A Review of General Topology. Part 6: Connectedness

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This document is the sixth part in a series which gives a review of the basics of general document. This installment covers the concept of connectedness in general topology. Topics, such as compactness and topological groups will be covered in follow-up documents.

There are several classes of readers that could benefit from this review. A reader who learned topology in the past but who has forgotten some details could use this as a summary of the key definitions and results. The proofs of many of the results are missing or are merely sketched, but enough details are given that a student comfortable with set-theoretic reasoning could supply the details. So a reader who has at least a causal familiarity of topology could use this series to systematically work through the subject, supplying the missing proofs along the way. The reader should be warned that this review is light on counter-examples and skips some less essential topics, so these notes are not a substitute for a more complete textbook. However, I have tried to hit all the really important elements. Can this series be used as a first introduction to general topology? I believe it can if used in conjunction with a knowledgeable instructor or knowledgeable friend, or if supplemented with other less concise sources that discuss additional examples and motivations.

For the reader who wants to systematically work through the material with full proofs, I mention that is a rigorous account in the sense that it only relies on results that can be fully proved by the reader without too much trouble given the outlines provided here. The reader is expected to be versed in basic logical and set-theoretic techniques employed in the upper-division curriculum of a standard mathematics major. But other than that, the subject is self-contained.[1] I have attempted to give full and clear statements of the definitions and results, with motivations provided where possible, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. So some of the proofs may be quite terse or missing altogether. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is

[1]Set theoretic reason here is taken to include not just ideas related to intersections, unions, and the empty set, but also complements, functions between arbitrary sets, images and preimages of functions, Cartesian products, relations such as order relations and equivalence relations, well-ordering and so on.

straightforward. Supplied proofs are sometimes just sketches, but I have attempted to be detailed enough that the prepared reader can supply the details without too much trouble. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant. There are several examples included and most of these require the reader to work out various details, so they provide additional exercise.

1 Logical Dependencies

This document assumes familiarity with some basic properties of $\mathbb{R}$ and its subfield $\mathbb{Q}$. For example the LUB property of $\mathbb{R}$ is critical to the material presented here. The related notions of Dedekind cuts and cut points is useful, but really only to motivate the definition of connected.

The basic topological notions from the first part of this series are used extensively in this document. Facts about product topologies are also used. Results in the other earlier documents in the series, covering Hausdorff spaces, sequences, and metric spaces, are only needed for examples, if at all.

2 Connected Spaces

The space of real numbers $\mathbb{R}$ has the property that any partition of $\mathbb{R}$ by two nonempty convex subsets S_1 and S_2 has a (unique) *cut point* x. The partition itself is called a *Dedekind cut* and the cut point is a point in the closure of both S_1 and S_2. The existence of cut points is traditionally described as a manifestation of the completeness of $\mathbb{R}$, but in this section we view it as related to the general topological phenomenon of *connectedness*. Note: it turns out we can drop the convexity condition and replace it with the condition that S_1 and S_2 be nonempty; cut points will still exist although we may loose the uniqueness of the cut point if the sets are not convex. The notion of *connection point* is a generalization of the notion of *cut point*.

Definition 1. Suppose S_1 and S_2 are two subsets of a topological space X. A point $x \in X$ is called a *connection point* for S_1 and S_2 if $x \in \overline{S_1}$ and $x \in \overline{S_2}$.

If $x \in \overline{S}$ the sometimes we say that x is a *contact point* of the subset S (this includes all limit points of S together with any point of S that is not a limit point of S). So a connection point for S_1 and S_2 is simply a point that is a contact point for both S_1 and S_2.

We proved earlier that if x is a contact point of S in a topological space X, then the image $f(x)$ is a contact point of $f[S]$ in the space Y for any continuous function $f\colon X \to Y$. This implies that being a connection point is preserved by continuous functions as well:

Proposition 1. *Suppose $f\colon X \to Y$ is continuous. If $x \in X$ is a connection point for subsets S_1 and S_2 then $f(x)$ is a connection point for $f[S_1]$ and $f[S_2]$.*

Intuitively we view a contact point as providing a "connection" or a "bridge" between two sets. This leads to the notion of *connectedness*:

Definition 2. A space X is said to be *connected* if, for all partitions of X by two nonempty subsets S_1 and S_2, there is a connection point for S_1 and S_2.

