A Review of General Topology. Part 7: Compactness

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March 2021 Edition*

This is the seventh part in a series which reviews the basics of general topology. This part is devoted to the concept of compactness and it is the last of the "core concepts" series. There are follow-up documents planned for topics such as topological groups, local compactness, manifolds, paracompactness, Tychonoff's Theorem, and so on, but they will not be part of a series and will be somewhat independent of each other.

There are several types of readers that I had in mind when writing this review. One type is a reader who learned topology in the past but who has forgotten the details and could use this as a summary of the key definitions and results and a description on how the theory is built up. (This type of reader might have even include the author before setting out to write this series). The proofs of many of the results are missing or are merely sketched, but enough details are given that a reader comfortable with set-theoretic reasoning could supply the details. So a reader who has at least a causal familiarity of topology could use this series to systematically work through the subject, supplying the missing proofs along the way. The reader should be warned that this review is light on counter-examples and skips some less essential topics, so these notes are not a substitute for a more complete textbook or reference text. However, I have tried to hit all the really important elements. What about readers without a past background in topology? Can this series be used as a first introduction to general topology? I believe it can if used in conjunction with a knowledgeable instructor or knowledgeable friend, or if supplemented with other less concise sources that discuss additional examples and motivations. But it does help if the reader has some exposure to the topology of $\mathbb{R}^n$ or, more generally, the topology of metric spaces.

For the reader who wants to systematically work through the material with full proofs, I mention that is a rigorous account in the sense that it only relies on results that can be fully proved by the reader without too much trouble given the outlines provided here. The reader is expected to be versed in basic logical and set-theoretic techniques employed in the upper-division curriculum of a standard mathematics major. But other than that, the subject is self-contained.[1] I have

[1]Set theoretic reason here is taken to include not just ideas related to intersections, unions, and the empty set, but also complements, functions between arbitrary sets, images and preimages of functions, Cartesian products, relations such as order relations and equivalence relations, well-ordering and so on.

attempted to give full and clear statements of the definitions and results, with motivations provided where possible, and give indications of any proof that is not straightforward. However, my philosophy is that, at this level of mathematics, straightforward proofs are best worked out by the reader. Working out proofs is a great way to sharpens ones topological understanding and intuitions. Whenever a proof is not given, this signals to the reader that they should work out the proof, and that the proof is straightforward. Supplied proofs are sometimes just sketches, but I have attempted to be detailed enough that the prepared reader can supply the details without too much trouble. Even when a proof is provided, I encourage the reader to attempt a proof first before looking at the provided proof. Often the reader's proof will make more sense because it reflects their own viewpoint, and may even be more elegant. There are several examples included and most of these require the reader to work out various details, so they provide additional exercise.

1 Logical dependencies and notation

In addition to requiring basic mathematical logic and set theory, this document builds on the theory developed in the six earlier documents in this series. It uses Part 6 (connectedness) only in a small way, for example briefly in Sections 5 and 8, and in Proposition 36. So a reader could study the current document before or alongside Part 6 without too much loss. Also, the Cartesian products considered here are typically finite products, so a reader does not have to be fully comfortable with infinite Cartesian products (Part 4 of this series). The last section does use infinite Cartesian products, but this section is not critical to the main thread of this document. (Tychonoff's Theorem about the compactness of products will be presented in a later document.)

The last section (specifically Subsection 13.1) uses a few standard results about ordinals to construct certain examples and counterexamples, but this is not essential to the main thread of this document.

Most terminology here is standard, but we will use terminology that, while used, is less universal in topology. For example, if X is a topological space with a designated basis $\mathcal{B}$ then we use the term *basic open sets* for elements of $\mathcal{B}$. We will make a distinction between "limit points" of a subset (where each neighborhood of a point has a point in the given set distinct from the given point) and "accumulation points" of a set (where each neighborhood of a point has an infinite number of points in the given set); for Hausdorff spaces, or even T_1-spaces, these two concepts are in fact the same.

2 Introduction: classic theorems from analysis

There are several classic theorem from analysis that require closed and bounded subsets of $\mathbb{R}^n$. For example, they all apply to closed intervals of $\mathbb{R}$, and are often introduced to students initially for such closed intervals.

Classic Theorem (Extreme Value Theorem). *Suppose that K is a closed and bounded subset of $\mathbb{R}^n$ and that $f\colon K \to \mathbb{R}$ is a continuous function. Then f has a maximum and a minimum.*

Classic Theorem (Bolzano-Weierstrass)**.** *Any bounded sequence in* $\mathbb{R}^n$ *has a convergent subsequence. In particular, if* K *is a closed and bounded subset of* $\mathbb{R}^n$ *then any sequence in* K *has a subsequence which converges to a point of* K*.*

The following theorem is useful for proving that continuous functions with nice domains are (Riemann) integrable, and for other applications as well.

Classic Theorem (Uniform Continuity)**.** *Suppose that* K *is a closed and bounded subset of* $\mathbb{R}^n$*. Then every continuous function* $K \to \mathbb{R}$ *is actually uniformly continuous.*

This theorem on uniform continuity is based on another useful theorem:

Classic Theorem (Heine-Borel)**.** *Suppose that* K *is a closed and bounded subset of* $\mathbb{R}^n$*. Then every cover of* K *by open subsets of* $\mathbb{R}^n$ *has a finite subcover.*

Finally nested sequences of closed and bounded sets have nonempty intersections (the reader may enjoy working out counter-examples for nested sets that are either not closed or not bounded).

Classic Theorem (Nested Intersection)**.** *Let* $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ *be a sequence of nonempty closed and bounded sets in* $\mathbb{R}^n$*. Then the intersection* $\cap_{i=1}^{\infty} K_i$ *is nonempty.*

All of these theorem can be extended to more general topological spaces once we identify a suitable generalization of "closed and bounded". Informally we call a set "compact" if it is "bounded" in the same manner as the sets described above: bounded in a local sense (they are closed, and so have their boundary within the set) and in a global sense (distance between points is bounded, there is no sequence that goes "off to infinity"). It turns out that "closed and bounded", while adequate for $\mathbb{R}^n$, is not sufficient for metric spaces in general, and the usual definition of bounded does not even make sense for more general topological spaces. So we would like to have a non-metric formulation of the idea. This challenge motivated topological investigations of the early 20th century, allowing the above classic theorems (which became commonplace in the late 19th century), to be generalized beyond $\mathbb{R}^n$.

In what follows, we develop the general modern notion of compactness which is based on the idea of an open cover (as in the Heine-Borel theorem). Along the way we will encounter fairly simple proofs for the above classic results and their generalizations.

3 Compactness in terms of covers

Historically, the first notion of compactness used in general topology was sequential compactness, which is closely tied the Bolzano-Weierstrass theorem in analysis. This was eventually replaced (around the 1920s) by a notion of compactness that is more linked to the Heine-Borel theorem. This later notion turned out to have better properties than sequential compactness; for example, the arbitrary Cartesian product of compact spaces is compact. Either notion of compactness, though, is suitable for generalizing the above theorems from analysis. Sequential compactness will be covered later in this document.

