of a non-Hausdorff space: let X be an infinite set with the finite complement topology (where a nonempty subset of X is open if and only if it has a finite complement). Then X is not Hausdorff. Note, however, that every one-point subset of X is closed, so X is an example of a T_1 space (in the terminology of Section 4 below).

The proof of the following is a pleasant exercise:

Proposition 9. *A topological space X is Hausdorff if and only if the diagonal*

$$\Delta = \{(x, x) \mid x \in X\}$$

is closed in $X \times X$.

3 Continuous Functions into Hausdorff Spaces

When we consider codomains that are Hausdorff spaces we find that continuous functions have several properties we expect from our experience with real-valued functions. We start with the set on which two continuous functions agree.

---

[3] Recall that if we fix a basis $\mathcal{B}$ then we adopted the term "basic open sets" for the members of $\mathcal{B}$.

[4] This generalizes to infinite products. We will postpone this until the future document where we consider arbitrary products of topological spaces. So far we have only worked with products of two spaces.

Proposition 10. *Let f and g be two continuous functions $X \to Y$ where Y is a Hausdorff space and X is any sort of topological space. Then*

$$\{x \in X \mid f(x) = g(x)\}$$

is closed in X.

Proof. It is enough to prove the following: if $x_0 \in X$ is such that $f(x_0) \neq g(x_0)$ then there is an open neighborhood U of x_0 such that $f(x) \neq g(x)$ for all $x \in U$.

So assume $x_0 \in X$ and $f(x_0) \neq g(x_0)$. Since Y is Hausdorff we have disjoint open subsets V_1 and V_2 of Y such that $f(x_0) \in V_1$ and $g(x_0) \in V_2$. Finally we confirm that the following U has the desired property:

$$U = f^{-1}[V_1] \cap g^{-1}[V_2].$$

□

Example 2. As a corollary to the above we have the closure of the zero set: if $f\colon X \to \mathbb{R}^n$ is a continuous function from a topological space X then the zero set

$$Z(f) = \{x \in X \mid f(x) = 0\}$$

is closed in X.

Proposition 11. *Let f be a continuous function $X \to Y$ where X and Y are topological spaces and where Y is Hausdorff. Then the graph of f*

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}$$

is closed in $X \times Y$.

Proof. Consider the pair of functions $g, h\colon X \times Y \to Y$ where $g(x, y) = y$ and where $h(x, y) = f(x)$. Our established results for functions on Cartesian products guarantee that these functions are continuous, even without the Hausdorff assumption for Y. Observe that the graph Γ_f is just the set where $g(x, y) = h(x, y)$. □

You might expect that a continuous function is determined by its values on any dense subset of the domain. If the codomain Y is Hausdorff this is in fact true.

Definition 2. A subset A of a topological space X is said to be *dense* if $\overline{A} = X$.

Lemma 12. *A subset A of a topological space X is dense if and only if every nonempty open subset of X contains points of A.*

The following expresses what we mean when we say that "a continuous function is determined by its values on a dense subset":

Proposition 13. *Let X be a topological space and let Y be a Hausdorff space. Suppose A is a dense subset of X. In other words, suppose $\overline{A} = X$. Given a function $f : A \to Y$, there is at most one continuous function $X \to Y$ extending f.*

Proof. Suppose g_1 and g_2 are continuous functions $X \to Y$ extending f, and consider the set $Z = \{x \in X \mid g_1(x) = g_2(x)\}$. Since Y is Hausdorff, Z is closed, and of course Z contains A. Thus $Z = X$, and so $g_1 = g_2$. □

Corollary 14. *Let X be a topological space and let Y be a Hausdorff space. Let A be a subset of X. Given a function $f : A \to Y$, there is at most one continuous function $\overline{A} \to Y$ extending f.*

Proof. We apply the previous result to the subspace X' where X' is the closure of A in X. Observe that the closure of A in X' is all of X', so A is dense in X'. □

4 T_1-Spaces (optional)

In what generality should one study topological spaces?[5] In other words, what is the optimal definition of a topological space? We want a notion of space that captures most if not all the spaces that we wish to consider in our mathematical work, while at the same time we want a notion specific enough to allow us to define the common concepts and prove some standard theorems. We want a framework that is general enough to allow us to see clearly the connections between various assumptions and their consequences.