Proposition 2. *Suppose $f: X \to Y$ is continuous and surjective. If X is connected then so is Y.*

Corollary 3. *Suppose $f: X \to Y$ is continuous then the image $f[X]$ of a connected space X is a connected subspace of Y.*

Corollary 4. *Suppose $f: X \to Y$ is a homeomorphism. Then X is connected if and only if Y is connected.*

Proposition 5. *Suppose a topological space X is partitioned by two sets S_1 and S_2. Then the following are equivalent:*

1. *There is no connection point for S_1 and S_2.*
2. *The subsets S_1 and S_2 are both closed in X.*
3. *The subsets S_1 and S_2 are both open in X.*

Proof. It is fairly straightforward to show (1) $\iff$ (2) since S_1 and S_2 are disjoint. Similarly (2) $\iff$ (3) is straightforward using complements. □

Proposition 6. *Let X be a topological space. The following are equivalent:*

1. *X is connected.*
2. *X cannot be partitioned into two nonempty closed sets.*
3. *X cannot be partitioned into two nonempty open sets.*

Example 1. In a later section we will see that $\mathbb{R}$ is connected but $\mathbb{Q}$ is not. The empty space is connected, as is any singleton space. A discrete space with more than one point is not connected.

Corollary 7. *A space X is connected if and only if the only clopen subsets of X are X and $\emptyset$.*

3 Connected Subsets

Now we consider the issue of connectedness for subsets of a fixed topological space X. A subset Y of a space X is said to be *connected* if Y is a connected space using the subspace topology.

Recall that the closure operation is well-behaved with respect to the subspace topology in the following sense: if Y is a subspace of X and if S is a subset of Y then the closure of S in Y is equal to $\overline{S} \cap Y$ where $\overline{S}$ is the closure of S in X. In other words, given a point y of Y and a subset $S \subseteq Y$, we have that y is a contact point of S in the topology of X if and only if y is a contact point of S in the subspace topology of Y. Since connection points are defined in terms of common contact points, we have the following:

Proposition 8. *Suppose that Y is a subset of X and that S_1 and S_2 are subsets of Y. Suppose that $y \in Y$. Then y is a connection point for S_1 and S_2 in the subspace topology on Y if and only if y is a connection point for S_1 and S_2 in the topology of X.*

We can restate Corollary 3 as follows:

Proposition 9. *The image of a connected subset under a continuous map is a connected subset of the codomain.*

Building on Proposition 6 we get the following:

Proposition 10. *Let Y be a connected subset of a topological space X. Suppose that A and B are open subsets of X such that (1) $Y \subseteq A \cup B$ and (2) $Y \cap A \cap B = \emptyset$. Then Y is a subset of either A or B.*

Similarly, suppose that A and B are closed subsets of X such that (1) $Y \subseteq A \cup B$ and (2) $Y \cap A \cap B = \emptyset$. Then Y is a subset of either A or B.

Most commonly this lemma is applied in the following special case:

Proposition 11. *Let Y be a connected subset of a topological space X. Suppose that A and B are open subsets of X that partition X. Then Y is a subset of either A or B.*

This lemma can be applied to the following two propositions (where we first replace X with the union of the subspaces).