Definition 1 (Compact)**.** A topological space X is *compact* if every cover of X by open sets possesses a finite subcover. A subset S of X is said to be *compact* if S is compact under the subspace topology.

Recall the definition of cover:

Definition 2 (Cover)**.** A *cover* of a topological space X is a collection $\mathcal{C}$ of subsets of X such that

$$X = \bigcup_{U \in \mathcal{C}} U.$$

A *cover* of a subset S of X is a collection $\mathcal{C}$ of subsets of X such that

$$S \subseteq \bigcup_{U \in \mathcal{C}} U.$$

We call such covers *open covers* if each subset in $\mathcal{C}$ is open in X. We define closed covers and so on in a similar manner. A *subcover* of a cover $\mathcal{C}$ is a subset of $\mathcal{C}$ that is itself a cover.

Proposition 1. *A subset S of a topological space X is compact if and only if every open cover of S has a finite subcover.*

Example 1. Finite subsets of a space are compact. A discrete space X is not compact if it is infinite. The real line is not compact. Open and half open intervals in $\mathbb{R}$ are not compact. Euclidean space $\mathbb{R}^n$ is not compact.

Theorem 2. *Suppose that $f\colon X \to Y$ is continuous. If K is a compact subset of X then $f[K]$ is a compact subset of Y.*

Corollary 3. *If X is a compact topological space then any space homeomorphic to X is also compact.*

Theorem 4. *Every closed subset Z of a compact space X is a compact subset of X.*

Proof. Incorporate the complement of Z to form a cover of the whole space X. □

Proposition 5. *The union $A \cup B$ of two compact subsets A and B of a topological space X is compact.*

Proposition 6. *Let X be a topological space with a designated basis. Then a subset S of X is compact if and only if each cover of S by basic open sets has a finite subcover.*

Proof. One direction is clear, so suppose any cover of S by basic opens sets has a finite subcover. Let $\mathcal{U}$ be a cover of S by general open subsets. Let $\mathcal{C}$ be the collection of all basic open sets B such that $B \subseteq U$ for some $U \in \mathcal{U}$, and apply the supposition to $\mathcal{C}$. □

We get a nested intersection theorem for nested collections of nonempty compact subsets:

Theorem 7. *Let X be a compact space, and let $\mathcal{N}$ be a nested collection of nonempty closed subsets (i.e., a collection totally ordered under inclusion: given two subsets of N one is a subset of the other). Then $\mathcal{N}$ has a non-empty intersection.*

Proof. Assume an empty intersection, and view complements as a cover of X. $\square$

In the above we assumed that X was itself compact, but of course the theorem is valid as long as one of the closed sets in $\mathcal{N}$ is compact. We can actually generalize this. In fact, we can reformulate compactness in terms of non-empty finite intersections:

Proposition 8. *A space X is compact if and only if the following holds: Let $\mathcal{C}$ be a collection of closed subsets with the property that any finite subcollection of $\mathcal{C}$ has a nonempty intersection; then $\mathcal{C}$ as a whole has a nonempty intersection.*

4 Compactness in Hausdorff spaces

Theorem 4 can be strengthened for Hausdorff spaces. First we prove the following key separation lemma.

Lemma 9. *Let X be a Hausdorff space. If $x \in X$ and if Y is a compact subset not containing x, then there are disjoint open subsets U and V of X such that $x \in U$ and $Y \subseteq V$.*

Proof. Use the Hausdorff property to form a suitable cover of Y by open subsets. $\square$

Theorem 10. *If X is a Hausdorff space, then every compact subset is closed.*

Proof. By Lemma 9 the complement is open. $\square$

Corollary 11. *Suppose X is a compact Hausdorff space. Then a subset S of X is compact if and only if S is closed in X.*

Corollary 12. *Suppose X is a Hausdorff space and $\mathcal{C}$ is a nonempty collection of compact subsets of X. Then the intersection $\bigcap_{K\in\mathcal{C}} K$ is compact.*

Corollary 13. *Suppose $f : X \to Y$ is a continuous map where X is compact and where Y is Hausdorff. Then f is a closed map: the image under f of every closed subset of X is a closed subset of Y.*

Corollary 14. *Suppose $f : X \to Y$ is a continuous bijection where X is compact, and Y is Hausdorff. Then f is an open map. So f is a homeomorphism.*

Example 2. Later we will show that the closed interval $[0, 1]$ is compact in $\mathbb{R}$. So if $f\colon [0, 1] \to \mathbb{R}^n$ is a path that is injective then its image is homeomorphic to $[0, 1]$.

Using Lemma 9 and the the idea behind its proof one gets the following separation proposition:

Proposition 15 (Separation of compact sets)**.** *Let X be a Hausdorff space. If Y and if Z are disjoint compact subspaces, then there are disjoint open sets U and V such that $Y \subseteq U$ and $Z \subseteq V$.*

Note that if a space X has such a separation property for compact subsets then the space is Hausdorff X (since singleton sets are compact). Thus the property of separating disjoint compact subsets characterizes Hausdorff spaces.

For Hausdorff spaces, you can think of compact subspaces as being "universally closed". In particular, no matter how a compact space K is embedded into a Hausdorff space, its image will be closed in that space:

Proposition 16. *Suppose K is a compact space. If X is a Hausdorff space with a subspace K' homeomorphic to K, then K' is closed in X.*

This proposition makes it easy to show that some spaces are not compact. For example, any open or half-open interval in $\mathbb{R}$ is not compact since it is not even closed in $\mathbb{R}$. As another example, $\mathbb{R}$ is not compact since it is homeomorphic to the interval $(0, 1)$ which is not closed in $\mathbb{R}$.

5 Compactness in ordered spaces

The most important property of compact sets in ordered spaces is that they have maxima and minima:

Theorem 17. *Every compact subset S of an ordered space X has a maximum and a minimum.*

Proof. Suppose S has no maximum, say. Consider the open cover of S by intervals of the form $(-\infty, b)$ with $b \in S$. □

From this we get an extreme value theorem for compact domains.

Corollary 18. *Let $f : A \to X$ be a continuous function where X is an ordered space. If A is compact, then f has a maximum and a minimum value.*

Recall that linear continua are linearly ordered sets with the density and LUB properties. Equivalently, they can be thought of as connected linearly ordered spaces. What if you only have the LUB property in a ordered space, but not necessarily the density property? (Recall that the LUB property is that any nonempty set with an upper bound has a least upper bound.)