In this series we have made a very standard choice for our definition of topological space. This was laid out in the first document of the series where we defined a topological space using a few standard axioms about the collection of subsets designated as open subsets. Since most spaces that mathematicians care about are Hausdorff, one might add the Hausdorff axiom and only study Hausdorff spaces. This has some benefits; for example, not needing to make a distinction between limit points and accumulation points, or being able to show that continuity and density interact as one expects from experience with real-valued function. However, there is some interest in non-Hausdorff spaces, so perhaps we should work with general spaces and only assume our spaces are Hausdorff when needed for a certain result. This has the theoretic advantage of making it plain when the Hausdorff axiom is required.

In his classic work [6], Pontrjagin took a middle path between starting with general topological spaces in the modern sense, and starting only with Hausdorff spaces from the start. He assumed as a basic principle that the closure of any point set $\{x\}$ is itself, but did not assume, at first, that his spaces are Hausdorff. Of course this is a reasonable assumption since most spaces of interest to mathematicians have this property, so to the extent it simplifies or clarifies the theory it is a reasonable assumption. This axiom is called the T_1 *axiom*, but it is usually not stated in terms of the closure of one-point subsets but rather in an equivalent form that more parallels the Hausdorff axiom discussed above:

Definition 3 (T_1 axiom, T_1 spaces)**.** A T_1 *space* is a topological space X such that the following T_1 axiom holds: *given distinct points $x, y \in X$, there is an open neighborhood of x not containing the point y.*

[5]This section will not be further used in our core series on topological spaces, and so I have labeled it as optional. A reader who wishes to quickly get to the more central topological concepts such as metric spaces, compactness, and connectedness can safely skip this section and the following sections. In my opinion, the separation axioms such as the T_0 and T_1 axioms are best left for serious study after the more essential core concepts have been learned: some readers will need them in their further mathematical work, others will not.

From this formulation it is clear that the Hausdorff axiom (sometimes called the T_2 axiom) is stronger than the T_1 axiom:

Proposition 15. *Every Hausdorff space is a T_1 space.*

As mentioned above, this can be reformulated in terms of one-point subsets:

Definition 4. Let X be a topological space. A *closed point* $x \in X$ is a point such that $\{x\}$ is closed.

Proposition 16. *Let X be a topological space. Then X is a T_1 space if and only if every point of X is a closed point.*

Remark. The Zariski topology of algebraic geometry yields topological spaces with nonclosed points. However, in most other applications of topology we only care about spaces where every point is a closed point. So assuming a space is T_1 when necessary is considered a mild restriction, and a theorem about T_1 spaces is considered a very general result.

Corollary 17. *Every finite subset of a T_1 space is closed.*

Corollary 18. *A finite T_1 space is discrete.*

Either formulation of the T_1 property yields the result that being a T_1 space is a topological property:

Proposition 19. *Every space homeomorphic to a T_1 space is also a T_1 space.*

Corollaries 17 and 18 are analogous to results about Hausdorff spaces. There are several other properties that T_1 spaces have in common with Hausdorff spaces:

Proposition 20. *Every subspace of a T_1 space is a T_1 space.*

Proposition 21. *The product of two T_1 spaces is a T_1 space.*[6]

Proposition 22. *Let S be a subset of a T_1 space X. A point $x \in X$ is a limit point of S if and only if every open neighborhood of x has an infinite number of points of S. (In particular, there is no need to make a distinction between limit points and accumulation points in T_1 spaces.)*

The following exercise gives another amusing characterization of T_1 spaces:

Exercise 1. Let X be a topological space. Show that X is a T_1 space if and only if the subspace topology on any two-point subspace of X is discrete.

5 Generic points and T_0-Spaces (optional)

The Zariski topology used extensively in algebraic geometry often uses nonclosed points which are sometimes called "generic points".[7] When nonclosed points exist the space cannot be Hausdorff, nor even T_1, so algebraic geometry supplies

[6] This generalizes to infinite products as we will see in a later document.

[7] This section is optional, and is designed only for readers who are interested in the possibility of nonclosed points, such as those studying the Zariski topology of algebraic geometry. This optional section builds on some ideas of the previous option section (Section 4).

topologies which fail to satisfy even the T_1 axiom. In this section we will motivate T_0 spaces as an interesting intermediate type between general topological spaces and T_1 spaces.

In the section we will write $a \lhd b$ if a is a point in the closure of $\{b\}$. In the Zariski topology we think of $a \lhd b$ (at least where a and b are distinct) as expressing the idea that b is a "generic" point that has as one of its specialization the point a. It is convenient to use this terminology more generally:

Definition 5. Let X be a topological space. If $a \in X$ is in the closure of $\{b\}$ then we say that *a is a specialization of b* and we write $a \lhd b$. A *generic point of X* is a nonclosed point.