Proposition 12. *Suppose a collection $\mathcal{C}$ of connected subsets of a topological space X has the property that each $C, D \in \mathcal{C}$ have a nonempty intersection $C \cap D$. Then the union of the subsets in $\mathcal{C}$ is a connected subset of X.*

Proposition 13. *Given a sequence of connected subsets $Y_1, Y_2, Y_3, \ldots$ of X such that adjacent terms Y_i and Y_{i+1} intersect. Then the union is connected. (A similar statement holds for a finite sequences).*

We can freely add contact points to a connected subset and the resulting subset remains connected:

Proposition 14. *Suppose C is a connected subset of X and suppose D is a subset of X such that $C \subseteq D \subseteq \overline{C}$. Then D is connected.*

Proof. Work in the subspace D. Start by observing that C is a connected subset of D. Consider a partition of D by nonempty open subsets A and B. $\square$

Proposition 15. *Suppose Y_1 and Y_2 are connected subsets of X and that $x \in X$ is a connection point for Y_1 and Y_2. Then $Y = Y_1 \cup Y_2 \cup \{x\}$ is connected.*

Proof. Let $Y_1' = Y_1 \cup \{x\}$ and $Y_2' = Y_2 \cup \{x\}$. Then the result follows from previous results. $\square$

Example 2. The finite union of two or more pairwise disjoint open sets, or the finite union of two or more pairwise disjoint closed sets is discontinuous. You can use this to produce interesting examples: for example, one can show that certain asymptotic curves are disconnected. For example, consider curves defined by $xy = 0$ and $xy = 1$ with $x \geq 0$. These are disjoint closed curves in $\mathbb{R}^2$, so their union must be disconnected (assume as known that the function $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $(x, y) \mapsto xy$ from is continuous, so the above two sets are closed).

4 Connected Ordered Sets

Let S be a subset of an ordered space X. Recall that S is said to be *convex* iff the following holds: if $a \leq x \leq b$ with $a, b \in S$ and $x \in X$ then $x \in S$. We say that S has the *density property* if the following holds: if $a < b$ with $a, b \in S$ then there is an element $c \in S$ such that $a < c < b$. There is a close connection between these properties and connectedness.

Proposition 16. *Suppose S is a connected subset of an ordered space X. Then S is convex.*

Proposition 17. *Suppose S is a connected subset of an ordered space X. Then S has the density property.*

Is convexity enough to guarantee that a subset is connected? It turns out that we need a LUB property as well ($\mathbb{Q}$ is convex in itself, but is not connected). We say that a linearly ordered set has the *LUB property* if every nonempty subset that has an upper bound also has a least upper bound in the set.

Theorem 18. *Suppose X is an ordered space with the LUB property and the density property. Then any convex subset of X is connected.*

Proof. Assume S is convex and that S is partitioned into two nonempty subsets A and B. Fix points $a \in A$ and $b \in B$. By symmetry of cases, we can assume $a < b$. Let A' be the intersection of A with the interval $[a, b]$. Since b is an upper bound of A', there is a least upper bound c of A'. Observe that $a \leq c \leq b$ so $c \in S$ by convexity. So either $c \in A$ or $c \in B$. If $c \in A$ then use density to observe that any open interval of X containing c contains a point of S strictly greater than c which must be in B. If $c \in B$ then use density to observe that any open interval of X containing c also contains a point of S strictly less than c, and in fact contains a point of A. Thus $c \in S$ is a connection point of A and B. □

Corollary 19. *Suppose X is an ordered space with the LUB property and the density property. Then X, with the order topology, is connected. In particular the space $\mathbb{R}$ is connected.*

Recall that intervals of an ordered space are convex. The converse is also true in spaces like $\mathbb{R}$ as we will see in the next two lemmas.

Lemma 20. *An ordered space has the LUB (least upper bound) property if and only if it has the GLB (greatest lower bound) property.*

Proof. Suppose, for example, that the LUB property holds. Let S be a nonempty set that is bounded below. Take the LUB of the set of lower bounds to S, and show it yields a GLB. $\square$

Lemma 21. *Suppose X is an ordered space with the LUB property. Then a subset S of X is convex if and only if S is an interval.*

Proof. We know from the first part of this series that all intervals are convex. So we assume S is convex with the goal of showing that it is an interval. The case where S is empty is trivial, so we assume S is nonempty as well.