Lemma 19. *Suppose that X is an ordered space with the LUB property. Then every closed interval $[a, b]$ is compact.*

Proof. Let $\mathcal{U}$ be an open cover of $[a, b]$. Let S be the set of $x \in [a, b]$ such that $[a, x]$ is covered by a finite subcover of $\mathcal{U}$. Clearly $a \in S$. Let c be the LUB of S. We can show that $c \in S$. From this it follows that $c = b$. □

For the other direction we have the following:

Lemma 20. *Suppose that X is an ordered space such that every closed interval is compact subspace. Then X has the LUB property.*

Proof. Suppose that S is a non-empty set with element a, with upper bound b, but with no LUB. Consider the open cover $\mathcal{U}$ of the subspace $[a, b]$ by sets of the form (1) $[a, x)$ where $x \in S$ and $a < x$, and (2) $(y, b]$ where y is an upper bound of S. Now consider a finite subcover of $\mathcal{U}$, and note that its union must be of the form $[a, x_0) \cup (y_0, b]$ for some $x_0 \in S$ and some y_0 that is an upper bound that is not a LUB. Since $x_0 \leq y_0$, this union excludes x_0. But the union must be all of $[a, b]$, a contradiction. □

From these two lemmas we get the following:

Theorem 21. *Suppose that X is an ordered space. Then X has the LUB property if and only if every closed interval is compact.*

We end the section with the Heine-Borel theorem for ordered spaces:

Theorem 22. *Suppose that X is an ordered space with the LUB property. Then a nonempty subset $S \subseteq X$ is compact if and only if it is closed and bounded (from above and below).*

Proof. Suppose S is compact. Then it has a maximum and minimum, so is bounded from above and below. Also, S is closed since it is compact and X is Hausdorff.

Suppose T is closed and bounded in X. Let $a \in X$ be a lower bound and $b \in X$ be an upper bound, so that $T \subseteq [a, b]$. By Lemma 19, we have $[a, b]$ is compact. Also T is closed in $[a, b]$, so is compact by Theorem 4. □

6 Compactness in products

The Cartesian product of two compact space is compact; the proof given below is fairly straightforward. This in turn implies that the finite Cartesian product of compact spaces is compact. Actually, the arbitrary Cartesian product of compact spaces is compact, but the proof is more difficult (and relies on the axiom of choice). It is called *Tychonoff's Theorem*, and will be discussed in another document.

Theorem 23. *The product $X \times Y$ of two compact spaces X and Y is compact.*

Proof. The proof is based on Proposition 6, so let $\mathcal{U}$ be a cover of $X \times Y$ by basic open sets (for the standard basis of $X \times Y$). Let $\mathcal{V}$ be the collection consisting of all open $V \subseteq X$ such that $V \times Y$ has a finite cover by sets in $\mathcal{U}$.

Let $x \in X$ and observe that $\{x\} \times Y$ is compact since it is homeomorphic to Y. Since $\mathcal{U}$ covers $\{x\} \times Y$, there is a finite subset of $\mathcal{U}$ that covers $\{x\} \times Y$. This finite subcover will actually cover $V \times Y$ for some open neighborhood V of $x \in X$. In other words, $V \in \mathcal{V}$.

We conclude that $\mathcal{V}$ is a cover of X. Now take a finite subcover $\mathcal{V}_0$ of $\mathcal{V}$. We can then produce a finite subcover of $\mathcal{U}$ that covers $X \times Y$. □

A similar proof yields a proof of the following "tube lemma":

Lemma 24 (Tube lemma)**.** *Assume Y is a compact space. Let W be an open subset of $X \times Y$ containing $\{x\} \times Y$ where x is a point of a space X. Then there is an open subset U of X such that $U \times Y$ (called a "tube") is contained in W.*

Recall that a closed map is one for which the image of any closed subset in the domain is closed in the codomain.

Corollary 25. *If Y is compact then the projection $X \times Y \to X$ is a closed map.*

Proof. Let Z be a closed subset of $X \times Y$, and suppose $x \in X$ is not in the image of Z under projection. Use the tube lemma applied to the complement W of Z. $\square$

Now we consider the issue of boundedness. In a metric space, we defined boundedness of a subset S in terms of the existence of an upper bound on distance between points. For $\mathbb{R}^n$ this notion of boundedness is independent of whether you choose the Euclidean metric, the product metric, or the taxi-cab metric. In $\mathbb{R}^n$ with the product metric, it is clear that this notion of boundedness is equivalent to the following notion:

Definition 3. Let $X_1 \times \cdots \times X_n$ be the finite Cartesian product ordered spaces. A subset of $X_1 \times \cdots \times X_n$ is *bounded* if the image under the projections to each X_i is bounded from below and from above.

Lemma 26. *Suppose $X_1, \ldots, X_n$ are linearly ordered sets. Then every compact subset of $X_1 \times \cdots \times X_n$ is closed and bounded.*

Proof. Ordered spaces are Hausdorff, and the product of Hausdorff spaces is Hausdorff. Now use Theorem 10 and Corollary 18. $\square$

Lemma 27. *Suppose A is a closed subspace of $X \times Y$, and suppose the images of A in X and Y are both contained in compact subspaces. Then A is compact.*

Proof. Let B be a compact subspace of X containing the image of A in X, and let C be a compact subspace of Y containing the image of A in Y. Then the subspace $B \times C$ is compact since B and C are compact. (Recall that the subspace topology on $B \times C$ agrees with the product topology of the subspace topologies on B and C individually.) Observe that $A \subseteq B \times C$. Since A is closed in $X \times Y$ it must be closed in the subspace $B \times C$. Now use Theorem 4. $\square$

Theorem 28. *Suppose $X_1, \ldots, X_n$ are linearly ordered sets with the LUB property. Then a subspace of $X_1 \times \cdots \times X_n$ is compact if and only if it is closed and bounded.*

Proof. Lemma 26 gives one direction. So we can assume A is a closed and bounded subspace of $X_1 \times \cdots \times X_n$. Since A is bounded, its image in X_i is contained in a closed interval $[a, b]$. By Lemma 19 the interval $[a, b]$ is a compact subspace of X_i. Now use Lemma 27. $\square$

Corollary 29 (Heine-Borel)**.** *A subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

7 Uncountability of perfect compact spaces

In this section we give an interesting proof of uncountability that is purely topological.

Definition 4. A subset A of a topological space X is *perfect* if A is equal to the set of limit points of A.

Remark. If X is Hausdorff, then A is perfect if and only if it is closed and every open subset of X intersecting A contains an infinite number of points of A. In particular, nonempty perfect subsets of Hausdorff spaces are infinite.

Example 3. The space $\mathbb{R}^n$ is perfect (in itself). Every closed interval of $\mathbb{R}$ with more than one point is perfect and compact. The subset $\mathbb{Q}$ is perfect in $\mathbb{Q}$ but is not compact.

Theorem 30. *Let X be a perfect compact Hausdorff space. Then X is uncountable.*

Proof. Suppose (x_i) is a (countable) sequence of points in X. Use the Hausdorff property to form a nested decreasing sequence (U_i) of nonempty open subsets of X such that x_i is not in in the closure K_i of U_i for each i. Consider the intersection of the nested compact subsets K_i. This intersection is nonempty, so contains a point x not in the sequence (x_i). (This construction uses the axiom of countable choice). $\square$

In particular, closed intervals in $\mathbb{R}$ with more than one point are uncountable. This ensures the uncountability of any set containing such a closed interval. In particular, we have the following:

Corollary 31. *Every interval of $\mathbb{R}$ with more than one point is uncountable. Every nonempty open subset of $\mathbb{R}$ is uncountable.*

8 The Cantor set

The Cantor set is one of the most important sets in analysis. It is an uncountable compact subset of $\mathbb{R}$ with zero length (i.e., it has zero Lebesgue measure), and is a simple example of a fractal. In this section we discuss the construction, and dervive key properties of, the Cantor set. (We do not require results on measure theory here).