In our terminology established in the first document of this series, a is a specialization of b if and only if a is a *contact point* of $\{b\}$. So intuitively we can paint the picture of a generic point b as a "fat" point, with certain small points a that touch b (but where we think of b as being too big to be in contact with a in all of its extent). Note also that a is a specialization of b if and only if $\overline{\{a\}} \subseteq \overline{\{b\}}$, so

$$a \lhd b \overset{\text{def}}{\iff} a \in \overline{\{b\}} \overset{\text{def}}{\iff} a \text{ is a contact point of } \{b\} \iff \overline{\{a\}} \subseteq \overline{\{b\}}.$$

From this we get the following:

Proposition 23. *Let X be a topological space. Then the relation $\lhd$ of specialization defined above is reflexive and transitive.*

Of course, if X is a T_1 space then the relation $\lhd$ reduces to the equality relation. The converse is true as well:

Proposition 24. *The relation $\lhd$ on a topological space X is equality if and only if every point of X is a closed point, which occurs if and only if X is a T_1 space.*

Proposition 23 asserts that $\lhd$ satisfies satisfies all the properties of a partial order except possibly the antisymmetric property (we have not shown that $a \lhd b$ and $b \lhd a$ together implies $a = b$). In the Zariski topology the relation $\lhd$ turns out to be antisymmetric and so is a true partial order on the space. We will skip the details on the Zariski topology here (mentioning it only for the benefit of readers interested in algebraic geometry), but we can address the following question: What type of spaces have the property that the relation $\lhd$ a partial order? In other words, when is $\lhd$ antisymmetric?

Proposition 25. *Let X be a topological space with the relation $\lhd$ defined above. The following are equivalent:*

1. *The relation $\lhd$ is a partial order.*
2. *The relation $\lhd$ is antisymmetric. In other words if $a, b \in X$ are such that both $a \lhd b$ and $b \lhd a$ then $a = b$.*
3. *For all $a, b \in X$ distinct, there is an closed set containing one but not both of a, b.*

4. *For all $a, b \in X$ distinct, there is an open set containing one but not both of a, b.*

5. *For all $a, b \in X$ distinct, the subset topology on $\{a, b\}$ is nontrivial in the sense that there exists a proper nonempty open subset.*

Any of the above conditions defines what is known as the T_0 axiom. It is customary to focus on open sets in the formal definition of this axiom since it best mirrors the T_1 and Hausdorff (T_2) axioms:

Definition 6 (T_0 axiom, T_0 spaces)**.** A T_0 *space* is a topological space X such that the following T_0 axiom holds: *given distinct points $a, b \in X$, there is an open set containing one but not both of a, b.*

Proposition 26. *Every T_1 space is a T_0 space. A topological space is a T_0 space if and only if the relation $\lhd$ defined above is a partial order, and is a T_1 space if and only if the relation $\lhd$ is just equality.*

The Zariski topology (not pursued here) provides good examples of T_0 spaces that are not T_1 spaces. It turns out that T_0 condition is useful in the theory of topological groups since to show a topological group is Hausdorff (and even regular), it is enough to check the T_0 condition.[8]

6 Further Separation Axioms (optional)

The Hausdorff axiom of Definition 1 is often called the "T_2 axiom", and can be thought of as an axiom describing the possibility of separating distinct points by disjoint open sets.[9] The axioms T_0 and T_1 are considered weaker forms of this separation property. In fact there are five common "separation axioms" T_0, T_1, T_2, T_3, and T_4 of increasing strength:

$$T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0.$$

According to Munkres [5, page 211 of §33], the use of the letter T here comes from "Trennungsaxiom" which is the German word for "separation axiom". Some authors go further and give separation axioms beyond T_0 to T_4.[10] We have described T_0, T_1, T_2 already; T_3 and T_4 are as follows:[11]

[8]Recall that Pontrjagin in his classic [6] on topological groups assumes T_1 for all spaces, which makes all topological group automatically Hausdorff, and even regular.

[9]This optional section builds on the other optional sections in this document.

[10]For example, [1, 425 Q] includes T_5 spaces (completely normal) and T_6 spaces (perfectly normal). Steen and Seebach [8, Section 2] even introduces intermediate axioms $T_{2\frac{1}{2}}$ and $T_{3\frac{1}{2}}$.

[11]In order to support the above chain of implications, we require that all one-point subsets be closed (the T_1 axiom) in our definitions of T_3 and T_4 spaces. But some authors drop the T_1 requirement in some definitions. For example, [8, Section 2] agrees with our definition of regular and normal spaces, but generalizes the notion of T_3 and T_4 spaces by dropping the T_1 requirement.