Let b be the LUB of S, or let $b = \infty$ if S has no upper bounds. Similarly let a be the GLB of S, or let $a = -\infty$ if S has no lower bounds. First we use convexity to establish that $(a, b) \subseteq S$. Next we establish that any element of S not in (a, b) must be a or b. Finally we define an interval using a and b, and show it is S. $\square$

The following follows from the results established above:

Theorem 22. *Suppose X is an ordered space with the LUB property and the density property, and suppose S is a subset of X. The following are equivalent:*

1. *S is connected.*
2. *S is convex.*
3. *S is an interval.*

In particular, a subset of $\mathbb{R}$ is connected if and only if it is an interval.

No subset of $\mathbb{Q}$ with more than one point is convex in $\mathbb{R}$. Thus we have the following:

Corollary 23. *The subset $\mathbb{Q}$ is not connected in $\mathbb{R}$. In fact, no subset of $\mathbb{Q}$ with more than one point is connected in $\mathbb{R}$.*

Remark. We can think of $\mathbb{Q}$ as a topological space in two natural ways: (1) as a subspace of $\mathbb{R}$, and (2) as an ordered space using the natural order on $\mathbb{Q}$. We leave it as an exercise that these are in fact the same topology. The above corollary implies that $\mathbb{Q}$ thought of as a subspace of $\mathbb{R}$ is not a connected topological space, and *ipso facto* $\mathbb{Q}$ is not connected when thought of as an ordered space. Since a subset of $\mathbb{Q}$ is connected in the subspace $\mathbb{Q}$ if and only if it is connected in the space $\mathbb{R}$, the only nonempty subsets of the space $\mathbb{Q}$ that are connected are the singleton sets.

Remark. Spaces whose connected nonempty subsets are all singleton sets are called *totally disconnected* spaces. These include discrete spaces (where all subsets are open) but, as the above remark shows, not all totally disconnected spaces are discrete.

Example 3. Any continuous function $f : X \to \mathbb{Q}$ where X is nonempty and connected must be a constant map. This generalizes when we replace $\mathbb{Q}$ with any totally discontinuous space.

Remark. Munkres defines a *linear continua* to be an ordered space with the LUB and density properties. So $\mathbb{R}$ is a typical example of a linear continua. Another interesting example is $I \times I$ where I is the interval $[0, 1] \subseteq \mathbb{R}$. This can be given the dictionary (lexigraphic) order. Observe that this topological space satisfies the density and the LUB property. So with this topology, $I \times I$ is connected by the above corollary.

The following converse holds:

Proposition 24. *Any connected linearly ordered space is a linear continuum.*

Proof. See Proposition 17 for the density property. Any bounded nonempty subset without a least upper bound gives a partition by nonempty open subsets: the set of upper bounds and the set of non-upper bounds. □

Every interval of $\mathbb{R}$ is convex, and so the subspace topology and the order topology agree. Since every interval is connected in $\mathbb{R}$, we get the following:

Corollary 25. *Any interval of $\mathbb{R}$ is a linear continuum (thought of as an ordered space).*

Theorem 26 (Intermediate Value Theorem)**.** *Suppose X is connected, and Y is linearly ordered. Suppose $a, b \in X$ and r is between $f(a)$ and $f(b)$ where f is a continuous function. Then r is in the image of f.*

Proof. The image of X is connected, so must be convex. □

Remark. A common setting for the intermediate value theorem is for continuous function $f\colon I \to \mathbb{R}$ where I is $\mathbb{R}$ or, more generally, an interval of $\mathbb{R}$.

The intermediate value theorem can be used to show that any continuous function from $[0, 1]$ to $[0, 1]$ has a fixed point (using the rule $x \mapsto f(x) - x$ to define a function $g\colon [0, 1] \to \mathbb{R}$).

5 Connected Product Spaces

Suppose that C_1 and C_2 are two connected spaces. If $c_1 \in C_1$ is fixed then $\{c_1\} \times C_2$ can be shown to be a connected subset of $C_1 \times C_2$. Similarly $C_1 \times \{c_2\}$ is connected for each $c_2 \in C_2$. We can use these ideas to show that $C_1 \times C_2$ itself is connected. This type of argument clearly extends to finite Cartesian products. With some extra work we can extend it to infinite products as well.