Let K_0 be the interval $[0, 1]$. From this we recursively form a subset $K_n \subseteq [0, 1]$ where each K_n is the union of 2^n disjoint closed intervals of length $1/3^n$. The construction is as follows. Let $[a, b]$ be one of the intervals defining K_n. Let $r = b-a$. Then K_{n+1} consists of the union of the 2^{n+1} intervals of the form $[a, a + r/3]$ or $[b - r/3, b]$.

By induction, K_n consists of 2^n disjoint closed of length $1/3^n$. So the total length of the intervals for K_n is $(2/3)^n$. The *Cantor set* C is defined to be the intersection of these compact sets K_n. We use the subspace topology for C (which is not the same as the order topology).

Each K_n is the finite union of compact subset of $\mathbb{R}$, so K_n is compact. So we get a nested sequence of compact sets. By the general theory of compact sets, the intersection C is a nonempty compact subset of $[0, 1]$. But also observe that any point that occurs as an endpoint of an interval defining K_n will also be an endpoint of an interval defining K_{n+1} so will be in the intersection. Thus C contains a countable number of such endpoints. So we have the following:

Proposition 32. *The Cantor set C is an infinite compact subset of $[0, 1]$.*

Proposition 33. *The Cantor set C is totally disconnection. In particular, C does not contain any intervals of $\mathbb{R}$ except for singleton sets, and has an empty interior.*

Proof. Let $x \in C$. Observe that the connected component of x in C must be a subset of the connected component of x in K_n. This means that the diameter of the connected component of x in C is at most $1/3^n$. This holds for all n, which forces the connected component to be a singleton. □

Proposition 34. *The Cantor set C is perfect: C equals the set of limit points of C.*

Proof. Since C is compact, it is closed. Thus the set of limit points of C is contained in C.

Now suppose $x \in C$. Let U be an open neighborhood of x. Note that for sufficiently large n, one of the defining intervals I of K_n will be contained in U, and both of the endpoints of I are in C. □

Proposition 35. *The Cantor set C is uncountable.*

Proof. Since C is compact, perfect, and Hausdorff. □

Observe that C contains points that do not occur as endpoints of the defining intervals of any K_n. In fact, most points of C are not such endpoints. However, such endpoints forms a dense subset of C.

9 Compactness in metric spaces

In this section we consider several applications of compactness to metric spaces.

9.1 Uncountability of a metric space

Proposition 36. *A connected metric space with more than one point is uncountable. More generally, a metric space with a component with more than one point is uncountable.*

Proof. Let X be a connected metric space with more than one point. Fix $x_0 \in X$ and consider the continuous map

$$x \mapsto d(x, x_0).$$

The image is a connected subspace of $\mathbb{R}$, so must be an interval. Also, the image contains 0 and at least one positive value, so we can appeal to Corollary 31. □

9.2 Diameter of a metric space

Throughout this section, X will refer to a metric space.

Recall that the *diameter* of a nonempty subset of a metric space X is the supremum of the following set of distances: $\{d(a_1, a_2) \mid a_1, a_2 \in S\}$.

Proposition 37. *Let S be a nonempty compact subset of a metric space X. Then there are points $x_1, x_2 \in S$ such that the diameter is $d(x_1, x_2)$. In other words, the diameter is the maximum, not just the supremum, of the distances between points of S.*

Proof. The distance function $d : X \times X \to \mathbb{R}$ is continuous and $S \times S$ is compact, so image $d[S \times S]$ is compact. Thus $d[S \times S]$ has a maximum. $\square$

9.3 Distance between a point and a subset

Let $x \in X$ and $S \subseteq X$ be a nonempty subset. Recall that the distance from x to S, written $d(x, S)$, is defined to be the infimum of $d(x, s)$ as s varies in S. If S is compact we have the following:

Proposition 38. *Let S be a compact nonempty subset of a metric space X, and let $x \in X$. Then there is a point $s \in S$ such that $d(x, S) = d(x, s)$. In other words, $d(x, S)$ is not just the infimum of $d(x, s)$ as s varies in S, but is the minimum.*

Proof. Use the fact that $s \mapsto d(x, s)$ is continuous for any fixed $x \in X$. $\square$

Similarly, we see that there is a farthest away point in S from x. We can use the Heine-Borel to extend the above proposition to closed subsets S of $\mathbb{R}^n$, but of course there might not be a farthest away point in S in this case.

Proposition 39. *Let S be a closed nonempty subset of $\mathbb{R}^n$, and let $x \in X$. Then there is a point $s \in S$ such that $d(x, S) = d(x, s)$. In other words, $d(x, S)$ is not just the infimum of $d(x, s)$ as s varies in S, but is the minimum.*

Proof. Fix a point $s_0 \in S$, and let $B = \overline{B}_r(x)$ be the closed ball of radius r equal to $d(x, s_0)$. Let $K = B \cap S$. By the Heine-Borel Theorem, K is compact. Observe that $d(x, S) = d(x, K)$, and then use the previous proposition. $\square$

9.4 Distance between subsets

Let A and B be nonempty subsets of X. Recall that the distance from A to B, written $d(A, B)$, is defined to be the infimum of $d(a, b)$ as (a, b) varies in $A \times B$. If A and B are compact we have the following:

Proposition 40. *Let A and B be nonempty compact subsets of a metric space X. Then there is a point $a \in A$ and a point $b \in B$ such that $d(a, b) = d(A, B)$.*

Proof. Use the continuity of $d\colon X \times X \to \mathbb{R}$ and the image of $A \times B$. $\square$

In particular, if A and B are disjoint, then the distance between A and B is strictly positive.

We cannot extend the above result to closed subsets: consider $X = \mathbb{R} - \{0\}$, where A is the set of positive real numbers and B is the set of negative real numbers. Or consider the set defined by $xy = 1$ and $y = 0$ in $\mathbb{R}^2$. However, we do have the following:

Proposition 41. *Suppose A and B are disjoint nonempty subsets of a metric space X where A is closed and where B is compact. Then $d(A, B) > 0$.*

Proof. Consider the continuous function $f\colon X \to \mathbb{R}$ defined by $x \mapsto d(x, A)$. The image $f[B]$ is compact, so f has a minimum. In other words, there is a point $b \in B$ such that $d(b, A) \leq d(x, A)$ for all $x \in B$. The distance $d(b, A)$ is positive since b is not in $\overline{A} = A$. Note that $d(b, A) \leq d(A, y) \leq d(x, y)$ for all $x \in A$ and $y \in B$. Thus $d(A, B) \geq d(b, A) > 0$. □

9.5 Contraction maps

Definition 5. Let X be a metric space. Then a *contraction* is a map $f : X \to X$ such that there is a constant $C < 1$ where $d\big(f(x), f(y)\big) \leq Cd(x, y)$ for all $x, y \in X$.