There are also disagreements in what "regular" and "normal" mean. Munkres [5, §31] agrees with our definition of regular and normal. Bourbaki [2, Chapter I, §8.4] also requires that regular spaces be Hausdorff, but Montgomery and Zippin [4, page 9 of §1.7] do not. Also [7, Chapter 17], which is largely devoted to separation axioms, agrees with our definitions of the T_i, but differs in the definition of regular and normal spaces by not requiring the T_1 axiom; this convention is adopted by Kelley [3] and Willard [9] as well.

Definition 7. A topological space X is said to be a T_3 *space*, or a *regular space*, if (1) all one-point subsets are closed and (2) given a point $x \in X$ and a closed subset $Z \subseteq X$ not containing x there are disjoint open subsets U and V such that $x \in U$ and $Z \subseteq V$. In other words, points and closed subsets can be separated by disjoint open subsets.

A topological space X is said to be a T_4 *space*, or a *normal space*, if (1) all one-point subsets are closed and (2) given disjoint closed subsets Z_1 and Z_2 of X there are disjoint open subsets U_1 and U_2 such that $Z_1 \subseteq U_1$ and $Z_2 \subseteq U_2$. In other words, disjoint closed sets can be separated by disjoint open subsets.

Metrizable spaces satisfy all these separation axioms, so these axioms can be thought of as conditions classifing nonmetrizable spaces. They are also used to provide convenient sufficient conditions for interesting results. For example, they naturally occur along with other assumptions in giving sufficient conditions for a space to be metrizable. Another example can be found in the theory of topological groups where it can be shown that a topological group that satisfies the T_0 axiom is Hausdorff, and even regular. These concerns are somewhat tangential to the immediate goals of this survey, so we are content to focus for now on the most important separation axiom: the Hausdorff axiom (T_2). Regular and normal spaces may be covered in more detail in a future document following the core documents.

Exercise 2 (Regularity equivalence)**.** Show that the second property (2) in the definition of *regular space* is equivalent to the following condition: given $x \in X$ and an open neighborhood U of x, there is an open subset V such that

$$x \in V \subseteq \overline{V} \subseteq U.$$

Exercise 3 (Normality equivalence)**.** Show that the second property (2) in the definition of *normal space* is equivalent to the following condition: given a closed subset Z and an open subset U with $Z \subseteq U$, there is an open subset V such that

$$Z \subseteq V \subseteq \overline{V} \subseteq U.$$

Historical Note. The identification and investigation of separation axioms seems to be largely a phenomenon of the 1920s, although the statement (but not the name) of the T_2 axiom dates earlier to Hausdorff (1914) who included this axiom as part of his early and influential definition of a topological space.

According to [1, 425 Q], the axiom T_0 is attributed to Kolmogorov and axiom T_1 is attributed to Fréchet. As mentioned, axiom T_2 is attributed to Hausdorff. Axiom T_3 is attributed to Vietoris (for a version given in a 1921 publication), and axiom T_4 is attributed to Tietze (for a version from a 1923 publication).

Willard [9] adds some details to this history. He attributes the T_1 axiom to either Fréchet or Riesz, and states that Tietze is responsible for the term *Trennnungsaxiom* (separation axiom) in 1923. He states that Vietoris is responsible for the notion of a regular space (1921). According to Kelley [3, Chapter 1, page 57] the T_i terminology is due to Alexandroff and Hopf (1935).

Bibliography

[1] *Encyclopedic Dictionary of Mathematics.* MIT Press, Cambridge, MA, second edition, 1987.

[2] Nicolas Bourbaki. *General Topology. Chapters 1–4.* Elements of Mathematics. Springer-Verlag, 1989. Translated from the French.

[3] John L. Kelley. *General Topology.* Springer-Verlag, 1975. Reprint of the 1955 edition (Van Nostrand), reprinted in the series Graduate Texts in Mathematics (No. 27).

[4] Deane Montgomery and Leo Zippin. *Topological Transformation Groups.* Dover, 2018. Reprint of the 1955 original, with updates in the 1974 edition.

[5] James R. Munkres. *Topology.* Prentice Hall, 2000. Second edition.

[6] J Pontrjagin. *Topological Groups.* Princeton University Press, 1939. First English Edition, translated by Emma Lehmer.

[7] Eric Schechter. *Handbook of Analysis and its Foundations.* Academic Press, 1997.

[8] Lynn Arthur Steen and J. Arthur Seebach, Jr. *Counterexamples in Topology.* Dover Publications, 1995. Reprint of the second (1978) edition.

[9] Stephen Willard. *General Topology.* Dover Publications, 2004. Reprint of the 1970 original: Addison-Wesley.