Theorem 27. *The product of connected spaces is connected.*

Proof. Let $P = \prod C_i$ be such a product, and assume we have a partition of P by two nonempty open subsets A and B. Fix a point (c_i) in A. Next we choose a point (c'_i) in B. Because B is open, and due to the definition of the standard basis, we see that we can actually choose $(c'_i) \in B$ so that $c'_i = c_i$ for all but a finite number of values. Let J be the finite set of indices j such that $c'_j \neq c_j$.

Choose k in this set J, and consider the injection $\iota_k\colon C_k \to P$ sending x to (d_i) where $d_k = x$ and $d_i = c'_i$ for $i \neq k$. Observe that ι_k is continuous, so the image is connected and thus lies completely in B. So we can replace c'_k in (c'_i) with any other value without leaving B. After such a replacement we can assume $c'_k = c_k$ (and assume that the other coordinates of (c'_i) are unchanged). Since J is finite, we can repeat this a finite number of times to show that (c_i) itself is in B, a contradiction. $\square$

Corollary 28. *The space $\mathbb{R}^n$ is connected.*

6 Path Connectedness

In this section $I = [0, 1]$ refers to the closed unit interval in $\mathbb{R}$. From the above we know that I is a connected topological space (using the order topology or equivalently the subspace topology).

Definition 3. A space X is *path connected* if for all $a, b \in X$ there is continuous map $[0, 1] \to X$ sending 0 to a and 1 to b. Such a map is called a *path* from a to b. If the image of the map is contained in S where $S \subseteq X$ then we say it is a *path in S from a to b.*.

Proposition 29. *If X is path connected then it is connected.*

Proof. Partition X into two open subsets, and show that any two points $a, b \in X$ must be in the same subset of this partition. $\square$

There are various examples of spaces that are connected but not path connected. Here is an interesting example:

Proposition 30. *The order space $I \times I$, ordered by the dictionary order, is connected but not path connected.*

Proof. We saw it was connected since it has the LUB and density properties. Claim: there is no path from the point $(0, 0)$ to the point $(1, 1)$. To see this first observe that such a path would be a surjective continuous function $I \to I \times I$ by the intermediate value theorem. However, $I \times I$ has an uncountable collection of (pairwise) disjoint open sets, but I does not. $\square$

As expected, a subset S of a topological space X is said to be path connected if it is path connected under the subset topology. This is seen to be equivalent to requiring that for any $a, b \in S$ there is a path $I \to X$ in S from a to b.

There are some similarities between connectedness and path connectedness. For example, it is a property that is preserved under continuous functions:

Proposition 31. *The image of a path connected space under a continuous map is a path connected subset of the codomain.*

Corollary 32. *Suppose $f\colon X \to Y$ is a homeomorphism. Then X is path connected if and only if Y is path connected.*

Given a path from a to b, and another from b to c, we can combine these paths to produce a path from a to c whose image is the union of the images of the given paths. This is a sort of "composition of paths". To define this we need to establish the continuity of linear functions $\mathbb{R} \to \mathbb{R}$. In the case of a map $x \mapsto ax+b$ with $a > 0$ this is just a special case of the following (proved in an earlier part of the series):

Proposition 33. *If $f : X \to Y$ is an order-preserving bijective function between order spaces then f is a homeomorphism.*

Constant maps are continuous, which covers the case $a = 0$ for linear maps. The case $a < 0$ is not too hard, given the result for $a > 0$:

Proposition 34. *Let $f : \mathbb{R} \to \mathbb{R}$ be given by the rule $x \mapsto ax + b$ where $a, b \in \mathbb{R}$. Then f is continuous. If $a \neq 0$ then f is a homeomorphism.*

Proof. If $a > 0$ then this map is order preserving with inverse $y \mapsto (y - b)/a$. So is a homeomorphism by Proposition 33. If $a = 0$ then f is a constant map, so is continuous.