We know that non-expansive maps are uniformly continuous, so contraction maps are of course continuous. Observe that if a metric space X has finite diameter D and if f is a contraction with factor C, then the image $f^n[X]$ of the nth iteration of f is a subset of diameter less than or equal to $C^n D$. So if the intersection of the spaces $f^n[X]$ is nonempty, it must consist of a single point. Recall that a fixed point for a function $f\colon X \to X$ is a point $x \in X$ where $f(x) = x$.

Proposition 42. *If X is a compact metric space then any contraction f has a unique fixed point x_0.*

Proof. Let K_n be $f^n[X]$, the image of X under the n-th iteration of f. As discussed above the intersection $\bigcap_{n=1}^{\infty} K_n$ has at most one point. Observe that each K_n is compact, and the sequence (K_i) is nested. So the intersection is nonempty. Let x_0 be the unique point of the intersection $\bigcap_{n=1}^{\infty} K_n$. Observe that $f(x_0) = x_0$. Observe also that any fixed point of f is in $\bigcap_{n=1}^{\infty} K_n$. □

9.6 Isometries

Definition 6. Let X, Y be metric spaces. Then an *isometry* is a map $f : X \to Y$ such that $d\big(f(x), f(y)\big) = d(x, y)$ for all $x, y \in X$.

Proposition 43. *If X is a compact metric space, then any isometry $f : X \to X$ is a homeomorphism.*

Proof. Injectivity is clear. We establish surjectivity by assuming x is not in the image of f, and derive a contradiction. Since $f[X]$ is compact, hence closed, the distance of x from $f[X]$ is some positive $\varepsilon > 0$. If a point y has distance r to a subset $S \subseteq X$, then $f(y)$ has distance r to the image $f[S]$ since f is an isometry.

Let f^n be the nth iteration of f, and observe that the distance from $x_n \stackrel{\text{def}}{=} f^n(x)$ to $f^{n+1}[X]$ is ε. If $i < j$, then $d(x_i, x_j) \geq \varepsilon$. Thus the set of x_i forms a closed, hence compact, subspace. But this space is infinite and discrete, which cannot be compact.

So we have established that f is a bijection. A bijective isometry has an isometry as an inverse, and so is a homeomorphism. □

Remark. Note that there are isometries of the interval $[0, \infty)$ to itself that are not bjective.

10 Uniform continuity

We start with the concept of a Lebesgue number, which a very useful tool in the topology of metric spaces and analysis.

Definition 7. Let S be a nonempty subset of a metric space X and let $\mathcal{U}$ be a cover of S. A *Lebesgue number* for $\mathcal{U}$ and S is a real number $\varepsilon > 0$ with the following property: if A is a bounded subset of X intersecting S and if the diameter of A is less than ε then $A \subseteq U$ for some $U \in \mathcal{U}$.

A key theorem of metric spaces is that there are Lebesgue numbers for open covers of compact subsets:

Theorem 44. *Let K be a nonempty compact subset of a metric space X. Then every open cover $\mathcal{U}$ of K has a Lebesgue number for $\mathcal{U}$ and K.*

Proof. Define a *half-size ball* for $\mathcal{U}$ to be an open ball $B_r(x)$ with the property that the ball of twice the radius $B_{2r}(x)$ is contained in some $U \in \mathcal{U}$. Let $\mathcal{V}$ be the collection of half-size balls for $\mathcal{U}$ with center in K. Since $\mathcal{U}$ covers K, it follows that $\mathcal{V}$ covers K as well. Let $\mathcal{V}_0$ be a finite subcover of $\mathcal{V}$ that covers K. For each $V_i \in \mathcal{V}_0$, write V_i as $B_{r_i}(x_i)$ with $x_i \in K$ and $r_i > 0$. Let $\varepsilon > 0$ be the minimum of these radii r_i.

Let A be a bounded subset of X intersecting K in a point x, and assume A has diameter less than ε. Let $B_{r_i}(x_i)$ be a ball in $\mathcal{V}_0$ that contains x. Claim: A is a subset of $B_{2r_i}(x_i)$. To see this, let $y \in A$ and observe that

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \varepsilon + r_i \leq r_i + r_i = 2r_i.$$

Observe that $A \subseteq B_{2r_i}(x_i) \subseteq U$ for some $U \in \mathcal{U}$ since $B_{2r_i}(x_i) \in \mathcal{V}_0 \subseteq \mathcal{V}$. □

A corollary to this is the uniform continuity theorem.

Theorem 45 (Uniform Continuity)**.** *Let $f : K \to Y$ be a continuous function between metric spaces where K is compact. Then f is uniformly continuous. In other words, given $\varepsilon > 0$ there is an $\delta > 0$ such that*

$$d(a, b) < \delta \implies d\big(f(a), f(b)\big) < \varepsilon$$

for all $a, b \in K$.

Proof. Consider the open cover $\mathcal{U}$ of K consisting of open sets of the form $f^{-1}[B_{\varepsilon/2}(y)]$ where $y \in Y$. Let $\delta > 0$ be a Lebesgue number for $\mathcal{U}$ and K.

Suppose that $a, b \in K$ are such that $d_K(a, b) < \delta$. The diameter of $\{a, b\}$ is $d_K(a, b)$ which is strictly less than δ. Thus

$$\{a, b\} \subseteq f^{-1}[B_{\varepsilon/2}(y)]$$

for some $y \in Y$. In other words, $f(a), f(b) \in B_{\varepsilon/2}(y)$. By the triangle inequality, we get the desired bound: $d_Y\big(f(a), f(b)\big) < \varepsilon$. $\square$

11 The Bolzano-Weierstrass theorem

The classical version of the Bolzano-Weierstrass theorem says that every bounded sequence in $\mathbb{R}^n$ has a converging subsequence. A variant of this theorem says that if K is a closed and bounded subset of $\mathbb{R}^n$ then every sequence in K has a subsequence that converges to a point in K. This proved to be a very useful theorem in analysis, and one of the original motivations to study compactness (culminating in the work of Fréchet in 1906) was to identify more general spaces where this result holds. Here we show that this property holds, almost, for compact spaces in general.

Suppose $(x_i)_{i\in I}$ is a sequence where the index set I is ordered such that I is order isomorphic to $\mathbb{N}$. We will take it as given that every infinite subset J of I is order isomorphic to $\mathbb{N}$ under the order of J inherited from I. The restriction of $(x_i)_{i\in I}$ to such an infinite subset J gives what we call a *subsequence* $(x_i)_{i\in J}$ of the given sequence $(x_i)_{i\in I}$.

When dealing with subsequences $(x_i)_{i\in J}$ of a sequence $(x_i)_{i\in I}$ it is common to replace the index set J with a standard indexing set such as $\mathbb{N} = \{0, 1, \ldots\}$ or the set of positive integers $\mathbb{N}_1 = \{1, 2, \ldots\}$ using the order isomorphism with J, but such replacement of index set results in a sequence that is essentially equivalent to the subsequence $(x_i)_{i\in J}$ from the point of view of convergence. For example, if $a_i = i + 1$ are terms of a sequence indexed by $I = \mathbb{N}$ then (a_i) gives a sequence of positive integers. If we take J to be the set of even natural numbers, then the corresponding subsequence would be a sequence of of odd positive integers. We can write this subsequence as $(i + 1)_{i\in J}$, but it is common to compose with the order isomorphism $\mathbb{N} \to J$ (here $n \mapsto 2n$) and write this sequence as $(2i + 1)_{i\in\mathbb{N}}$.

Suppose that S is a subset of a topological space X. Recall that a *limit point* of S is a point $x \in X$ such that every neighborhood of x contains a point of S not equal to x. Similarly a *contact point* of S is a point $x \in X$ such that every neighborhood of x contains a points of S including x itself if $x \in S$; in other words x is in the closure of S. It will be convenient here to define a third type of nearby point: call a point $x \in X$ an *accumulation point of* S if every neighborhood of x contains an infinite number of points of S. Thus every accumulation point of S is a limit point of S, and every limit point is a contact point of S. As we have seen, if X is a Hausdorff space, then limit points and accumulation points of a subset S are the same thing. Recall that we have also defined accumulation points of sequences. These two notions are closely related but differ in an essential point: a sequence (x_i) can repeat terms infinitely often resulting in an accumulation point that might not be an accumulation of the image $S = \{x_i\}$ of the sequence. However, we do have the following two lemma:

Lemma 46. *Let (a_i) be a sequence of points in a topological space X. If $x \in X$ is an accumulation point of the set $S = \{a_i\}$ of values of the sequence, then x is an accumulation point of (a_i).*

Lemma 47. *Let X be a topological space. Then the following are equivalent:*

1. *Every infinite subset of X has an accumulation point.*
2. *Every sequence in X has an accumulation point.*

Proof. First assume (1). Let (a_i) be a sequence in X. If the image $\{a_i\} \subseteq X$ is finite, then some value occurs for an infinite set of indices, and the existence of an accumulation point of (a_i) follows. If $\{a_i\} \subseteq X$ is infinite, then let x be an accumulation point of the set $\{a_i\}$ and use the previous lemma.

Now assume (2). Let S be an infinite subset of X. Use the axiom of countable choice to produce an injective map $f\colon \mathbb{N} \to S$. Note that f is a sequence in X so has an accumulation point x. Observe that x is an accumulation point of the image of f, and so is an accumulation point of S. □

Accumulation points of sequences and limits of subsequences are closely related as we will see in the next two lemmas:

Lemma 48. *Let (a_i) be a sequence of points in a topological space X. If (a_i) has a converging subsequence with limit x, then x is an accumulation point of the sequence (a_i).*

Lemma 49. *Let $(a_i)_{i\in I}$ be a sequence of points in a topological space X. If x is an accumulation point of the sequence (a_i) and if x has a countable neighborhood basis then (a_i) has a subsequence converging to x.*

Proof. Let $\{B_1, B_2, \ldots\}$ be the set of basic open sets in the neighborhood basis. For $k \geq 1$, define i_k recursively: let i_k be the smallest element of I such that (i) $a_{i_k} \in B_1 \cap B_2 \cap \cdots \cap B_k$, and (ii) such that $i_k > i_{k-1}$ if $k > 1$. Let J be the collection of these i_k. Then $(a_i)_{i\in J}$ is the desired subsequence. □

This gives us the following:

Proposition 50. *Let X be a topological space such that every element of X has a countable neighborhood basis. Then the following are equivalent:*

1. *Every infinite subset of X has an accumulation point.*
2. *Every sequence in X has an accumulation point.*
3. *Every sequence in X has a convergent subsequence.*

We can now give two version of the Bolzano-Weierstrass theorem:

Theorem 51 (Bolzano-Weierstrass version 1)**.** *Let X be a compact topological space. Then every infinite subset of X has an accumulation point, and every sequence in X has an accumulation point.*

Proof. Let S be an infinite subset of X. If x is not an accumulation point then there is an open neighborhood U of x that intersects S in a finite set. Suppose S has no accumulation points. Then we can cover X by open subsets whose intersections with S are all finite. Use a finite subcover to derive a contradiction. The second claim now follows from the first. $\square$

Corollary 52 (Bolzano-Weierstrass version 2). *Let X be a compact topological space such that every point has a countable neighborhood basis. Then every sequence in X has a convergent subsequence.*

Remark. Spaces such that every point has a countable neighborhood basis are sometimes called “first countable spaces”. They include metric spaces, so we have a Bolzano-Weierstrass theorem for metric spaces. We further explore this idea in the context of metric spaces in the next section along with “second countable spaces”. A second countable space is a topological space that has a countable basis, and so is automatically a first countable space.

12 Sequential compactness

The notion of compactness we use today is not the original notion of compactness. Fréchet in 1906 formulated an idea of compactness in terms of the existence of converging subsequences of every sequence. In other words, compact spaces were essentially the spaces possessing a Bolzano-Weierstrass theorem. Today we use the term *sequential compactness* for this earlier version of compactness. The current open cover notion of compactness has its origins in Heine's proof that each continuous function on a closed interval is uniformly continuous, which led to the Heine-Borel theorem in $\mathbb{R}^n$. Trying generalize the ideas of the Heine-Borel theorem led to the open cover notion of compactness, which was used by Alexandroff and Urysohn in the 1920s under the name “bicompactness”. In the 1930s Tychonoff proved that the arbitrary product of compact spaces is compact under the open cover definition of compactness, but this does not hold using the sequential definition of compactness. This was one of the major reasons why the open cover notion of compactness won out over the sequential notion of compactness. (In the next section we will see that the countable product of sequential compact spaces is sequentially compact.)

Definition 8. Let X be a topological space. If every sequence of X has a convergent subsequence then we say that X is *sequentially compact*. A subset S of X is *sequentially compact* if it is sequentially compact in the subspace topology.

Many of the important theorems about compact spaces were originally proved in terms of sequentially compact spaces. For example, the Bolzano-Weierstrass theorem for sequentially compact spaces holds just by definition. We now consider the sequential compact versions of some of the other basic theorems.

Proposition 53. *If $f\colon X \to Y$ is a surjective continuous function, and if X is sequentially compact space, then Y is sequentially compact.*

Proof. Recall that if a is the limit of sequence (a_i) then $f(a)$ is the limit of the image sequence $f(a_i)$. From this fact, and the axiom of countable choice, we obtain sequential compactness for Y. $\square$

Corollary 54. *If X is sequentially compact, then any space homeomorphic to X is sequentially compact.*

The next two propositions have fairly straightforward proofs.

Proposition 55. *A closed subset of a sequentially compact space is a sequentially compact subset.*

Proposition 56. *The product of two sequentially compact spaces is sequentially compact.*

As remarked above, the above generalizes to arbitrary products for compact spaces, but not for arbitrary products of sequentially compact spaces.

Some results for sequential compactness require a extra countability assumption when compared to the analogous result for compactness. The following is an example where we require the nesting be countable:

Proposition 57. *Let X be a compact space and let $(N_1, N_2, \ldots)$ be a decreasing nested sequences of nonempty closed subsets of X. Then $\bigcap_{i=1}^{\infty} N_i$ is nonempty.*

Proof. Use the axiom of countable choice to form a suitable sequence with a convergent subsequence. $\square$

Remark. The above result is enough to prove uncountability of any perfect sequentially compact Hausdorff space. The prove is similar to the proof for perfect compact Hausdorff spaces.