So we can now focus on the case $a < 0$. We start with the function $h : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto -x$. The preimage of the interval (c, d) is $(-d, -c)$. This means that h is continuous. Since h is its own inverse, h is in fact a homeomorphism.

The map $h \circ f$ is given by $x \mapsto (-a)x - b$. This is a homeomorphism when $-a > 0$ by the earlier case. So $f = h \circ h \circ f$ is a homeomorphism when $a < 0$. □

Using linear functions we can paste together paths. We also need a pasting lemma. Recall that if Y is an open subspace of X, then a subset of Y is open in Y if and only if it open in X. Similarly if Y is a closed subspace of X, then a subset of Y is closed in Y if and only if it closed in X. We will use these facts this in the proof of the following:

Lemma 35 (Pasting lemma)**.** *Suppose that A_1 and A_2 are subspaces of a topological space X. Suppose $f_1 : A_1 \to Y$ and $f_2 : A_2 \to Y$ are continuous functions that agree on $A_1 \cap A_2$. Finally suppose A_1 and A_2 are both open in X or both closed in X. Then there is a unique continuous function $f : A \to Y$, where $A = A_1 \cup A_2$, whose restriction to A_1 is f_1 and whose restriction to A_2 is f_2.*

Proof. Existence and uniqueness of a function f is straightforward; the challenge is to show that such f is continuous. If A_1 and A_2 are open, then consider the preimage of an open subset of Y. If A_1 and A_2 are closed, then consider the preimage of a closed subset of Y. □

Example 4. Consider the intervals $A = [0, 1], A_1 = [0, 1/2]$ and $A_2 = [1/2, 1]$. Note that A_1 and A_2 are closed in the topology of A and $A_1 \cup A_2 = A$. So if we have a continuous functions $g_1 : A_1 \to X$ and $g_2 : A_2 \to X$, then we can paste these together to form a continuous function $g : A \to X$ in the sense of the above lemma if and only $g_1(1/2) = g_2(1/2)$ (since $A_1 \cap A_2 = \{1/2\}$).

Definition 4. Let $a, b, c \in X$ where X is a topological space. Let $g_1 : [0, 1] \to X$ be a path from a to b and let $g_2 : [0, 1] \to X$ be a path from b to c. Then the *composition of g_1 followed by g_2* is the unique path $[0, 1] \to X$ obtained by pasting together $g_1 \circ f_1 : [0, 1/2] \to X$ and $g_2 \circ f_2 : [1/2, 1] \to X$. Here $f_1 : [0, 1/2] \to [0, 1]$ is the continuous map defined by $x \mapsto 2x$, and $f_2\colon [1/2, 1] \to [0, 1]$ is the continuous map defined by $x \mapsto 2x - 1$.

We can use composition of paths to help prove the following:

Proposition 36. *Suppose a collection $\mathcal{C}$ of path connected subsets of a topological space X has the property that each pair $A, B \in \mathcal{C}$ has a nonempty intersection $A \cap B$. Then the union T of the subsets in $\mathcal{C}$ is a path connected subset of X.*

Proof. Let $a, c \in T$. Then $a \in A$ and $c \in C$ for some A and C in $\mathcal{C}$. Choose a point $b \in A \cap C$. Consider a path g_1 in A from a to b, and a path g_2 in C from b to c. Then take the composition of these to get a path in T from a to c. □

Similarly we can prove the following:

Proposition 37. *Given a sequence of path connected subsets of X such that adjacent terms intersect. Then the union is path connected.*

Example 5. Intervals of $\mathbb{R}$ are path connected; this can be shown with linear maps (Proposition 34). Star-convex subsets of $\mathbb{R}^n$ are path connected, and so are connected. This can be shown by pasting together straight paths. (Star-convex subsets are subsets S with a distinguished point p, such that for all $q \in S$, the line segment with endpoints p and q lies in S.) In particular, open and closed disk in $\mathbb{R}^n$ are path-connected since they are star-convex. By composing straight paths, it is not hard to show that $\mathbb{R}^n$ minus a point is path connected if $n > 1$. This shows that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic if $n > 1$.