For Hausdorff spaces we have the following, at least if there is a countable neighborhood basis for each point:

Proposition 58. *If Y is a sequentially compact subset of a Hausdorff space X, and if every point of X has a countable neighborhood basis, then Y is closed in X.*

Proof. Start with a limit point x of Y and form a sequence of points of Y converging to x (using the axiom of countable choice). Now take a subsequences of this sequence converging to a point in Y. $\square$

In many standard spaces, sequential compactness is equivalent to compactness:

Theorem 59. *Let X be a topological space with a countable basis. Then X is sequentially compact if and only if X is compact.*

Proof. If X is compact, it is sequentially compact by the Bolzano-Weierstrass theorem (Corollary 52).

Suppose X is sequentially compact. Fix a countable basis of X. Suppose $\mathcal{U}$ is a cover of X by basic open sets. Since $\mathcal{U}$ is countable, we can write its members in a sequence $(B_1, B_2, \ldots)$. We claim that $B_1, \ldots, B_k$ covers X for sufficiently large k. Suppose not, and for each k let x_k be a point of X not in $B_1 \cup \cdots \cup B_k$. By Proposition 50 there is an accumulation point $x \in X$ of (x_k). In other words, every neighborhood of x contains infinitely many x_k. Since $x \in B_l$ for some l, we have that B_l contains infinitely many x_k. However, if $k \geq l$ then x_k is not in B_l, a contradiction.

Since every cover of X by basic open sets has a finite subcover, we conclude that X is compact (Proposition 6). □

We can adapt this theorem to metric spaces, even though not all metric spaces have countable basis. The key is that compact metric spaces do in fact have countable basis.

Lemma 60. *Let X be a metric space that is either compact or sequentially compact. For any $\varepsilon > 0$, there is a finite cover of X by open balls of radius ε.*

Proof. The proof for compact is straightforward: take a finite subcover of a cover of X by open balls of radius ε. So assume X is sequentially compact. We can also assume that X is nonempty, since the empty set is covered by the empty cover. We recursively form a sequence (using the axiom of countable choice) as follows: let x_1 be any point of X. Assuming $x_1, \ldots, x_k$ have been choosen, let x_{k+1} to be outside of $B_\varepsilon(x_i)$ for $i \leq k$. If no such point exists we have our finite cover, so assume this recursion gives an infinite sequence.

By Proposition 50 there is an accumulation point $x \in X$ of the sequence (x_k). In other words, every neighborhood of x contains infinitely many x_k. Take an open ball of radius $\varepsilon/2$ around x and derive a contradiction. □

Remark. In the above lemma and following proposition we require "compact or sequentially compact". This can simply be replaced by "compact" after proving the equivalence of compact and sequentially compact.

Proposition 61. *Let X be a metric space that is either compact or sequentially compact. Then X has a countable basis.*

Proof. By the above lemma we can form a finite cover $\mathcal{B}_n$ of X by open balls of radius $r = 1/n$. Let $\mathcal{B}$ be the union $\bigcup_{n=1}^{\infty} \mathcal{B}_n$ (we need the axiom of countable choice to form the sequence $\mathcal{B}_n$). Observe that $\mathcal{B}$ is a countable basis for X. □

Theorem 62. *Let X be a metric space. Then X is sequentially compact if and only if X is compact.*

Proof. Suppose X is sequentially compact. Then X has a countable basis by the above proposition. Thus X is compact by Theorem 59

Similarly if X is compact, then X has a countable basis by the above proposition. Thus X is sequentially compact by Theorem 59 □

Remark. From this theorem we know we can extend any result about compact metric spaces to sequentially compact metric spaces simply because the two notions are identical for metric spaces. For example, the results of Section 9 extend to sequentially compactness. Similarly, a Heine-Borel theorem holds for sequentially compact metric spaces since such a result holds, by definition, for any compact space. Since $\mathbb{R}$ is a metric space, we have the following extreme value theorem (proved similarly to Corollary 18):

Proposition 63. *Let $f\colon A \to \mathbb{R}$ be a continuous function. If A is sequentially compact then f has a maximum and a minimum value.*

What if $\mathbb{R}$ is replaced by another ordered space? This is more problematic since a sequentially compact subset of an ordered set does not necessarily have a maximum or minimum. See Section 13.1 for an example of a sequentially compact ordered space that has no maximum.

13 Further Examples and Counterexamples (Optional)

13.1 A sequential compact that is not compact

We start by giving an example of a sequentially compact space that is not compact. This example assumes some knowledge of the class of ordinals. Recall that the class of ordinals is well-ordered. In particular, there is a first ordinal, written 0. There is also a smallest ordinal ω_1 such that the interval $[0, \omega_1)$ is uncountable. We call ω_1 the *first uncountable ordinal*. We also consider the closed interval $[0, \omega_1]$, and ω_1 is also the smallest ordinal such that $[0, \omega_1]$ is uncountable since it is obtained by adding one element to $[0, \omega_1)$.

Lemma 64. *The ordered space $[0, \omega_1)$ has no maximum.*

Proof. Suppose γ is a maximum. By the definition of ω_1 the interval $[0, \gamma)$ is countable since $\gamma < \omega_1$. So $[0, \omega_1) = [0, \gamma] = [0, \gamma) \cup \{\gamma\}$ is countable, a contradiction. □

Lemma 65. *The ordered space $[0, \omega_1)$ is first-countable: every point has a countable neighborhood basis.*

Proof. Let $x \in [0, \omega_1)$ and let x' be the first ordinal larger than x. There are only a countable number of $y < x$ in $[0, \omega)$. Observe that if $x > 0$ then the collection of open intervals $\{(y, x') \mid y < x\}$ is a countable neighborhood basis of x. Note also that 0 is an isolated point, so $\{0\}$ is open and $\{\{0\}\}$ is a neighborhood basis of 0. □

Lemma 66. *Every sequence in the ordered space $[0, \omega_1)$ has an accumulation point.*

Proof. Let $(x_i)_{i \in I}$ be a sequence in $[0, \omega_1)$. Note that by definition of ω_1, we have the interval $[0, x_i)$ is countable for each $i \in I$. Thus the union

$$A = \bigcup_{i \in I} [0, x_i)$$

is countable (we may need the axiom of countable choice here). Since $[0, \omega_1)$ is uncountable, there must be an element in the complement of A, and such an element is an upper bound of $\{x_i\}$. Let x be the least upper bound of $\{x_i\}$. Observe that x is a limit point of $\{x_i\}$. Since $[0, \omega_1)$ is an ordered space, it is Hausdorff. Thus x is an accumulation point of $\{x_i\}$, and so an accumulation point of (x_i). □

Proposition 67. *The ordered space $[0, \omega_1)$ is sequentially compact but not compact.*

Proof. Sequential compactness follows from the above lemmas and Proposition 50. Failure of compactness follows from Lemma 64 and Theorem 17. □

Now we show that the hypothesis "every point of X has a countable neighborhood basis" cannot be dropped from Proposition 58.