The $n - 1$ sphere is the continuous image of $\mathbb{R}^n$ minus the origin (assume for now that $x \mapsto x/\|x\|$ is known to be continuous). Thus the $n - 1$ sphere is also path connected if $n > 1$.

Example 6. We saw above that $I \times I$ with the dictionary order is connected, but not path connected. Are there subspaces of $\mathbb{R}^n$ with this behavior? Munkres gives a *comb example*. Consider the subspace C of $\mathbb{R}^2$ given by the union of $\{1, 1/2, 1/3, \ldots\} \times I$ with $I \times \{0\}$. Let D be the union of C with the singleton $\{(0, 1)\}$.

Proposition 38. *The subspace D of $\mathbb{R}^2$ defined above is connected but not path connected.*

Proof. By composing paths, we see that C is path connected, and so is connected. Since $(0, 1)$ is a contact point of C we have D is connected by Lemma 14. Next we show that D is not path connected by showing that all paths in D starting at $(0, 1)$ must be constant.

Let $g\colon [0, 1] \to \mathbb{R}^2$ be a path in D from $(0, 1)$ to $(u, v) \in D$. Let U be the preimage under g of $\{(0, 1)\}$, and let V be the complement of U in I. Since V is the preimage of the open subset C of D, we know V is open. We claim that U is also open.

To see this, let $t \in U$. In other words, assume $g(t) = (0, 1)$. Let A be the intersection of D with the open rectangle $(-1, 1) \times (1/2, 3/2)$. By continuity of g there is an interval W containing t whose image is contained in A. For each $(x, y) \in A$ with $x > 0$, choose an irrational number $0 < \alpha < x$. We use α to form an open partition A_1, A_2 of A where we have $(0,1) \in A_1$ and $(x,y) \in A_2$. Since W is connected, its image must be contained totally in A_1. This shows that (x,y) is not in $g[W]$. In particular, no point of C is in $g[W]$. In other words, $W \subseteq U$. Since t is an arbitrary point of U, this means U is open.

The sets U and V give an open partition of I. This means that V must be empty since I is connected. So all paths in D that start with $(0,1)$ must be constant. In particular, the space D is not path connected. □

Graphs of continuous functions give a handy collection of examples of path connected spaces:

Proposition 39. *A continuous function from an interval of* $\mathbb{R}$ *to* $\mathbb{R}$ *has a path connected graph.*

Proof. Recall that the graph of a function is homeomorphic to $\mathbb{R}$. □

Example 7. Suppose we have established the continuity of the sine function. Then let S be the closure of the graph of $y = \sin(1/x)$ where $x > 0$. By Proposition 14, the subspace S is connected. However, it is not path connected (using an argument similar to that of C of Proposition 38 above).

Finally we consider products:

Proposition 40. *The product of path connected sets is path connected.*

Proof. Given two points in the product, construct a path out of component functions. You may have to use the axiom of choice for general products. □

7 Components

Definition 5. Let x be a point in a topological space X. The *connected component* of x is defined to be the union of all connected subsets of X containing x. A *connected component* of X, or *component* for short, is a subset of X that is the connected component of x for some $x \in X$.

The *path connected component* of x is defined to be the union of all path-connected subsets of X. A *path connected component* of X, or *path component* for short, is a subset of X that is the path connected component of x for some $x \in X$.

Proposition 41. *Let x be a point in a topological space X. The connected component of x is connected. In fact it is the largest connected subset of X containing x. Similarly, the path-connected component of x is path-connected, and so is connected. In fact it is the largest path-connected subset of X containing x.*

Proof. Start with the definition and Propositions 12 and 36. □

Lemma 42. *Let x and y be points in a topological space X. If y is in the connected component of x, then the connected components of x and y are equal. Similarly, if y is in the path connected component of x, then the path connected components of x and y are equal.*

Proof. Let C_x be the connected component of x and let C_y be the connected component of y. If $y \in C_x$ then, since C_x is connected and contains y, we have $C_x \subseteq C_y$. This means that $x \in C_y$, so $C_y \subseteq C_x$ by a similar argument.

The second claim is justified in a similar manner. □

Proposition 43. *Let x be a point in a topological space X. Then x is contained in exactly one connected component of X. Thus X is partitioned by its components. Similarly, x is contained in exactly one path connected component of X. Thus X is also partitioned by its path components.*

Proof. Clearly $x \in C_x$ where C_x is the connected component of x. Suppose $x \in C_y$ where C_y is the connected component of $y \in X$. Then $C_x = C_y$ by the previous result. □

Proposition 44. *Every nonempty connected subset of a topological space X is a subset of a unique component of X. Every nonempty path connected subset of a topological space X is a subset of a path component of X.*

Corollary 45. *Every path component of a topological space X is contained in a unique component of X. Thus every component is partitioned by path components.*

Example 8. A space is connected if and only if it has one component. Thus $\mathbb{R}$ has one component, and that component is $\mathbb{R}$ itself. In totally discontinuous space X, the components are just the singleton subsets of X. Thus every singleton subset of $\mathbb{Q}$ is a component of $\mathbb{Q}$. This shows that components of a space need not be open subsets. However, we have the following:

Proposition 46. *Every component of a topological space is a closed subset.*

Proof. This is a consequence of Proposition 14. □

Remark. The above does not hold for path components. For example, for C and D in Proposition 38 the set C is a path component of D but is not closed in D.

8 Local Path Connectedness

There is a common class of spaces used in mathematics, namely the local path connected spaces, where the theory of components is greatly simplified.

Definition 6. A space X is *locally path connected* if X has a basis of path connected sets.

Proposition 47. *Suppose X is a locally path connected space. Then every path component of X is open in X. In fact, the path components of X are clopen in X.*

Proof. Let C be a path component of X. Let $x \in C$. By assumption there is an open neighborhood B of x that is also path connected. Thus $B \subseteq C$. Such B exists for all $x \in C$ so C must be open.

Since every path component is the complement of the union of path components, which we now know is open, every path component is closed as well. □

Corollary 48. *Suppose X is a locally path connected space. Then every component of X is open in X. In fact, the components of X are clopen in X.*

Proof. This follows from the fact that every component is the union of path components. ☐

Remark. We can define a notion of locally connected, and and use it to prove that components of a locally connected space are open. This gives a generalization of the above result. We don't really need this result since our main application is for open subspaces of $\mathbb{R}^n$ which are already locally path connected.

Lemma 49. *The space $\mathbb{R}$ is locally path connected.*

Proof. Recall that every open interval is path connected. ☐

Lemma 50. *The finite product of locally path connected spaces is locally path connected. In particular, $\mathbb{R}^n$ is locally path connected.*

Remark. We can generalize this to infinite products if t all but a finite number of terms of the product are path connected.

Lemma 51. *Open subspaces of a locally path connected space are locally path connected.*

Corollary 52. *Every open subspace of $\mathbb{R}^n$ is locally path connected.*

Remark. We have seen examples of subsets of $\mathbb{R}^2$ that are not locally path connected. (Any space that is connected but not path connected cannot be locally path connected as we shall see).

Proposition 53. *If X is locally path connected, then any space homeomorphic to X is locally path connected.*

Another nice feature of locally path connected spaces is that path connected and connected align:

Proposition 54. *Let X be a locally path connected. Then X is connected if and only it is path connected.*

Proof. One direction has been established, so we can assume that X is connected with a goal to show X is path connected. In this case any path component and its complement gives a partition of X into open subsets, which gives us our result. ☐

Corollary 55. *If a space is locally path connected, then each of its components is path connected. Hence each component is a path component.*