Proposition 68. *The subspace* $[0, \omega_1)$ *is a sequentially compact subspace of the ordered space* $[0, \omega_1]$ *that is not closed in* $[0, \omega_1]$.

Proof. Note that the subspace topology and the order topology of $[0, \omega_1)$ agree since $[0, \omega_1)$ is a convex subset of $[0, \omega_1]$. So $[0, \omega_1)$ is a sequentially compact subset of $[0, \omega_1]$. Also, observe that ω_1 is a limit point of $[0, \omega_1)$ by Lemma 64. □

Finally we note the following:

Proposition 69. *The ordered space* $[0, \omega_1]$ *is compact.*

Proof. This follows from Theorem 21. □

13.2 Countable products of sequentially compact spaces

Suppose $(X_1, X_2, \ldots)$ is a sequence of sequentially compact spaces. Here the index set is $I = \mathbb{N}_1 = \{1, 2, \ldots\}$. In this section we will see a proof that the Cartesian product $\prod_{i=1}^{\infty} X_i$ is also sequentially compact (using the axiom of choice).

Proposition 70. *Suppose* $(X_1, X_2, \ldots)$ *is a sequence of sequentially compact spaces. Then the Cartesian product*

$$\prod_{i=1}^{\infty} X_i$$

is also sequentially compact (using the product topology).

Proof. Let $(\alpha_j)_{j \in \mathbb{N}}$ be a sequence of points of the product $\prod X_i$. Our goal is to establish that $(\alpha_j)_{j \in \mathbb{N}}$ has a converging subsequence. Note that each α_j is itself a sequence $(a_{j,i})_{i \in I}$, where here $a_{j,i} \in X_i$.

We now construct a sequence of infinite sets $J_0 \supseteq J_1 \supseteq J_2 \supseteq \cdots$. Set $J_0 = \mathbb{N}$. Define J_{n+1} recursively in terms of J_n as follows. Consider the sequence $(a_{j,n+1})_{j \in J_n}$ in X_{n+1}. Since X_{n+1} is sequentially compact, there is an infinite subset $J_{n+1} \subseteq J_n$ such that $(a_{j,n+1})_{j \in J_{n+1}}$ converges in X_{n+1}. This gives a nested decreasing sequence $(J_i)_{i \in \mathbb{N}}$ of infinite subsets of $\mathbb{N}$ (using the axiom of choice).

Let n_1 be the smallest element of J_1, and for each $k \geq 1$ let n_{k+1} be defined recursively as the smallest element of J_{k+1} such that $n_{k+1} > n_k$. Let J_∞ be the set $\{n_1, n_2, n_3, \ldots\}$.

We would like to show that $(\alpha_j)_{j \in J_\infty}$ is a convergent subsequence of $(\alpha_j)_{j \in \mathbb{N}}$ in the product $\prod X_i$. To do so first fix $i_0 \in I$ and consider the projection map $\pi_i \colon \prod X_i \to X_{i_0}$ and the associated projected sequence $(\pi_{i_0}(\alpha_j))_{j \in J_\infty}$. This sequence is just $(a_{j,i_0})_{j \in J_\infty}$. Note that $J'_{i_0} \stackrel{\text{def}}{=} J_\infty - \{n_1, \ldots, n_{i_0 - 1}\}$ is a subset of J_{i_0}. Since $(a_{j,i_0})_{j \in J_{i_0}}$ converges in X_{i_0} by definition of J_{i_0}, it follows that the subsequence $(a_{j,i_0})_{j \in J'_{i_0}}$ converges in X_{i_0} as well. Since J_∞ is just J'_{i_0} with a finite set of integers added, $(a_{j,i_0})_{j \in J_{i_\infty}}$ must also converges in X_{i_0}. In other words, the sequence $(\pi_{i_0}(\alpha_j))_{j \in J_\infty}$ coverges in X_{i_0}.

This convergence holds for each $i_0 \in I$. It is a basic property of product spaces that $(\alpha_j)_{j \in J_\infty}$ converges in $\prod X_i$. This establishes the sequential compactness for $\prod X_i$. $\square$

13.3 Uncountable products

Let $\underline{2}$ be the set $\{0, 1\}$ with the discrete topology. Note that $\underline{2}$ is both compact and sequentially compact. (Also note that if we use von Neumann's definition of ordinal numbers then $\underline{2}$ is just 2). Let $\Lambda = \underline{2}^{\mathbb{N}}$ be the set of functions $\mathbb{N} \to \underline{2}$. The goal of this section is to show that the Cartesian product $\underline{2}^\Lambda$ is not sequentially compact. It is in fact compact by Tychonoff's theorem (which we will not show here, but note that it uses the axiom of choice), so $\underline{2}^\Lambda$ gives an example of a sequentially compact space that is not compact. This will also establish that Λ is uncountable (a well-known fact, but this gives another proof) since countable products of sequentially compact spaces are sequentially compact.

Proposition 71. *The space $\underline{2}^\Lambda$ is not sequentially compact.*

Proof. We prove $\underline{2}^\Lambda$ is not sequentially compact by exhibiting a sequence in $\underline{2}^\Lambda$ with no converging subset. First observe that $\underline{2}^\Lambda$ can be thought of as the set of functions $\Lambda \to \underline{2}$. The sequence we exhibit is $(h_k)_{k \in \mathbb{N}}$ where $h_k \colon \Lambda \to \underline{2}$ is defined by the rule $\lambda \mapsto \lambda(k)$. Recall that elements of Λ are functions $\lambda \colon \mathbb{N} \to \underline{2}$, so this definition makes sense. We suppose $(h_k)_{k \in \mathbb{N}}$ has a convergent subsequence $(h_k)_{k \in I}$ and derive a contradiction.

Let $\lambda \in \Lambda$ be a given function $\lambda \colon \mathbb{N} \to \underline{2}$. Let $\pi_\lambda \colon \underline{2}^\Lambda \to \underline{2}$ be the λ-projection map. By continuity of π_λ the image of the converging sequence $(h_k)_{k \in I}$ gives a converging sequence $(\pi_\lambda(h_k))_{k \in I}$ in $\underline{2}$. Note that the λ-component of h_k is $h_k(\lambda)$, and recall that $h_k(\lambda) = \lambda(k)$. So the sequence $(\pi_\lambda(h_k))_{k \in I}$ is just $(\lambda_k)_{k \in I}$. Hence $(\lambda_k)_{k \in I}$ converges in $\underline{2}$. Since $\underline{2}$ is discrete, this just means that there exists an integer $N_\lambda \in I$ such that $\lambda_i = \lambda_{N_\lambda}$ for all $i \geq N_\lambda$ with $i \in I$. In other words, $(\lambda_k)_{k \in I}$ is eventually stable on I.

This stability on I would have to hold for all functions $\lambda \colon \mathbb{N} \to \underline{2}$. However, we can define a function λ such that (1) $\lambda(k) = 0$ for all $k \notin I$ and (2) the values of λ alternate on I. In other words, the sequence of values on I is $(0, 1, 0, 1, 0, 1, \ldots)$. This violates that stability claim, giving us our contradiction. $\square$